The number of independent sets in bicyclic graphs*

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Abstract: In this paper, we determine upper and lower bounds for the number of independent sets in a bicyclic graph in terms of its order. This gives an upper bound for the total number of independent sets in a connected graph which contains at least two cycles. In each case, we characterize the extremal graphs.

Key Words: Bicyclic graphs; Independent sets; Bounds; Fibonacci numbers

AMS subject classification: 05C69, 05C05

1. Introduction

Let G be a graph on n vertices. Two vertices of G are said to be *independent* if they are not adjacent in G. A *k-independent set* of G is a set of k-mutually independent vertices. Denote by $i(G,k)$ the number of the k-independent sets of G. For convenience, we regard the empty vertex set as an independent set. Then $i(G,0) = 1$ for any graph G. The total number of the independent sets of G, denoted by $i(G)$, is defined as

$$i(G) = \sum_{k=0}^{n} i(G,k).$$

The first papers about counting maximal independent sets in a graph are those of Miller and Muller [21] and Moon and Moser [22]. For a survey see [4, 5]. The set of independent sets in G is denoted by $I(G)$.

In chemical literature the graph parameter $i(G)$ is referred to as the *Merrifield-Simmon index*, it is one of the most popular topological indices in chemistry, which was extensively studied in a monograph [20]. There

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Merrifield and Simmons showed the correlation between this index and boiling points. After that some results on this topological index appeared (e.g., see, [3, 9–12]).

The total number of independent sets of a graph G is also called the *Fibonacci number* of the graph G. It was introduced in 1982 in a paper of Prodinger and Tichy [24]. Recently, there have been many papers studying the Fibonacci number for a graph. In [1], Alameddine studied bounds for the Fibonacci number of a maximal outer planar graph. Gutman [13], Zhang and Tian [29, 30] studied the Fibonacci number for hexagonal chains and catacondensed systems, respectively. In [14], Li et al., characterized the tree with the maximal Fibonacci number among the trees with a given diameter. In [15], Zhao and Li investigated the orderings of two classes of trees by their Fibonacci numbers, and used these orderings to determine the unique tree with the second (and respectively the third) smallest Fibonacci number among all trees with n vertices. In [23], Pedersen and Vestergaard studied the Fibonacci number for the unicyclic graphs. In [27], Yu and Tian studied the Fibonacci numbers of the graphs with given edge-independence number and cyclomatic number. Yu and Lv [19, 28] studied the Fibonacci numbers of trees with maximal degree and given pendent vertices, respectively. Ye et al., ordered the unicyclic graphs with given girth according to the Fibonacci numbers in [26].

The problem of counting the number of independent sets in a graph is NP-complete (see for instance Roth [25]). When dealing with a graph parameter for which the value is NP-complete to determine, it is often useful to find bounds for its values. Chou and Chang [6] gave an upper bound on the number of maximal independent set in graphs with at most one cycle. In 2005, Pedersent and Vestergarrd gave upper and lower bounds for the total number of independent sets in unicyclic graphs [23]. In this paper, we consider the total number of independent sets in bicyclic graphs. In particular, we prove that every bicyclic graph G on n vertices satisfies $5F_{n-3} \leqslant i(G) \leqslant 5 \cdot 2^{n-4} + 1$ for $n \geqslant 6$ and we characterize the extremal graphs for these inequalities, where F_{n-3} is the $(n-3)$-th Fibonacci number. We also determine the extremal bicyclic graphs on n vertices for $n = 4, 5$.

In order to state our results, we introduce some notation and terminology. Other undefined notation may refer to Bondy and Murty [2]. We only consider finite, undirected and simple graphs. For a vertex v of a graph G, we denote $N(v) = \{u|uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. A *pendent vertex* is a vertex of degree 1. A *bicyclic graph* is a connected graph with n vertices and $n+1$ edges.

If $W \subseteq V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, if $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively. We denote by P_n, C_n and $K_{1,n-1}$ the path, the cycle and the star on n vertices, respectively.

We list some lemmas that will be used in this paper.

Lemma 1.1 ([8]). *Let $G = (V, E)$ be a graph.*

(i) *If $uv \in E(G)$, then $i(G) = i(G - uv) - i(G - (N[u] \cup N[v]))$;*
(ii) *If $v \in V(G)$, then $i(G) = i(G - v) + i(G - N[v])$;*
(iii) *If $G_1, G_2, \ldots, G_t$ are the components of the graph G, then $i(G) = \prod_{j=1}^{t} i(G_j)$.*

Lemma 1.2 ([16]). *For any tree T on n vertices, $i(T) \geqslant F_{n+1}$, the equality holds if and only if $T \cong P_n$.*

Recall that $H_{n,k}$ is a k-cycle with $n - k$ leaves attached to one of its vertices.

Lemma 1.3 ([23]). *Let G denote a unicyclic graph with n vertices. If $G \ncong H_{n,3}$, then $i(G) \leqslant 5 \cdot 2^{n-4} + 2$. Equality occurs if and only if* (i) *$G \cong H_{n,4}$ or* (ii) *G is obtained from a C_3 by attaching one leaf to one of its vertices and $n - 4$ leaves to another of its vertices.*

2. Lemmas and main results

For a graph G, according to the definitions of $i(G)$, by Lemma 1.1, if v is a vertex of G, then $i(G) > i(G - v)$. In particular, when v is a pendent vertex of G and u is the unique vertex adjacent to v, we have $i(G) = i(G-v) + i(G - \{u, v\})$. So it is easy to see that $i(P_0) = 1, i(P_1) = 2$ and $i(P_n) = i(P_{n-1}) + i(P_{n-2})$ for $n \geqslant 2$. Denote by F_n the n-th Fibonacci number. Recall that $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 1$ and $F_1 = 1$. We have

$$i(P_n) = F_{n+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+2} \right].$$

Note that $F_{n+m} = F_n F_m + F_{n-1} F_{m-1}$, for convenience, we let $F_n = 0$, if $n < 0$. By Lemma 1.1, we can also obtain an important fact, i.e., $i(G) < i(G - e)$ for any $e \in E(G)$.

We shall in the following give both lower and upper bounds for the total number of independent sets in bicyclic graphs.

2.1. The lower bound for $i(G)$

Given integers n and k with $3 \leqslant k \leqslant n$, the *lollipop* $L_{n,k}$ is the unicyclic graph of order n obtained from the two vertices disjoint graphs C_k and P_{n-k} by adding an edge joining a vertex of C_k to an endpoint, say u, of P_{n-k}.

Lemma 2.1. $i(L_{n,k}) = F_{k-2}F_{n-k} + F_kF_{n-k+1}$.

Proof. By Lemma 1.1, in $L_{n,k}$ the number of independent sets which contain u is equal to $i(L_{n,k} - N[u]) = i(P_{k-3}) \cdot i(P_{n-k-1})$, and the number of independent sets of $L_{n,k}$ which do not contain u is equal to $i(L_{n,k} - u) = i(P_{k-1}) \cdot i(P_{n-k})$. Therefore,

$$i(L_{n,k}) = i(P_{k-3}) \cdot i(P_{n-k-1}) + i(P_{k-1}) \cdot i(P_{n-k}) = F_{k-2}F_{n-k} + F_kF_{n-k+1}.$$

This completes the proof. □

In C_n, choose two vertices such that the distance between them is 2, then connect them by an edge, denote the resulted graph by G_n^1. In $L_{n,4}$, there exist two non-adjacent vertices of degree 2 in the subgraph C_4, then connect them by an edge, denote the resulted graph by $G_n^{1'}$. G_n^1 and $G_n^{1'}$ are depicted in Figure 1.

Lemma 2.2. *If G is a bicyclic graph of order n which contains exactly three cycles, then $i(G) \geqslant 2F_{n-1}$ and equality occurs if and only if $G \cong G_n^1$ or $G \cong G_n^{1'}$.*

Proof. It is easy to see that

$$i(G_n^1) = i(G_n^{1'}) = 2F_{n-1}.$$

We apply induction on the order of the graph. The statement is easily verified for $n \in \{4, 5, 6, 7, 8\}$. Hence we may assume $n \geqslant 9$. Among the bicyclic graphs of order n with exactly three cycles, let G denote the one whose total number of independent vertex subsets is minimum.

Case 1. G is a bicyclic graph without trees attached.

Choose a cycle C_k with minimal length among the three cycles. If $k = 3$, then G is G_n^1 and we are done. If not, then each cycle of G has more than three vertices. Let w be a vertex of degree 3 in G, then choose two vertices $v, u \in V(G) \setminus V(C_k)$ such that v is adjacent to w and u is adjacent to v. By Lemma 1.1, the number of independent sets of G which contain u is equal to $i(G - N[u]) = i(L_{n-3,k})$, and the number of independent sets of G which do not contain u is equal to $i(G - u)$. Furthermore, the number of independent sets of $G - u$ which contain v is equal to $i(G - u - N[v])$, and the number of independent sets of $G - u$ which do not contain v is equal

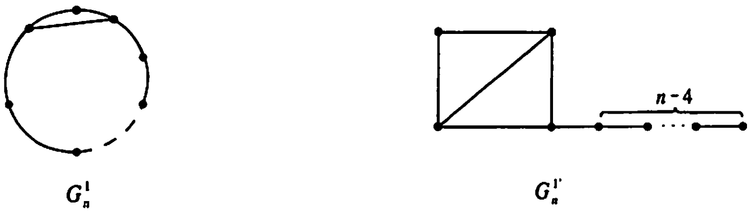

Figure 1: Graphs G_n^1 and $G_n^{1'}$.

to $i(G-u-v)=i(L_{n-2,k})$. By Lemma 1.2, $i(G-u-N[v])\geqslant i(P_{n-3})$. Thus,

$$\begin{aligned}
i(G) &\geqslant i(L_{n-3,k})+i(P_{n-3})+i(L_{n-2,k})\\
&= F_{k-2}F_{n-3-k}+F_kF_{n-3-k+1}+F_{n-2}+F_{k-2}F_{n-2-k}+F_kF_{n-2-k+1}\\
&= F_{k-2}F_{n-k-3}+F_kF_{n-k-2}+F_{n-2}+F_{k-2}F_{n-k-2}+F_kF_{n-k-1}\\
&= F_{k-2}F_{n-k-3}+F_{k-1}F_{n-k-2}+F_{k-2}F_{n-k-2}+F_{n-2}\\
&\quad +F_{k-2}F_{n-k-2}+F_{k-1}F_{n-k-1}+F_{k-2}F_{n-k-1}\\
&= F_{n-3}+F_{k-2}F_{n-k-2}+F_{n-2}+F_{n-2}+F_{k-2}F_{n-k-1}\\
&= F_n+F_{k-2}F_{n-k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
i(G)-2F_{n-1} &\geqslant F_n+F_{k-2}F_{n-k}-2F_{n-1}\\
&= F_{k-2}F_{n-k}-F_{n-3}\\
&= F_{k-3}F_{n-k}+F_{k-4}F_{n-k}-F_{n-k+k-3}\\
&= F_{k-3}F_{n-k}+F_{k-4}F_{n-k}-F_{n-k}F_{k-3}-F_{n-k-1}F_{k-4}\\
&= F_{k-4}F_{n-k-2}.
\end{aligned}$$

Note that $k\geqslant 4$ and $k\leqslant n-2$ is obvious, otherwise it is a contradiction to the assumption that C_k is minimal. Therefore, $F_{k-4}F_{n-k-2}>0$, i.e., $i(G)>2F_{n-1}$.

Case 2. G is a bicyclic graph with trees attached.

Choose two cycles, say C_{k_1}, C_{k_2}, in G. Let x denote a vertex of G having maximum distance to $C_{k_1}\cup C_{k_2}$.

Subcase 2.1. Suppose that $\mathrm{dist}(C_{k_1}\cup C_{k_2},x)\geqslant 2$. The number of independent sets of G which contain x is equal to $i(G-N[x])$. The maximality of $\mathrm{dist}(C_{k_1}\cup C_{k_2},x)$ and the assumption that $\mathrm{dist}(C_{k_1}\cup C_{k_2},x)\geqslant 2$ imply that $G-N[x]$ consists of one component with precisely three cycles and possibly a number of isolated vertices, say $G-N[x]=H\cup\overline{K}_s$, where H is a bicyclic graph of order $n-2-s$. By induction on n,

$i(G-N[x]) \geqslant 2^s \cdot 2F_{n-3-s} \geqslant 2F_{n-3}$. In fact, we have $i(G-N[x]) > 2F_{n-3}$ if $s \geqslant 1$. The number of independent sets of G which do not contain x is equal to $i(G-x)$ and by induction on n, $i(G-x) \geqslant 2F_{n-2}$. Together these two inequalities imply $i(G) \geqslant 2F_{n-3} + 2F_{n-2} = 2F_{n-1}$. If $i(G) = 2F_{n-1}$, then we must have $s = 0$, $i(G-x) = 2F_{n-2}$ and $i(G-N[x]) = 2F_{n-3}$. Moreover, since $G-x$ is not the graph G^1_{n-1}, the induction on n implies that $G - x \cong G^{1'}_{n-1}$ and consequently $G \cong G^{1'}_n$.

Subcase 2.2. Assume $\text{dist}(C_{k_1} \cup C_{k_2}, x) = 1$. Then we shall show that this assumption leads to a contradiction. Let $|V(C_{k_1})| \geqslant |V(C_{k_2})|$, then it suffices to consider the following three cases.

(i) Suppose that some vertex on a cycle of G has more than one leaf attached, say that v_1 has at least two leaves x and y. For convenience, let $N(v_1) \cap C_{k_1} = \{v_2, v_k\}$. Define $H := (G - \{v_1y, v_1v_2\}) \cup \{xy, yv_2\}$. Now H is a bicyclic graph on n vertices. We define a mapping ϕ from $I(H)$ to $I(G)$. Let B denote an independent set in H and let $\phi(B)$ be defined by Table 1. The number beneath each vertex indicates whether or not the vertex is considered to be in the independent set B. For instance, the second row reads 0001, which meas that v_2 is in B while neither v_1, x nor y is in B.

Table 1: Definition of the mapping $\phi : I(H) \to I(G)$.

v_1	x	y	v_2	$\phi(B)$
0	0	0	0	B
0	0	0	1	B
0	0	1	0	B
0	1	0	0	B
1	0	0	0	B
0	1	0	1	B
1	0	0	1	$(B - \{v_1\}) \cup \{x, y\}$
1	0	1	0	$(B - \{v_1\}) \cup \{x\}$

The mapping ϕ is injective. Moreover, $\{x, y, v_k\} \in I(G)$, but there exists no independent set $B \in I(H)$ with $\phi(B) = \{x, y, v_k\}$. Hence ϕ is also non-surjective. It follows that $i(G) > i(H)$, which contradicts the minimality of $i(G)$.

(ii) Suppose that every vertex on the cycles of G either has exactly one leaf attached or has no leaf attached. We may w.l.o.g. assume that G has three succeeded vertices, say v_1, v_2, v_3 on a cycle such that v_1 has no leaf attached while one of its neighbor v_2 has exactly one leaf attached, say x. Define $H := (G - \{v_1v_2\}) \cup \{xv_1\}$. The graph H has order n and is bicyclic. We define a mapping ϕ from $I(H)$ to $I(G)$. Let B denote an independent set in H and let $\phi(B)$ be defined by Table 2.

Table 2: Definition of the mapping $\phi : I(H) \to I(G)$.

v_1	x	v_2	$\phi(B)$
0	0	0	B
0	0	1	B
0	1	0	B
1	0	0	B
1	0	1	$(B - \{v_2\}) \cup \{x\}$

The mapping ϕ is injective. It is easy to see that $\{x, v_1, v_3\} \in I(G)$. But there exists no independent set $B \in I(H)$ with $\phi(B) = \{x, v_1, v_3\}$. Hence ϕ is also non-surjective. It follows that $i(G) > i(H)$, which contradicts the minimality of $i(G)$.

(iii) Suppose that every vertex on the cycles of G has exactly one leaf attached. We may assume that G has three succeeded vertices, say v_1, v_2, v_3, on a cycle such that v_1 has exactly one leaf attached say x while v_2 is y. Define $H := (G - \{v_1v_2\}) \cup \{xy\}$. The graph H has order n and is bicyclic. We define a mapping ϕ from $I(H)$ to $I(G)$. Let B denote an independent set in H and let $\phi(B)$ be defined by Table 3.

Table 3: Definition of the mapping $\phi : I(H) \to I(G)$.

v_1	x	y	v_2	$\phi(B)$
0	0	0	0	B
0	0	0	1	B
0	0	1	0	B
0	1	0	0	B
1	0	0	0	B
0	1	0	1	B
1	0	0	1	$(B - \{v_1, v_2\}) \cup \{x, y\}$
1	0	1	0	B

The mapping ϕ is injective. It is easy to see that $\{x, y, v_3\} \in I(G)$. But there exists no independent set $B \in I(H)$ with $\phi(B) = \{x, y, v_3\}$. Hence ϕ is also non-surjective. It follows that $i(G) > i(H)$, which contradicts the minimality of $i(G)$.

This completes the proof. □

Lemma 2.3. *If G is a unicyclic graph of order n obtained from the two vertices disjoint graphs C_k and T by adding an edge joining a vertex of C_k to a vertex of T, where T is a tree on $(n-k)$-vertex. Then $i(G) \geqslant i(L_{n,k})$ and equality holds if and only if $G \cong L_{n,k}$.*

Proof. Let $e = vw$ be the edge connecting C_k and T to create the graph G, then v is a vertex of C_k and w is a vertex of T. Note that the number of independent sets of G which contain v is equal to $i(G - N[v]) = i(P_{k-3}) \cdot i(T - w)$ and the number of independent sets of G which do not contain v is equal to $i(G - v) = i(P_{k-1}) \cdot i(T)$. Then

$$i(G) = i(P_{k-3})i(T - w) + i(P_{k-1})i(T).$$

By Lemma 2.1,

$$i(L_{n,k}) = i(P_{k-3})i(P_{n-k-1}) + i(P_{k-1})i(P_{n-k}).$$

By lemma 1.2, we know $i(T - w) \geqslant i(P_{n-k-1})$ and $i(T) \geqslant i(P_{n-k})$, then $i(G) \geqslant i(L_{n,k})$, the equality holds if and only if $G \cong L_{n,k}$.

This completes the proof. □

We show in the following that G_n^0 (as shown in Figure 2) is the bicyclic graph on n vertices whose total number of independent sets is minimal.

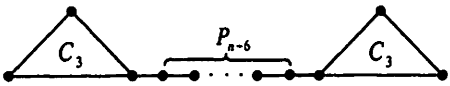

Figure 2: Graph G_n^0

Lemma 2.4. *Let G be a bicyclic graph with n vertices having exactly two cycles, say C_{k_1}, C_{k_2}. If both C_{k_1} and C_{k_2} have no trees attached, then $i(G) \geqslant i(G_n^0)$ and equality holds if and only if $G \cong G_n^0$.*

Proof. It is easy to see that there exist exactly one vertex of degree 3 on each cycle of G, choose one of such vertex, say w in C_{k_2}. Choose other two vertices v, u in $V(C_{k_2})$ such that w, v, u are three succeeded vertices in C_{k_2}. Then the number of independent sets of G which contain u is equal to $i(G - N[u]) \geqslant i(L_{n-3,k_1})$, the equality holds if and only if $G - N[u] \cong L_{n-3,k_1}$. The number of independent sets of G which do not contain u is equal to $i(G - u)$. By lemma 2.3, $i(G - u) \geqslant i(L_{n-1,k_1})$, the equality holds if and only if $G - u \cong L_{n-1,k_1}$, i.e., $k_2 = 3$ and no trees are attached to the path connecting C_{k_1} and C_{k_2}. And so

$$\begin{aligned} i(G) &\geqslant i(L_{n-3,k_1}) + i(L_{n-1,k_1}) \qquad (2.1)\\ &= F_{k_1}F_{n-3-k_1+1} + F_{k_1-2}F_{n-3-k_1} + F_{k_1}F_{n-1-k_1+1} + F_{k_1-2}F_{n-1-k_1}\\ &= F_{k_1}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1-3} + F_{k_1}F_{n-k_1} + F_{k_1-2}F_{n-k_1-1}\\ &= F_{k_1-1}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1-3} + F_{k_1-1}F_{n-k_1}\\ &\quad +F_{k_1-2}F_{n-k_1} + F_{k_1-2}F_{n-k_1-1}\\ &= F_{n-3} + F_{k_1-2}F_{n-k_1-2} + F_{n-1} + F_{k_1-2}F_{n-k_1}\\ &= 3F_{n-3} + F_{n-4} + F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1}, \end{aligned}$$

the equality holds in (2.1) if and only if $k_2 = 3$ and no trees are attached to the path connecting C_{k_1} and C_{k_2}.

On the other hand, we can, similarly, obtain

$$i(G_n^0) = i(L_{n-4,3}) + i(P_2)i(L_{n-3,3}) = 5F_{n-3}. \qquad (2.2)$$

Hence

$$\begin{aligned} i(G) - i(G_n^0) &= i(G) - 5F_{n-3}\\ &\geqslant 3F_{n-3} + F_{n-4} + F_{k_1-2}F_{n-k_1-2} \qquad (2.3)\\ &\quad +F_{k_1-2}F_{n-k_1} - 5F_{n-3}\\ &= F_{n-4} + F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1} - 2F_{n-3}\\ &= F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1} - F_{n-3} - F_{n-5}\\ &= F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1} - F_{n-k_1-1+k_1-2} - F_{n-5}\\ &= F_{k_1-2}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1} - F_{n-k_1-1}F_{k_1-2}\\ &\quad -F_{n-k_1-2}F_{k_1-3} - F_{n-5}\\ &= F_{k_1-4}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1-2} - F_{n-k_1-2+k_1-3}\\ &= F_{k_1-4}F_{n-k_1-2} + F_{k_1-2}F_{n-k_1-2} - F_{n-k_1-2}F_{k_1-3}\\ &\quad -F_{n-k_1-3}F_{k_1-4}\\ &= F_{k_1-4}(F_{n-k_1-4} + F_{n-k_1-2}). \end{aligned}$$

The equality holds in (2.3) if and only if $k_2 = 3$ and no trees are attached to the path connecting C_{k_1} and C_{k_2}. Note that $F_{k_1-4} \geqslant 0, F_{n-k_1-4} \geqslant 0$ and $n - k_1 - 2 \geqslant 0$, we have $F_{n-k_1-4} + F_{n-k_1-2} > 0$. Hence

$$F_{k_1-4}(F_{n-k_1-4} + F_{n-k_1-2}) \geqslant 0,$$

the equality holds if and only if $k_1 - 4 < 0$, i.e., $k_1 = 3$. Thus, we obtain $i(G) \geqslant i(G_n^0)$, the equality holds if and only if $k_1 = k_2 = 3$ and no trees are attached to the path connecting C_{k_1} and C_{k_2}, i.e., $G \cong G_n^0$. □

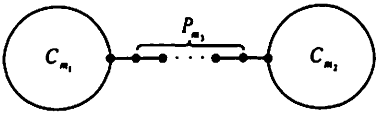

Figure 3: The arrangement of two cycles in G

Lemma 2.5. *Let G be a bicyclic graph with n vertices having exactly two cycles, then $i(G) \geqslant 5F_{n-3}$ and equality occurs if and only if $G \cong G_n^0$.*

Proof. By (2.2),

$$i(G_n^0) = 5F_{n-3}.$$

By Lemma 2.4 we should only consider the bicyclic graph with trees attached to at least one of its cycles. We apply induction on the order of the graph. The statement is easily verified for $n \in \{5, 6, 7, 8\}$. Hence we may assume $n \geqslant 9$. Among the bicyclic graphs of order n with exactly two cycles, let G denote the one whose total number of independent vertex subsets is minimum.

Assume that the two cycles in G are C_{m_1}, C_{m_2}, and C_{m_1} connects C_{m_2} by a path; see Figure 3. Let x denote a leaf of some attached trees of C_{m_1} or C_{m_2} such that it has maximal distance to $C_{m_1} \cup C_{m_2}$.

Case 1. Suppose that $\mathrm{dist}(C_{m_1} \cup C_{m_2}, x) \geqslant 2$. The number of independent sets of G which contain x is equal to $i(G - N[x])$. The maximality of $\mathrm{dist}(C_{m_1} \cup C_{m_2}, x)$ and the assumption that $\mathrm{dist}(C_{m_1} \cup C_{m_2}, x) \geqslant 2$ imply that $G - N[x]$ consists of one component with precisely two cycles and possibly a number of isolated vertices, say $G - N[x] = H \cup \overline{K}_s$, where H is a bicyclic graph of order $n - 2 - s$. By induction on n, $i(G - N[x]) > 2^s \cdot (5F_{n-5-s}) \geqslant 5F_{n-5}$. The number of independent sets of G which do not contain x is equal to $i(G - x)$ and by induction on n, $i(G - x) > 5F_{n-4}$. Together these two inequalities imply $i(G) > 5F_{n-3}$.

Case 2. Assume $\mathrm{dist}(C_{m_1} \cup C_{m_2}, x) = 1$. The proof of Case 2 is almost the same as that of Lemma 2.2, so we omit the procedure here.

From above we obtain that if the bicyclic graph G with trees attached to at least one of its cycles, then $i(G) > 5F_{n-3}$. By Lemma 2.4, for any bicyclic graph G with n vertices, $i(G) \geqslant 5F_{n-3}$ and equality holds if and only if $G \cong G_n^0$. □

Lemma 2.6. $i(G_n^1) \geqslant i(G_n^0), \quad i(G_n^{1'}) \geqslant i(G_n^0)$.

Proof. Note that $i(G_n^1) = 2F_{n-1}, i(G_n^0) = 5F_{n-3}$, and so

$$i(G_n^1) - i(G_n^0) = 2F_{n-1} - 5F_{n-3} = F_{n-6} \geqslant 0.$$

It is easy to check that $i(G_n^1) = i(G_n^0)$ when $n = 5$ and $i(G_n^1) > i(G_n^0)$ for $n \geqslant 6$. Similarly, we can also show that $i(G_n^{1'}) \geqslant i(G_n^0)$. □

Summarizing Lemmas 2.2, 2.5 2.6 and the fact that there exists exactly one bicyclic graph on 4 vertices, we arrive at:

Theorem 2.7. *Let G be a bicyclic graph of order n.*

(a) *If $n = 4$, then G_4^1 is the unique bicyclic graph and $i(G_4^1) = 6$.*
(b) *If $n = 5$, then $i(G) \geqslant 10$ and equality holds if and only if $G \in \{G_5^0, G_5^1, G_5^{1'}\}$.*
(c) *If $n \geqslant 6$, then $i(G) \geqslant 5F_{n-3}$, the equality holds if and only if $G \cong G_n^0$.*

2.2. The upper bound for $i(G)$

In this subsection, we shall determine the upper bound for $i(G)$, although the upper bound has been obtained in [7], our proof is much more concise than that in [7].

Let G_n^* be a bicyclic graph of order n formed by adding two edges to join one leaf to other two different leaves of $K_{1,n-1}$. G_n^* is depicted in Figure 4.

Lemma 2.8. $i(G_n^*) = 5 \cdot 2^{n-4} + 1$.

Proof. Denote the vertex of degree $n-1$ in G_n^* by u. Then the number of independent sets of G_n^* which contain u is equal to 1 and the number of independent sets of G_n^* which do not contain u is equal to $i(G_n^* - u) = i(P_3) \cdot 2^{n-4} = 5 \cdot 2^{n-4}$. Therefore, $i(G) = 5 \cdot 2^{n-4} + 1$. □

Theorem 2.9. *Let G be a bicyclic graph of order n.*

(a) *If $n = 5$, then $i(G) \leqslant 11$ and the equality holds if and only if $G \in \{B_{5,2}, G_5^*\}$.*
(b) *If $n \geqslant 6$, then $i(G) \leqslant 5 \cdot 2^{n-4} + 1$ and equality occurs if and only if $G \cong G^*$.*

Proof. (a) When $n = 5$, the set of tricyclic graphs with 5 vertices is $\{G_4^*, B_{5,1}, B_{5,2}, B_{5,4}, B_{5,6}\}$. By directed calculation, we obtain $i(G) \leqslant 11$ and the equality holds if and only if $G \in \{B_{5,2}, G_5^*\}$.

(b) If $G = G_n^*$, then by Lemma 2.8 $i(G) = 5 \cdot 2^{n-4} + 1$ and we are done. Otherwise, we distinguish two cases to prove this result.

Case 1. The longest length of the cycle in G is no less than 4. Then choose an edge $e \in E(G)$ such that $G - e$ contains exactly one cycle of length at least 4. By Lemma 1.3 we have

$$i(G - e) \leqslant i(H_{n,4}).$$

Note that either $G - e \ncong H_{n,4}$, or $G - e \cong H_{n,4}$. If $G - e \ncong H_{n,4}$, then

$$i(G) < i(G - e) < i(H_{n,4}) = 5 \cdot 2^{n-4} + 2,$$

that is to say, $i(G) < 5 \cdot 2^{n-4} + 1$.

If $G - e \cong H_{n,4}$, then $G \in \{B_{n,1}, B_{n,2}, B_{n,3}, B_{n,4}\}$; see Figure 4. By

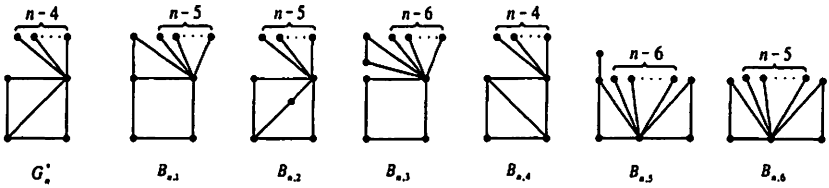

Figure 4: Graphs $G_n^*, B_{n,1}, B_{n,2}, B_{n,3}, B_{n,4}, B_{n,5}$ and $B_{n,6}$.

directed calculation, we obtain

$$i(B_{n,1}) = 4 \cdot 2^{n-4} + 2, \quad i(B_{n,2}) = 9 \cdot 2^{n-5} + 2,$$
$$i(B_{n,3}) = 15 \cdot 2^{n-6} + 2, \quad i(B_{n,4}) = 4 \cdot 2^{n-4} + 2.$$

Therefore, $i(G) < 5 \cdot 2^{n-4} + 1$ for G in Case 1 and $G \ncong G_n^*$.

Case 2. The longest length of the cycle in G is 3. Then G possesses exactly two cycles, each of which is of length 3. Choose an edge $e \in E(G)$ such that $G - e$ is unicyclic. If $G - e \cong H_{n,3}$, then $G \cong B_{n,6}$. Hence,

$$i(G) = i(B_{n,6}) = 9 \cdot 2^{n-5} + 1 < 5 \cdot 2^{n-4} + 1.$$

If $G - e \ncong H_{n,3}$, then by (ii) of Lemma 1.3 and the assumption of G, we have

$$i(G - e) \leqslant i(\tilde{G}),$$

where $\tilde{G}$ is obtained from a C_3 by attaching one leaf to one of its vertices and $n - 4$ leaves to another of its vertices.

If $G - e \ncong \tilde{G}$, then

$$i(G) < i(G - e) < i(\tilde{G}) = 5 \cdot 2^{n-4} + 2,$$

that is to say, $i(G) < 5 \cdot 2^{n-4} + 1$.

If $G - e \cong \tilde{G}$, then together with the assumption of G we obtain $G \cong B_{n,5}$; see Figure 4. So we obtain

$$i(G) = i(B_{n,5}) = 15 \cdot 2^{n-6} + 2 < 5 \cdot 2^{n-4} + 1.$$

By Cases 1 and 2, we completes the proof. □

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