# On bipartite factorization of complete bipartite multigraphs*

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**Abstract**

Let $\lambda K_{m,n}$ be a complete bipartite multigraph with two partite sets having $m$ and $n$ vertices, respectively. A $K_{p,q}$-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint $K_{p,q}$-factors of $\lambda K_{m,n}$ which is a partition of the set of edges of $\lambda K_{m,n}$. When $\lambda = 1$, Martin, in paper [Complete bipartite factorisations by complete bipartite graphs, Discrete Math., 167/168 (1997), 461-480], gave simple necessary conditions for such a factorization to exist, and conjectured those conditions are always sufficient. In this paper, we will give similar necessary conditions for $\lambda K_{m,n}$ to have a $K_{p,q}$-factorization, and prove the necessary conditions are always sufficient in many cases.

**Keywords**: Complete bipartite graph, complete bipartite multigraph, factorization

## 1 Introduction

Let $K_{m,n}$ be a complete bipartite graph with two partite sets having $m$ and $n$ vertices, respectively. $\lambda K_{m,n}$ is the disjoint union of $\lambda$ graphs, each of which is isomorphic to $K_{m,n}$. A subgraph $F$ of $\lambda K_{m,n}$ is called a spanning subgraph of $\lambda K_{m,n}$ if $F$ contains all the vertices of $\lambda K_{m,n}$. Let $p$ and $q$ are positive integers. A $K_{p,q}$-factor of $\lambda K_{m,n}$ is a spanning subgraph $F$ of $\lambda K_{m,n}$ such that every component of $F$ is a $K_{p,q}$ and every pair of $K_{p,q}$ has no vertex in common. A $K_{p,q}$-factorization of $\lambda K_{m,n}$ is a set of edge-disjoint $K_{p,q}$-factors of $\lambda K_{m,n}$ which is a partition of the set of edges

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of $\lambda K_{m,n}$. The graph $\lambda K_{m,n}$ is called $K_{p,q}$-factorizable whenever it has a $K_{p,q}$-factorization. In paper [13] a $K_{p,q}$-factorization of $\lambda K_{m,n}$ is defined as a resolvable $(m, n, p+q, \lambda)$ $K_{p,q}$-design. For graph theoretical terms, see [1].

The $K_{p,q}$-factorization of a complete bipartite multigraph $\lambda K_{m,n}$ has been studied by many researchers. When $\lambda = 1$, Martin, in paper [6], investigated the $K_{p,q}$-factorization of $K_{m,n}$ and gave simple necessary conditions for such a factorization to exist and conjectured that the conditions are also sufficient. Martin's conjecture, from then on, has drawn focus from many researchers. Martin's conjecture has been proved in many cases. The case $p = 1$ and $q = 2$ was first proved in [12]. The case $p = 2$ and $q = 3$ was proved in [16] and for $p = 1$ and $q = 3$ in [10]. The balanced case $(m = n)$ was proved in a series of papers [6, 7, 8, 9]. The general case for $q = p + 1$ was solved in [11, 5]. Very recently, Martin [11] showed that the conjecture is true when $gcd(q - p, x + y) = 1$.

In this paper, we pay attention to the existence for the $K_{p,q}$-factorization of a complete bipartite multigraph $\lambda K_{m,n}$. Assume that a $K_{p,q}$-factorization of $\lambda K_{m,n}$ is given, certain integers are defined as followings:

$t =$ the number of copies of $K_{p,q}$ in any factor,

$x =$ the number of copies of $K_{p,q}$ with its partite set of size $p$ in $X$ in a particular $K_{p,q}$-factor,

$y =$ the number of copies of $K_{p,q}$ with its partite set of size $p$ in $Y$ in a particular $K_{p,q}$-factor,

$f =$ the number of $K_{p,q}$-factors in the factorization,

$r_1 =$ the number of $K_{p,q}$-factors which contribute $p$ edges for any vertex $v$ in $X$,

$s_1 =$ the number of $K_{p,q}$-factors which contribute $q$ edges for any vertex $v$ in $X$,

$r_2 =$ the number of $K_{p,q}$-factors which contribute $p$ edges for any vertex $v$ in $Y$,

$s_2 =$ the number of $K_{p,q}$-factors which contribute $q$ edges for any vertex $v$ in $Y$.

Similar to Theorem 2.5 in Ref [6], we give the necessary conditions for $K_{p,q}$-factorization of $\lambda K_{m,n}$.

**Theorem 1.1** Let $\lambda, p, q, m$ and $n$ be positive integers with $pq \neq 1$. If

$\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then the following expressions are all integers:

$$t = \frac{m+n}{p+q}, x = \frac{pm-qn}{p^2-q^2}, y = \frac{pn-qm}{p^2-q^2}, f = \frac{\lambda mn(p+q)}{pq(m+n)},$$
$$r_1 = \frac{\lambda n(pn-qm)}{p(p-q)(m+n)}, r_2 = \frac{\lambda m(pm-qn)}{p(p-q)(m+n)},$$
$$s_1 = \frac{\lambda n(pm-qn)}{q(p-q)(m+n)}, s_2 = \frac{\lambda m(pn-qm)}{q(p-q)(m+n)}.$$

In [15] Wang and Du proved these conditions are sufficient when $p = 1$ and $q = 2$. Some further work was done in [2, 3, 4]. The aim of this paper is to prove these necessary conditions are sufficient when $gcd(q-p, x+y) = 1$.

# 2 Preliminaries

In this section we shall give some preliminary results. The strategy and methods in this paper are similar to paper [11]. We first need the following Lemmas whose proof are easy.

**Lemma 2.1** If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $s\lambda K_{m,n}$ has a $K_{p,q}$-factorization for any positive integer $s$.

**Lemma 2.2** If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $\lambda K_{ms,ns}$ has a $K_{p,q}$-factorization for any positive integer $s$.

Combining Lemmas 2.1 and 2.2, we have a corollary as follows.

**Corollary 2.3** $\lambda K_{sp,sq}$ has a $K_{p,q}$-factorization for any positive integer s.

**Lemma 2.4** If $\lambda K_{m,n}$ has a $K_{p,q}$-factorization, then $\lambda K_{ms,ns}$ has a $K_{ps,qs}$-factorization for any positive integer s.

Given $p, q, m$ and $n$, there is a least integer $\lambda_0$ satisfying necessary conditions with respect to $K_{p,q}$-factorization of $\lambda K_{m,n}$. We say $\lambda_0$ the base case.

**Lemma 2.5** Given $p, q, m$ and $n$, any integer $\lambda$ satisfying necessary conditions with respect to $K_{p,q}$-factorization of $\lambda K_{m,n}$ is an integer multiple of the base case $\lambda_0$.

**Proof** Let $\lambda_0$ be the base case of $\lambda$. Then if $\lambda$ is another integer satisfying necessary conditions, there will be some rational number $k$ for which $\lambda = k\lambda_0$. Suppose $k$ is not an integer, then we can define $\lambda_1 = \lambda - \lfloor k \rfloor \lambda_0$ so that $0 < \lambda_1 < \lambda_0$. Consider

$$f = \lambda mn(p+q)/[pq(m+n)]$$

which is an integer. Similarly we have

$$f_0 = \lambda_0 mn(p+q)/[pq(m+n)]$$

is an integer. Hence, writing $k = \lfloor k \rfloor + \alpha$ where $\alpha$ is a rational between 0 and 1,

$$\begin{aligned} f &= \lambda mn(p+q)/[pq(m+n)] \\ &= (\lambda_1 + \lfloor k \rfloor \lambda_0) mn(p+q)/[pq(m+n)] \\ &= \lambda_1 mn(p+q)/[pq(m+n)] + \lfloor k \rfloor \lambda_0 mn(p+q)/[pq(m+n)] \\ &= \lambda_1 mn(p+q)/[pq(m+n)] + \lfloor k \rfloor f_0. \end{aligned}$$

Thus $\lambda_1 mn(p+q)/[pq(m+n)]$ must be an integer. Similar arguments show that all other necessary conditions are also verifiable for $\lambda_1$. Since $0 < \lambda_1 < \lambda_0$, this contradicts the minimality of $\lambda_0$. □

Assume $p < q$. From the expressions $x$ any $y$ in Theorem 1.1 we may assume $pm \leq qn$ and $pn \leq qm$, and Corollary 2.3 implies that $\lambda K_{m,n}$ has a $K_{p,q}$-factorization when $pm = qn$ and $pn = qm$. So we only need to treat the case $pm < qn$ and $pn < qm$. In this case, we use the ratio $x : y = \alpha : \beta$ to class $m$ and $n$ satisfying necessary conditions. For any fixed ratio $\alpha : \beta$ there is a least pair of integers $m_0, n_0$ satisfying necessary conditions. We call this $m_0, n_0$ the base pair for the ratio $\alpha : \beta$.

The proof of Lemma 2.6 and Lemma 2.7 below is similar to that of Theorem 2.8 and Theorem 2.11 in [6], respectively.

**Lemma 2.6** Given a fixed ratio $x : y$, any pair of integers $m$ and $n$ satisfying necessary conditions is an integer multiple of the base pair for that ratio.

**Lemma 2.7** Let $k, p$ and $q$ be positive integers where $p$,$q$ are coprime. If $m_0, n_0$ is the base pair for some ratio $x : y$ with respect to $K_{p,q}$-factorization of $\lambda K_{m,n}$, then $km_0, kn_0$ is the base pair for some ratio $x : y$ with respect to $K_{kp,kq}$-factorization of $\lambda K_{m,n}$.

Lemmas 2.5-2.7 imply that we only need consider the case $p$ and $q$ are coprime, the base case $\lambda_0$ and the base pair $m_0, n_0$. Now we discuss how to calculate $m_0, n_0$ and $\lambda_0$ with fixed case.

The necessary conditions for $K_{p,q}$-factorization of $\lambda K_{m,n}$ can be rewritten in terms of $x, y, p$ and $q$ as follows:

$$t = x + y,$$

$$\begin{aligned} f &= \lambda x + \lambda y + \lambda(p-q)^2 xy/[pq(x+y)], \\ r_1 &= \lambda y - \lambda(p-q)xy/[p(x+y)], \\ s_1 &= \lambda x + \lambda(p-q)xy/[q(x+y)], \\ r_2 &= \lambda x - \lambda(p-q)xy/[p(x+y)], \\ s_2 &= \lambda y + \lambda(p-q)xy/[q(x+y)]. \end{aligned}$$

The following is then indicated:

**Lemma 2.8** Let $p < q$ and $x, y$ be coprime pairs of positive integers ($p \neq q$), the base pair for the $K_{p,q}$-factorization of $\lambda K_{m,n}$ with balance ratio $x : y$ is given by $m = d(qx + py)$ and $n = d(px + qy)$, where $d$ is the denominator of the (reduced) fraction $\lambda(q-p)xy/[pq(x+y)]$.

We set $p_1 = gcd(p, x)$, $p_2 = gcd(p, y)$, $q_1 = gcd(q, x)$ and $q_2 = gcd(q, y)$ so that $p = p_0p_1p_2$, $q = q_0q_1q_2$, $x = p_1q_1x_0$ and $y = p_2q_2y_0$. From the assumption that $gcd(p, q) = gcd(x, y) = 1$, it is straightforward to show that the quantities $p_1, p_2, q_1, q_2, x_0, y_0, p_0, q_0$ are all pairwise coprime apart from possibly $(p_1, x_0), (q_1, x_0), (p_2, y_0), (q_2, y_0), (p_i, p_0)$ and $(q_i, q_0)$ $(i = 1, 2)$. Then $\lambda(q-p)xy/[pq(x+y)] = \lambda(q-p)x_0y_0/[p_0q_0(x+y)]$. If gcd $(q-p, x+y) = 1$, we get the denominator of the (reduced) fraction $\lambda(q-p)xy/[pq(x+y)]$ is

$$d = \frac{p_0q_0(x+y)}{gcd(\lambda, p_0q_0(x+y))}.$$

By simple calculation, the base pair $m_0, n_0$ is as following:

$$m_0 = \frac{p_0q_0(x+y)(qx+py)}{gcd(\lambda, p_0q_0(x+y))},$$

$$n_0 = \frac{p_0q_0(x+y)(px+qy)}{gcd(\lambda, p_0q_0(x+y))}.$$

Thus the base case of $\lambda$ is $\lambda_0 = gcd(\lambda, p_0q_0(x+y))$, since $\lambda_0$, $m_0$ and $n_0$ satisfy the necessary conditions for $K_{p,q}$-factorization, and any integer $\lambda$ $(0 < \lambda < \lambda_0)$ doesn't satisfy.

The construction for our main result is similar to Martin's (factor matrix).

**Lemma 2.9** A $K_{p,q}$-factorization of $\lambda K_{m,n}$ with $f$ $K_{p,q}$-factors, equates to an $m \times n$ array in which

(1) each cell contains $\lambda$ distinct symbols in $\{1, 2, \cdots, f\}$,

(2) each symbol in $\{1, 2, \cdots, f\}$ occurs in every column and every row, and

(3) those cells with the integer $i$, say, correspond to the edges in the $i$-th $K_{p,q}$-factor.

We call this array the factor array of the $K_{p,q}$-factorization. An example factor array is shown in table 1.

| | | |
|---|---|---|
| 1,4,8,9 | 2,5,7,9 | 3,6,7,8 |
| 1,5,6,7 | 2,4,6,8 | 3,4,5,9 |
| 2,3,4,7 | 1,3,5,8 | 1,2,6,9 |

Table 1: the $K_{1,2}$-factorization of $4K_{3,3}$

# 3 Main Result

**Theorem 3.1** Given coprime pairs $(p, q)$ and $(x, y)$. The necessary conditions in Theorem 1.1 are sufficient when $gcd(q-p, x+y) = 1$.

**Proof** Without loss of generality, we assume that $p < q$, with $gcd(p, q) = gcd(x, y) = 1$. As before, set $p = p_0p_1p_2$, $q = q_0q_1q_2$, $x = p_1q_1x_0$ and $y = p_2q_2y_0$, where $p_1 = gcd(p, x)$, $p_2 = gcd(p, y)$, $q_1 = gcd(q, x)$ and $q_2 = gcd(q, y)$, then we have a least integer $\lambda_0 = gcd(\lambda, p_0q_0(x+y))$ and the base case

$$m_0 = \frac{1}{\lambda_0}p_0q_0(x+y)(qx+py) = \frac{1}{\lambda_0}p_0q_0(x+y)p_1q_2\mu,$$

where $\mu = q_0 q_1^2 x_0 + p_0 p_2^2 y_0$ and

$$n_0 = \frac{1}{\lambda_0} p_0 q_0 (x+y)(px+qy) = \frac{1}{\lambda_0} p_0 q_0 (x+y) p_2 q_1 \nu,$$

where $\nu = p_0 p_1^2 x_0 + q_0 q_2^2 y_0$. The factor size is $p_0 q_0 (x+y)^2 pq/\lambda_0$, and the number of factors is $f = \mu\nu$.

The details of the construction for $K_{p,q}$-factorization of $\lambda K_{m,n}$ are similar to Martin's (Theorem 6 in [11]). So we only get the amendments and show that these factor pieces satisfy our required.

**The definition of the vertical pieces**

Let $J$ be a $q_2\mu \times p_2\nu$ matrix with general term $J_{\alpha\beta}$. We can express $\alpha = (a-1)q_2 + c$ and $\beta = (b-1)p_2 + d$ uniquely for $a, b, c, d$ where $1 \le a \le \mu$, $1 \le b \le \nu$, $1 \le c \le q_2$ and $1 \le d \le p_2$. Set $J_{\alpha\beta} = (a-1)\nu + b$ and $J$ is then decomposable as a $\mu \times \nu$ array of rectangular $q_2 \times p_2$ blocks (called microblocks), where each microblock has a single factor label and the labels read in the natural order across $J$ from left to right and from top to bottom.

$J$ is a model with rotational variants $J(i,j)$ called miniblocks. $J(i,j)$ is obtained by rotating rows of $J$ cyclically downwards $iq_2$ places and the columns cyclically to the right $jp_2$ places. The effect is to leave a microblock structure with labels shifted cyclically down $i$ places and to the right $j$ places. Note that $J = J(0,0)$.

Using these, we next construct the $p_1 q_2 \mu \times p_2 q_1 \nu$ matrix $H$ as a $p_1 \times q_1$ block array of miniblock variants of $J$. Specifically, if $1 \le \gamma \le p_1$ and $1 \le \delta \le q_1$, then the miniblock of $H$ of row index $\gamma$ and the column index $\delta$ is defined to be $J = J(\delta - 1, \gamma - 1)$.

Note that $p_1 < \nu$ and $q_1 < \mu$, so that all the miniblocks comprising $H$ are distinct. This ensures that $H$ has the following properties:

(1) In every column and every row of miniblocks in $H$, a given factor label is associated with at most one microblock.

(2) Within every column (resp. row) of miniblocks in $H$, a given factor label is associated with a cyclically contiguous set of $p_1$ (resp. $q_1$) microblock columns (resp. rows).

$H$ also has rotational variants $H(i,j)$. For $0 \le i < q_1 q_0 x_0$ and $0 \le j < p_0 p_1 x_0$, $H(i,j)$ is the array of miniblocks where, for $1 \le \alpha \le p_1$ and $1 \le \beta \le q_1$, the miniblock with row index $\alpha$ and column index $\beta$ is $H(i,j)_{\alpha\beta} = J(iq_1 + \beta - 1, jp_1 + \alpha - 1)$.

Note that $iq_1 + \beta - 1 < \mu$ and $jp_1 + \alpha - 1 < \nu$. If $i \neq i'$, then in any fixed microblock row the set of factor labels occurring in $H(i,j)$ is disjoint from the set of factor labels occurring in $H(i',j)$. Similarly, if $j \neq j'$, in any fixed microblock column the set of factor labels occurring in $H(i,j)$ is disjoint from the set of factor labels occurring in $H(i,j')$.

Now we can place the vertical factor pieces in $F$. Notice that all the calculations are taken modulo $\frac{1}{\lambda_0}p_0q_0(x+y)$ in the range $1, 2, \cdots, \frac{1}{\lambda_0}p_0q_0(x+y)$. First decompose $F$ as $\frac{1}{\lambda_0}p_0q_0(x+y) \times \frac{1}{\lambda_0}p_0q_0(x+y)$ array $G$ of blocks of size $p_1q_2\mu \times p_2q_1\nu$ (of size equal to $H$). We first assign to the partial row $G_{1,1}, \cdots, G_{1,p_0q_0x}$. Recall $p_0q_0x = p_0q_0p_1q_1x_0$. For $1 \leq j \leq p_0q_0x$ write $j - 1 = rp_1p_0 + s$ uniquely for $0 \leq s < p_1p_0$ and $0 \leq r < q_1q_0x_0$, and write $r = tq_1q_0 + u$ uniquely for $0 \leq u < q_1q_0$ and $0 \leq t < x_0$. Then define $G_{1j} = H(r, s+tp_1p_0)$. Finally, for $2 \leq i \leq \frac{1}{\lambda_0}p_0q_0(x+y)$ and $1 \leq v \leq p_0q_0x$, define $j \equiv i+\nu-1$ $(mod\ \frac{1}{\lambda_0}p_0q_0(x+y))$ in the range $1, 2, \cdots, \frac{1}{\lambda_0}p_0q_0(x+y)$. This has effect of copying the assignments of the top row of $G$ by a process of diagonal replication into the other rows.

The same as those of Martin's discussion, this definition determines an assignment of vertical copies of $K_{p,q}$. So we have the following properties:

(3) This definition determines an assignment of vertical copies of $K_{p,q}$, so that no two copies with the same factor label overlap in a column or in a row.

(4) Within every column (resp. row) of miniblocks in $G$ thus defined, a given factor is associate with a cyclically contiguous set of $p_0p_1^2x_0$ (resp. $q_0q_1^2x_0$) microblock columns (resp. rows).

(5) $p_0p_1^2x_0 < \nu$ and $q_0q_1^2x_0 < \mu$, so there is no danger of the resulting contiguous sets of rows and columns rotating back onto themselves.

This complete the definition of the vertical factor pieces.

## The definition of the horizontal pieces

Our standard model miniblock M is a $q_2\mu \times p_2\nu$ array as follows: for $1 \leq s \leq \mu$ and $1 \leq t \leq \nu$ assign the label $\nu(s-1)+t$ to the entries $M_{ij}$, where $i = q_2(s-1) + \alpha$ and $j = p_2(t-1) + \alpha$ for $1 \leq \alpha \leq p_2q_2$, where the subscripts are reduced module $q_2\mu$ and $p_2\nu$ respectively into the correct range.

We construct cyclic variants $M(i,j)$ of $M$, by rotating columns $p_0p_1^2x_0p_2 + ip_2q_2$ places cyclically to the right and rows $q_0q_1^2x_0q_2 + jp_2q_2$ places cyclically down. The effect of the summands $p_0p_1^2x_0p_2$ and $q_0q_1^2x_0q_2$ is to ensure that we avoid these in anything previously defined. The other two summands ensure that varying $i$ and $j$ avoids partial overlaps with the diagonals (of length $q_2p_2$).

From each miniblock $M(i,j)$, we construct a large block $L(i,j)$ as a $p_1 \times q_1$ array of miniblocks all equal to $M(i,j)$. The following properties are satisfied.

(1) Within every column (resp. row) of miniblocks in $L(i,j)$, a given factor label is associated with a cyclically contiguous set of $p_2q_2$ columns (resp. rows).

(2) For each factor label $\phi$, the entries with that label contribute a total of $p_2q_2$ subarrays of size $p_1 \times q_1$ with non-overlapping rows and columns, but covering the contiguous sets as described above.

We now use these to fill in the remaining $G$-blocks of $F$ and to define all the required horizontal factors. The approach is to complete the first row of $G$ and copy this over the remaining rows by diagonal replication.

So we define the $G_{1,p_0q_0x+j}$, $1 \le j \le p_0q_0y = p_0q_0p_2q_2y_0$, working as in the vertical case. Given $j$, we define unique integers $r, s, t$ where $j-1 = rq_2q_0+s$, $r = tp_2p_0+u$, $0 \le s < q_2q_0$, $0 \le u < p_2p_0$ and $0 \le t < y_0$, and then set $G_{1,p_0q_0x+j} = L(s+tq_2q_0, r)$. We then extend this over the remainder of $G$ by assigning $G_{\alpha\beta} = G_{1,\beta-\alpha+1}$ for $2 \le \alpha \le \frac{1}{\lambda_0}p_0q_0(x+y)$ and $p_0q_0x+1 \le \beta-\alpha+1 \le p_0q_0(x+y)$, where all the calculations are taken modulo $\frac{1}{\lambda_0}p_0q_0(x+y)$ in the range $1, 2, \cdots, \frac{1}{\lambda_0}p_0q_0(x+y)$.

Similarly vertical analysis, we have the following:

(3) This definition determines an assignment of horizontal copies of $K_{p,q}$ so that no two copies with the same factor label overlap in a column or in a row.

(4) Within every column (resp. row) of miniblocks in $G$ defined in the second stage, a given factor label is associated with a cyclically contiguous set of $q_0q_2^2y_0$ (resp. $p_0p_2^2y_0$) microblock columns (resp. rows).

(5) $q_0q_2^2y_0 < \nu$ and $p_0p_2^2y_0 < \mu$. So there is no danger of the resulting contiguous sets of rows and columns rotating back onto themselves.

Finally, we notice that $F$ is a $\frac{1}{\lambda_0}p_0q_0(x+y) \times \frac{1}{\lambda_0}p_0q_0(x+y)$ array $G$ of blocks of size $p_1q_2\mu \times p_2q_1\nu$, and for each row $\alpha$, we have filled the $p_0q_0(x+y)$ blocks $G_{\alpha\beta}$ $(1 \le \beta \le p_0q_0(x+y))$. So each cell of $F$ is covered by $\lambda_0$ distinct $G_{\alpha\beta}$ and $F$ is indeed the factor array of a $K_{p,q}$-factorization of $\lambda_0K_{m_0,n_0}$.

So the proof of the theorem is completed. □

**Corollary 3.2** For all $p \geq 1$, the necessary conditions for $K_{p,p+1}$-factorization of $\lambda K_{m,n}$ are sufficient.

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