Indecomposable tournaments and their indecomposable subtournaments on 5 and 7 vertices

Houmem Belkhechine
Faculté des Sciences de Gabès
Cité Riadh, Zirig
6072 Gabès
Tunisie
houmem@gmail.com

Imed Boudabbous
Institut Préparatoire aux Études d'Ingénieurs de Sfax
Route Menzel Chaker Km 0.5
3018 Sfax
Tunisie
imed.boudabbous@gmail.com

Abstract

Given a tournament $T = (V, A)$, a subset X of V is an interval of T provided that for every $a, b \in X$ and $x \in V - X$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\emptyset$, $\{x\}(x \in V)$ and V are intervals of T, called trivial intervals. A tournament, all the intervals of which are trivial, is indecomposable; otherwise, it is decomposable. A critical tournament is an indecomposable tournament T of cardinality ≥ 5 such that for any vertex x of T, the tournament $T - x$ is decomposable. The critical tournaments are of odd cardinality and for all $n \geq 2$ there are exactly three critical tournaments on $2n + 1$ vertices denoted by T_{2n+1}, U_{2n+1} and W_{2n+1}. The tournaments T_5, U_5 and W_5 are the unique indecomposable tournaments on 5 vertices. We say that a tournament T embeds into a tournament T' when T is isomorphic to a subtournament of T'. A diamond is a tournament on 4 vertices admitting only one interval of cardinality 3. We prove the following theorem: if a diamond and T_5 embed into an indecomposable tournament T, then W_5 and U_5 embed into T. To conclude, we prove the following: given an indecomposable tournament T, with

ARS COMBINATORIA 108(2013), pp. 493-504

$|V(T)| \geq 7$, T is critical if and only if only one of the tournaments T_7, U_7 or W_7 embeds into T.

Key words: Tournament; Indecomposable; Critical; Embedding.

1 Basic definitions

A *tournament* $T = (V(T), A(T))$ or (V, A) consists of a finite *vertex* set V with an *arc* set A of ordered pairs of distinct vertices satisfying: for $x, y \in V$, with $x \neq y$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The *cardinality* of T is that of $V(T)$ denoted by $|V(T)|$. For two distinct vertices x and y of a tournament T, $x \longrightarrow y$ means that $(x, y) \in A(T)$. For $x \in V(T)$ and $Y \subset V(T)$, $x \longrightarrow Y$ (rep. $Y \longrightarrow x$) signifies that for every $y \in Y$, $x \longrightarrow y$ (resp. $y \longrightarrow x$). Given a vertex x of a tournament $T = (V, A)$, $N_T^+(x)$ denotes the set $\{y \in V : x \longrightarrow y\}$. The *score* of x (in T), denoted by $s_T(x)$, is the cardinality of $N_T^+(x)$. A tournament is *regular* if all its vertices share the same score. A *transitive* tournament or *total order* is a tournament T such that for x, y, $z \in V(T)$, if $x \longrightarrow y$ and $y \longrightarrow z$, then $x \longrightarrow z$. For two distinct vertices x and y of a total order T, $x < y$ means that $x \longrightarrow y$. We write $T = a_0 < \cdots < a_n$ to mean that T is the total order defined on $V(T) = \{a_0, \dots, a_n\}$ by $A(T) = \{(a_i, a_j) : i < j\}$.

The notions of isomorphism, of subtournament and of embedding are defined in the following manner. First, let $T = (V, A)$ and $T' = (V', A')$ be two tournaments. A one-to-one correspondence f from V onto V' is an *isomorphism* from T onto T' provided that for $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The tournaments T and T' are then said to be *isomorphic*, which is denoted by $T \simeq T'$. Moreover, an isomorphism from a tournament T onto itself is called an *automorphism* of T. The automorphisms of T form a subgroup of the permutation group of $V(T)$, called the *automorphism group* of T. Second, given a tournament $T = (V, A)$, with each subset X of V is associated the *subtournament* $T(X) = (X, A \cap (X \times X))$ of T *induced* by X. For $x \in V$, the subtournament $T(V - \{x\})$ is denoted by $T - x$. For tournaments T and T', if T' is isomorphic to a subtournament of T, then we say that T' *embeds* into T. Otherwise, we say that T *omits* T'. The *dual* of a tournament $T = (V, A)$ is the tournament obtained from T by reversing all its arcs. This tournament is denoted by $T^\star = (V, A^\star)$, where $A^\star = \{(x, y) : (y, x) \in A\}$. A tournament T is then said to be *self-dual* if T and $T^\star$ are isomorphic.

The indecomposability plays an important role in this paper. Given a tournament $T = (V, A)$, a subset I of V is an *interval* ([4], [7], [10]) (or a *clan* [3] or an *homogeneous subset* [5]) of T provided that for every $x \in V - I$, $x \rightarrow I$ or $I \rightarrow x$. This definition generalizes the notion of interval of a total order. Given a tournament $T = (V, A)$, $\emptyset$, V and $\{x\}$, where $x \in V$, are clearly intervals of T, called *trivial* intervals. A tournament is then said to be *indecomposable* ([7], [10]) (or *primitive* [3]) if all of its intervals are trivial, and is said to be *decomposable* otherwise. For instance, the 3-cycle $C_3 = (\{0,1,2\},\{(0,1),(1,2),(2,0)\})$ is indecomposable whereas a total order of cardinality ≥ 3 is decomposable. Let us mention the following relationship between indecomposability and duality. The tournaments T and T^* have the same intervals and, thus, T is indecomposable if and only if T^* is indecomposable.

2 The critical tournaments

An indecomposable tournament $T = (V, A)$ is said to be *critical* if $|V| > 1$ and for all $x \in V$, $T - x$ is decomposable. In order to present our main results and to present the characterization of the critical tournaments due to J.H. Schmerl and W.T. Trotter [10], we introduce the tournaments T_{2n+1}, U_{2n+1} and W_{2n+1} defined on $2n+1$ vertices, where $n \geq 2$, as follows:

- The tournament T_{2n+1} is the tournament defined on $\mathbb{Z}/(2n+1)\mathbb{Z}$ by $A(T_{2n+1}) = \{(i,j) : j - i \in \{1,\dots,n\}\}$, so that, $T_{2n+1}(\{0,\dots,n\}) = 0 < \cdots < n$, $T_{2n+1}(\{n+1,\dots,2n\}) = n+1 < \cdots < 2n$ and for $i \in \{0,\dots,n-1\}$, $\{i+1,\dots,n\} \longrightarrow i+n+1 \longrightarrow \{0,\dots,i\}$ (see Figure 1).
- The tournament U_{2n+1} is obtained from T_{2n+1} by reversing the arcs of $T_{2n+1}(\{n+1,\dots,2n\})$. Therefore, U_{2n+1} is defined on $\{0,\dots,2n\}$ as follows: $U_{2n+1}(\{0,\dots,n\}) = 0 < \cdots < n$, $U^*_{2n+1}(\{n+1,\dots,2n\}) = n+1 < \cdots < 2n$ and for $i \in \{0,\dots,n-1\}$, $\{i+1,\dots,n\} \longrightarrow i+n+1 \longrightarrow \{0,\dots,i\}$ (see Figure 2).
- The tournament W_{2n+1} is defined on $\{0,\dots,2n\}$ in the following manner: $W_{2n+1} - 2n = 0 < \cdots < 2n-1$ and $\{1,3,\dots,2n-1\} \longrightarrow 2n \longrightarrow \{0,2,\dots,2n-2\}$ (see Figure 3).

Theorem 1 ([10]) *Up to isomorphism, the critical tournaments of cardinality ≥ 5 are the tournaments T_{2n+1}, U_{2n+1} and W_{2n+1}, where $n \geq 2$.*

Notice that the critical tournaments are self-dual.

3 The tournaments T_5, U_5 and W_5 in an indecomposable tournament

We study the indecomposable tournaments according to their indecomposable subtournaments on 5 vertices. A recent result on our topic is a

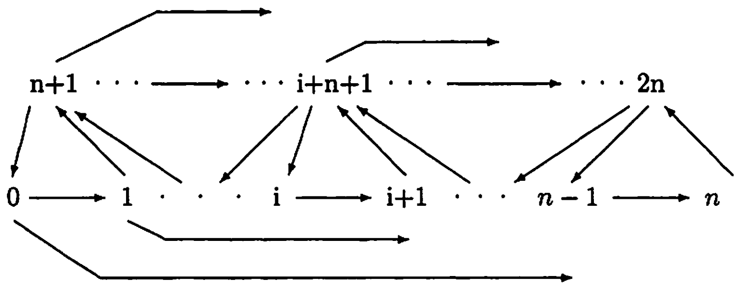

Figure 1: T_{2n+1}.

characterization of the indecomposable tournaments omitting W_5 obtained by B.J. Latka [8]. In order to recall this characterization, we introduce the *Paley* tournament P_7 defined on $\mathbb{Z}/7\mathbb{Z}$ by $A(P_7) = \{(i,j) : j-i \in \{1,2,4\}\}$. Notice that the tournaments obtained from P_7 by deleting one vertex are isomorphic and denote $P_7 - 6$ by B_6.

Theorem 2 **([8])** *Given a tournament T of cardinality ≥ 5, T is indecomposable and omits W_5 if and only if T is isomorphic to an element of $\{B_6, P_7\} \cup \{T_{2n+1} : n \geq 2\} \cup \{U_{2n+1} : n \geq 2\}$.*

A *diamond* is a tournament on 4 vertices admitting only one interval of cardinality 3. Up to isomorphism, there are exactly two diamonds D_4 and $D_4^\star$, where D_4 is the tournament defined on $\{0,1,2,3\}$ by $D_4(\{0,1,2\}) = C_3$ and $3 \longrightarrow \{0,1,2\}$.

The following theorem is the main result. This theorem is presented in [1] without a detailed proof.

Theorem 3 *Given an indecomposable tournament T, if a diamond and T_5 embed into T, then U_5 and W_5 embed into T.*

C. Gnanvo and P. Ille [6] and G. Lopez and C. Rauzy [9] characterized the tournaments omitting diamonds. In the indecomposable case they obtained the following characterization.

Proposition 1 **([6, 9])** *Given an indecomposable tournament T of cardinality ≥ 5, T omits the diamonds D_4 and $D_4^\star$ if and only if T is isomorphic to T_{2n+1} for some $n \geq 2$.*

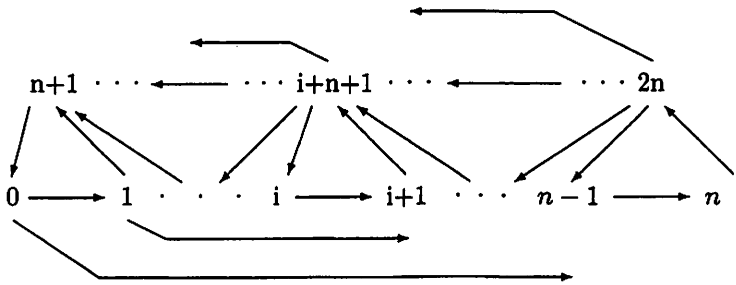

Figure 2: U_{2n+1}.

4 Proof of Theorem 3

Before proving Theorem 3, we introduce some notations and definitions.

Definition 1 *Given a tournament $T = (V, A)$, with each subset X of V, such that $|X| \geq 3$ and $T(X)$ is indecomposable, are associated the following subsets of $V - X$.*

- $Ext(X) = \{x \in V - X : T(X \cup \{x\})$ *is indecomposable*$\}$.
- $[X] = \{x \in V - X : x \to X$ *or* $X \to x\}$.
- *For every* $u \in X$, $X(u) = \{x \in V - X : \{u, x\}$ *is an interval of* $T(X \cup \{x\})\}$.

Lemma 1 ([3]) *Let $T = (V, A)$ be a tournament and let X be a subset of V such that $|X| \geq 3$ and $T(X)$ is indecomposable.*

1. *The family $\{X(u) : u \in X\} \cup \{Ext(X), [X]\}$ constitutes a partition of $V - X$.*
2. *Given $u \in X$, for all $x \in X(u)$ and for all $y \in V - (X \cup X(u))$, if $T(X \cup \{x, y\})$ is decomposable, then $\{u, x\}$ is an interval of $T(X \cup \{x, y\})$.*
3. *For every $x \in [X]$ and for every $y \in V - (X \cup [X])$, if $T(X \cup \{x, y\})$ is decomposable, then $X \cup \{y\}$ is an interval of $T(X \cup \{x, y\})$.*
4. *Given $x, y \in Ext(X)$, with $x \neq y$, if $T(X \cup \{x, y\})$ is decomposable, then $\{x, y\}$ is an interval of $T(X \cup \{x, y\})$.*

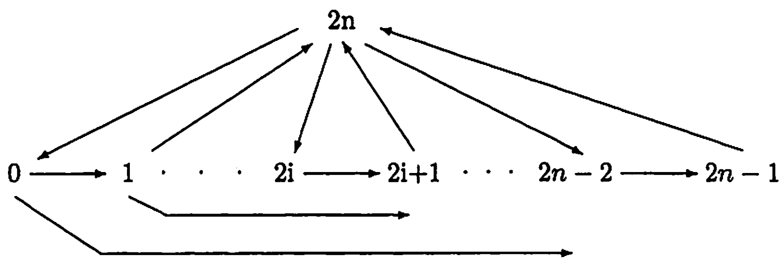

Figure 3: W_{2n+1}.

The below result follows from Lemma 1.

Proposition 2 **([3])** *Let $T = (V, A)$ be an indecomposable tournament. If X is a subset of V, such that $\mid X \mid \geq 3$, $\mid V - X \mid \geq 2$ and $T(X)$ is indecomposable, then there are distinct elements x and y of $V - X$ such that $T(X \cup \{x, y\})$ is indecomposable.*

Corollary 1 *Let $T = (V, A)$ be an indecomposable tournament such that $\mid V \mid$ is even and $\mid V \mid \geq 6$. For each $x \in V$, there is $y \in V - \{x\}$ such that $T - y$ is indecomposable.*

PROOF. As T is indecomposable, there is $X \subset V$ such that $x \in X$ and $T(X) \simeq C_3$. Otherwise, $N_T^+(x)$ or $V - (\{x\} \cup N_T^+(x))$ would be non trivial intervals of T. Since $\mid V \mid$ is even, by applying several times Proposition 2 from the indecomposable subtournament $T(X)$, we get a vertex $y \in V - X$ such that $T - y$ is indecomposable. □

The 3-cycle C_3 is indecomposable and embeds into any indecomposable tournament of cardinality ≥ 3 as observed in the preceding proof. It follows, by Proposition 2, that any indecomposable tournament T of cardinality ≥ 5, admits an indecomposable subtournament on 5 vertices. The indecomposable tournaments on 5 vertices are critical because the four tournaments on 4 vertices are decomposable. So let us mention the following facts.

Remark 1

- *The indecomposable tournaments on 5 vertices are, up to isomorphism, the three critical tournaments T_5, U_5 and W_5.*

- *There is no indecomposable tournament of cardinality* ≥ 5 *omitting each of the tournaments* T_5, U_5 *and* W_5.

The tournaments T_{2n+1} play an important role in the proof of Theorem 3. We recall some of their properties.

Remark 2

- *The tournaments* T_{2n+1} *are regular: for all* $i \in \{0, \ldots, 2n\}$, $s_{T_{2n+1}}(i) = n$;

- *For* $0 \leq i \leq 2n$, *the unique non trivial interval of* $T_{2n+1} - i$ *is* $\{i+n, i+n+1\}$;

- *The automorphism group of* T_{2n+1} *is generated by the permutation* $\sigma : i \mapsto i+1$;

- *The permutation* $\pi : i \mapsto -i$, *is an isomorphism from* T_{2n+1} *onto its dual.*

Now we are ready to prove Theorem 3.

PROOF OF THEOREM 3. Let $T = (V, A)$ be an indecomposable tournament into which a diamond and T_5 embed. Consider a minimal subset X of V such that $T(X)$ is indecomposable and a diamond and T_5 embed into $T(X)$. Now, let Y be a maximal subset of X such that $T(Y) \simeq T_{2n+1}$ for some $n \geq 2$. We establish that $\mid X \mid = 6$ by using the following observation. Consider a subset Z of X such that $T(Z) \simeq T_{2n+1}$ and assume that $Ext(Z) \cap X \neq \emptyset$. Let $x \in Ext(Z) \cap X$. We have $T(Z \cup \{x\})$ is indecomposable. Furthermore, as $\mid Z \cup \{x\} \mid$ is even, a diamond embeds into $T(Z \cup \{x\})$ by Proposition 1. Since T_5 embeds into $T(Z \cup \{x\})$ as well, it follows from the minimality of X that $X = Z \cup \{x\}$. As an immediate consequence, we have: if Z is a subset of X such that $T(Z) \simeq T_{2n+1}$ and $\mid X - Z \mid \geq 2$, then $Ext(Z) \cap X = \emptyset$. By Lemma 1, for every $x \in X - Z$, either $x \in [Z]$ or there is $u \in Z$ such that $x \in Z(u)$.

For a contradiction, suppose that $Ext(Y) \cap X = \emptyset$. By Proposition 2, there are $x \neq y \in X - Y$ such that $T(Y \cup \{x, y\})$ is indecomposable. Clearly, if $\{x, y\} \subseteq [Y]$, then Y would be a non trivial interval of $T(Y \cup \{x, y\})$. For instance, assume that there is $v \in Y$ such that $y \in Y(v)$. By Lemma 1, either there is $u \in Y$ such that $x \in Y(u)$ or $x \in [Y]$. In each of both instances, we obtain a contradiction.

First, suppose that there is $u \in Y$ such that $x \in Y(u)$. We have $u \neq v$, otherwise $\{u, x, y\}$ would be a non trivial interval of $T(Y \cup \{x, y\})$. By Remark 2, the automorphism group of T_{2n+1} is generated by $\sigma : i \mapsto i + 1$. Therefore, by interchanging x and y, we can denote the element of

Y by $0,\dots,2n$ in such a way that $T(Y) = T_{2n+1}$, $u = 0$ and $1 \leq v \leq n$. Since $T(Y \cup \{x,y\})$ is indecomposable and $0 \longrightarrow v$, we get $y \longrightarrow x$ by Lemma 1. Consider $Z = (Y - \{0\}) \cup \{x\}$. We have $T(Z) \simeq T_{2n+1}$ and, by the preceding observation, either $y \in [Z]$ or there is $w \in Z$ such that $y \in Z(w)$. The first instance is not possible because $\{v-2, v-1\} \cap Z \neq \emptyset$ and $\{v-2, v-1\} \longrightarrow y \longrightarrow x$. So assume that there is $w \in Z$ such that $y \in Z(w)$. As $y \longrightarrow x \longrightarrow v$, $w \neq v$. Moreover, if $w = x$, then $\{x, y\}$ is an interval of $T(Z \cup \{y\})$. Since $\{v, y\}$ is an interval of $T((Z \cup \{y\}) - \{x\})$, we would obtain that $\{x, y, v\}$ is an interval of $T(Z \cup \{y\})$ so that $\{x, v\}$ would be an interval of $T(Z)$. Therefore, $w \notin \{v, x\}$ and hence $\{v, w\}$ is an interval of $T(Z) - x$. As $x \in Y(0)$, it follows from Remark 2 that $\{v, w\} = \{n, n+1\}$ so that $v = n$ and $n \longrightarrow y$. By considering the automorphism σ^{n+1} of $T(Y)$ defined by $\sigma^{n+1}(i) = i + n + 1$, we obtain that $y \in Y(0)$ and $x \in Y(n+1)$. By considering $T^\star$ instead of T, we get $y \in Y(0)$ and $x \in Y(n)$ because the permutation $\pi : i \mapsto -i$ is an isomorphism from $T(Y)$ onto $T(Y)^\star$ by Remark 2. Lastly, by interchanging x and y in the foregoing, we obtain $n \longrightarrow x$ in $T^\star$ which means that initially $x \longrightarrow 0$ in T. It follows that the function $Y \cup \{x, y\} \longrightarrow \{0, \dots, 2n+2\}$, defined by $x \mapsto 2n+2$, $y \mapsto n+1$, $i \mapsto i$ for $0 \leq i \leq n$ and $i \mapsto i+1$ for $n+1 \leq i \leq 2n$, realizes an isomorphism from $T(Y \cup \{x, y\})$ onto T_{2n+3}. Consequently, $T(Y \cup \{x, y\}) \simeq T_{2n+3}$, with $Y \cup \{x, y\} \subseteq X$, which contradicts the maximality of Y.

Second, suppose that $x \in [Y]$. By interchanging T and $T^\star$, assume that $y \longrightarrow x \longrightarrow Y$. Consider $Z = (Y - \{v\}) \cup \{y\}$. We have $T(Z) \simeq T_{2n+1}$ and, by the previous observation, either $x \in [Z]$ or there is $w \in Z$ such that $x \in Z(w)$. The first instance is not possible because $y \longrightarrow x \longrightarrow Z - \{y\}$. Since $y \longrightarrow x \longrightarrow Z - \{y\}$ and hence $s_{T(Z \cup \{x\})}(x) = 2n$, the second is not possible either. Indeed, given $w \in Z$, if $x \in Z(w)$, then $s_{T(Z \cup \{x\})}(x) \in \{n, n+1\}$ because $s_{T(Z)}(w) = n$.

It follows that $Ext(Y) \cap X \neq \emptyset$. Set $T(Y) = T_{2n+1}$. By the preceding observation, $X = Y \cup \{x\}$, where $x \in Ext(Y) \cap X$. As $\mid X \mid$ is even, it follows from Corollary 1 that there is $j \in X - \{x\}$ such that $T(X) - j$ is indecomposable. By considering the automorphism σ^{2n+1-j} of $T(Y)$, we can assume that $j = 0$. For a contradiction, suppose that $T(X) - 0 \simeq T_{2n+1}$. We would have $s_{(T(X)-0)}(x) = n$. Since $s_{(T(Y)-0)}(i) = n$ for $1 \leq i \leq n$ and $s_{(T(Y)-0)}(i) = n-1$ for $n+1 \leq i \leq 2n$, we would obtain that $N^+_{(T(X)-0)}(x) = \{1, \dots, n\}$ so that $\{0, x\}$ would be a non trivial interval of $T(X)$. Consequently, $T(X) - 0$ is not isomorphic to T_{2n+1}. By Proposition 1, a diamond embeds into $T(X) - 0$. It follows from the minimality of $T(X)$ that $T(X) - 0$ and hence $T(Y) - 0$ omit T_5. As T_5 embeds into $T_{2m+1} - 0$ for $m \geq 3$, we get $n = 2$.

It remains to verify that U_5 and W_5 embed into $T(X)$. Since $x \notin [Y]$, $s_{T(X)}(x) \in \{1, 2, 3, 4\}$. By interchanging T and $T^\star$, assume that $s_{T(X)}(x) =$

1 or 2. First, assume that there is $i \in \mathbb{Z}/5\mathbb{Z}$ such that $N^+_{T(X)}(x) = \{i\}$. By considering the automorphism $j \mapsto j - i$ of T_5, assume that $i = 0$. The function $\mathbb{Z}/5\mathbb{Z} \longrightarrow X - \{3\}$, which fixes 0, 1, 2, 4 and which maps 3 to x, is an isomorphism from U_5 onto $T(X) - 3$. Furthermore, the function $\mathbb{Z}/5\mathbb{Z} \longrightarrow X - \{2\}$, defined by $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto x$, $3 \mapsto 0$ and $4 \mapsto 1$, is an isomorphism from W_5 onto $T(X) - 2$. Finally, assume that there is $i \in \mathbb{Z}/5\mathbb{Z}$ such that $N^+_{T(X)}(x) = \{i, i+1\}$ or $\{i, i+2\}$. If $N^+_{T(X)}(x) = \{i, i+1\}$, then $\{i-1, x\}$ would be an interval of $T(X)$. So, by considering the automorphism $k \mapsto k - i$ of T_5, assume that $N^+_{T(X)}(x) = \{0, 2\}$. The function $\mathbb{Z}/5\mathbb{Z} \longrightarrow X - \{0\}$, defined by $0 \mapsto 2$, $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto x$ and $4 \mapsto 1$, is an isomorphism from U_5 onto $T(X) - 0$. Furthermore, the function $\mathbb{Z}/5\mathbb{Z} \longrightarrow X - \{2\}$, defined by $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto x$, $3 \mapsto 0$ and $4 \mapsto 1$, is an isomorphism from W_5 onto $T(X) - 2$. □

5 A new characterization of the critical tournaments

In this section we discuss some other questions concerning the indecomposable subtournaments on 5 and 7 vertices of an indecomposable tournament. In particular, we obtain a new characterization of the critical tournaments. In that order, we recall the following two results concerning the critical tournaments.

Lemma 2 ([10]) *The indecomposable subtournaments of T_{2n+1} on at least 5 vertices, where $n \geq 2$, are isomorphic to T_{2m+1}, where $2 \leq m \leq n$. The same holds for the indecomposable subtournamants of U_{2n+1} and of W_{2n+1}.*

Lemma 3 ([2]) *Given an indecomposable tournament T of cardinality $\geq$ 5, T is critical if and only if T omits any indecomposable tournament on six vertices.*

Let T be an indecomposable tournament of cardinality ≥ 5. We denote by $I_5(T)$ the set of the elements of $\{T_5, U_5, W_5\}$ embedding in T. By Remark 1, $I_5(T) \neq \emptyset$. By Theorem 3, $I_5(T) \neq \{T_5, U_5\}$ and $I_5(T) \neq \{T_5, W_5\}$. We characterize the indecomposable tournaments T such that $I_5(T) = \{T_5\}$ (resp. $I_5(T) = \{U_5\}$). The following remark completes this discussion.

Remark 3 *For $J = \{W_5\}$, $\{U_5, W_5\}$ or $\{T_5, U_5, W_5\}$ and for $n \geq 6$, there exists an indecomposable tournament T of cardinality n such that $I_5(T) = J$.*
For $n \geq 5$, the tournaments E_{n+1}, F_{n+1} and G_{n+1} defined below on $\{0, \ldots, n\}$ are indecomposable and satisfy $I_5(E_{n+1}) = \{T_5, U_5, W_5\}$, $I_5(F_{n+1}) = \{W_5\}$ and $I_5(G_{n+1}) = \{U_5, W_5\}$.

- $E_{n+1}(\{0,\ldots,4\}) = T_5$ *and, for all* $5 \leq k \leq n$, $N^+_{E_{n+1}(\{0,\ldots,k\})}(k) = \{k-1\}$;
- $A(F_{n+1}) = \{(i,j) : i+1 < j$ *or* $i = j+1\}$;
- $G_n(\{0,\ldots,n-1\}) = F_n$ *and* $N^+_{G_{n+1}}(n) = \{0\}$.

The following is an easy consequence of Theorem 2 and of Lemma 2.

Corollary 2 *The next two assertions are satisfied by any indecomposable tournament* T *of cardinality* ≥ 5.

1. T *is isomorphic to* T_{2n+1} *for some* $n \geq 2$ *if and only if the indecomposable subtournaments of* T *on 5 vertices are isomorphic to* T_5.
2. T *is isomorphic to* B_6, P_7 *or to* U_{2n+1} *for some* $n \geq 2$ *if and only if the indecomposable subtournaments of* T *on 5 vertices are isomorphic to* U_5.

For all $n \geq 6$, the tournament F_n defined in Remark 3 is an indecomposable non critical tournament all the indecomposable subtournaments of which are isomorphic to W_5. This leads us to the following characterization of the tournaments W_{2n+1} and to the problem below.

Proposition 3 *Given an indecomposable tournament* T *of cardinality* ≥ 7, T *is isomorphic to* W_{2n+1} *for some* $n \geq 3$ *if and only if the indecomposable subtournaments on 7 vertices of* T *are isomorphic to* W_7.

PROOF. By Lemma 2, if $T \simeq W_{2n+1}$, where $n \geq 3$, then the indecomposable subtournaments of T on 7 vertices are isomorphic to W_7. Conversely, assume that the indecomposable subtournaments of T on 7 vertices are isomorphic to W_7. By Lemma 2, it suffices to show that T is critical. Clearly, if $|V(T)| = 7$, then $T \simeq W_7$. So assume that $|V(T)| \geq 8$. For a contradiction, suppose that T is not critical. It follows from Lemma 3 that there exists $X \subset V(T)$ such that $|X| = 6$ and $T(X)$ is indecomposable. By Proposition 2, there is $Y \subseteq V(T)$ such that $X \subset Y$, $|Y| = 8$ and $T(Y)$ is indecomposable. As $|Y|$ is even, $T(Y)$ is not critical. Consider $x \in Y$ such that $T(Y) - x$ is indecomposable. We have $T(Y) - x \simeq W_7$ and hence we can denote the elements of Y by $0,\ldots,7$ in such a way that $x = 7$ and $T(Y) - 7 = W_7$. By Corollary 1, there is $y \in \{0,\ldots,6\}$ such that $T(Y) - y$ is indecomposable and thus $T(Y) - y \simeq W_7$. To obtain a contradiction, we verify that $\{y, 7\}$ would be a non trivial interval of $T(Y)$. By interchanging T and $T^\star$, we can assume that $y \in \{0,1,2\} \cup \{6\}$ because the permutation of $\mathbb{Z}/7\mathbb{Z}$, which fixes 6 and which exchanges i and $5-i$ for $0 \leq i \leq 5$, is an isomorphism from W_7 onto its dual. First, assume that $y = 6$. We have

$T(Y) - \{6, 7\} = 0 < \cdots < 5$. Since $\{1, \ldots, 5\} \cup \{7\}$ is not an interval of $T(Y) - 6$, $7 \longrightarrow 0$. As $\{i, i+1\}$ is not an interval of $T(Y) - 6$ for $0 \leq i \leq 4$, we obtain successively that $1 \longrightarrow 7$, $7 \longrightarrow 2$, $3 \longrightarrow 7$, $7 \longrightarrow 4$ and $5 \longrightarrow 7$. Second, assume that $y \in \{0, 1, 2\}$. For $z \in \{0, \ldots, 7\} - \{y, 6\}$, C_3 embeds into $T(Y) - \{y, z\}$ because $T(\{2i, 2i+1, 6\}) \simeq C_3$ for $i \in \{0, 1, 2\}$. It follows that the isomorphism from W_7 onto $T(Y) - y$ fixes 6. Consequently, $T(Y) - \{y, 6\}$ is transitive. We have only to check that $T(Y) - \{y, 6\}$ is obtained from the usual total order on $\{0, \ldots, 5\}$ by replacing y by 7. If $y = 0$, then $7 \longrightarrow 1$ because $1 \longrightarrow \{2, \ldots, 6\}$. Thus $T(Y) - \{y, 6\} = 7 < 1 < \cdots < 5$. If $y = 1$ or 2, then $\{y - 1, y + 1\}$ is an interval of $T(Y) - \{y, 7\}$. Therefore, $\{y - 1, y + 1\}$ is not an interval of $T(\{y - 1, y + 1, 7\})$ and hence $T(Y) - \{y, 6\} = \cdots < y - 1 < 7 < y + 1 < \cdots < 5$.

□

From Corollary 2 and Proposition 3, we obtain the following recognition of the critical tournaments from their indecomposable subtournaments on 7 vertices.

Corollary 3 *Given an indecomposable tournament T, with $|V(T)| \geq 7$, T is critical if and only if the indecomposable subtournaments on 7 vertices of T are isomorphic to only one of the tournaments T_7, U_7 or W_7.*

Problem 1 *Characterize the indecomposable tournaments all of whose indecomposable subtournaments on 5 vertices are isomorphic to W_5.*

References

[1] H. Belkhechine and I. Boudabbous, Tournois indécomposables et leurs sous-tournois indécomposables à 5 sommets, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 685-688.

[2] Y. Boudabbous, J. Dammak and P. Ille, Indecomposability and duality of tournaments, Discrete Math. 223 (2000) 55-82.

[3] A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, Theoret. Comput. Sci. 3 (70) (1990) 343-358.

[4] R. Fraïssé, L'intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d'une relation, in: M. Pouzet, D. Richard (Eds.), Orders, Description and Roles, North-Holland, Amsterdam, 1984, pp. 313-342.

[5] T. Gallai, Transitiv orientierbare Graphen, Acta. Math. Acad. Sci. Hungar. 18 (1967) 25-66.

[6] C. Gnanvo and P. Ille, la reconstruction des tournois sans diamants, Z. Math. Logik Grundlag. Math. 38 (1992) 283-291.

[7] P. Ille, Indecomposable graphs, Discrete Math. 173 (1997) 71-78.

[8] B.J. Latka, Structure Theorem for tournaments omitting N_5, Journal of Graph Theory 42 (2003) 165-192.

[9] G. Lopez and C. Rauzy, Reconstruction of binary relations from their restrictions of cardinality 2, 3, 4 and (n-1). I. Z. Logik Grundlag. Math. 38 (1992) 27-37.

[10] J.H. Schmerl and W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, Discrete Math. 113 (1993) 191-205.