The scrambling indices of primitive digraphs with exactly two cycles*

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Abstract

In 2009, Akelbek and Kirkland introduced a useful parameter called the scrambling index of a primitive digraph D, which is the smallest positive integer k such that for every pair of vertices u and v, there is a vertex w such that we can get to w from u and v in D by walks of length k. In this paper, we study and obtain the scrambling indices of all primitive digraphs with exactly two cycles.

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Keywords: Primitive digraph; scrambling index; cycle.

1 Introduction

For terminology and notation used here we follow [1, 5]. Let $D = (V, E)$ denote a digraph (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$ and order n. Loops are permitted but multiple arcs are not. A digraph D is called *primitive* if for some positive integer k, there is a walk of length exactly k from each vertex u to each vertex v (possibly u again). It is well known that D is primitive if and only if D is strongly connected and the greatest common divisor of all the cycle lengths of D is 1.

The *distance* from vertex u to vertex v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. For a vertex u and a vertex set X of D, the distance from u to X in D, denoted by $d(u, X)$, is $d(u, X) = \min\{d(u, x) | x \in X\}$. If $u \in X$, we define $d(u, X) = 0$. The notation $u \xrightarrow{k} v$ is used to indicate that there is a walk of length k from u to v.

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In [5], Akelbek and Kirkland introduced a useful parameter called the *scrambling index* of a primitive digraph D, denoted by $k(D)$, which is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. For $u, v \in V(D)(u \neq v)$, the *local scrambling index* of u and v is the number

$$k_{u,v}(D) = \min\{k \mid u \xrightarrow{k} w \text{ and } v \xrightarrow{k} w, \text{ for some } w \in V(D)\},$$

then

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\}.$$

Akelbek and Kirkland's definition of the scrambling index is the same as Cho and Kim's [3] definition of the competition index in the case of primitive digraphs. The two research groups started from different points, but got the results nearly simultaneously. Their achievements are widely applied to stochastic matrices and food webs. For details, see, e.g. [2, 3, 4, 5].

Recently, some papers on the scrambling indices have been published ([4, 5, 6, 7, 8]). In this work, we consider the scrambling indices of all primitive digraphs of order $n \geq 4$ with exactly two cycles. It is clear that any strongly connected digraph of order $n \geq 4$ with exactly two cycles is isomorphic to either $D_{s,n}$ (as given in Figure 1) or $D_{t,s,n}$ (as given in Figure 2). Note $D_{s,n}$ is primitive if and only if $\gcd(n, s) = 1$, likewise $D_{t,s,n}$ is primitive if and only if $\gcd(n - t, s) = 1$.

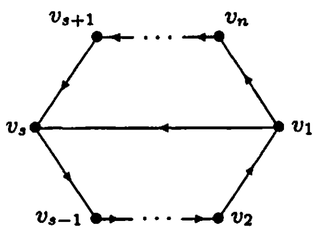


Figure 1 Digraphs $D_{s,n}$ ($n \geq 4, 1 \leq s \leq n-1$).

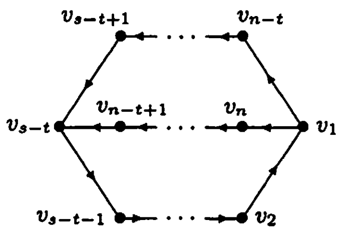


Figure 2 Digraphs $D_{t,s,n}$ ($n \geq 4, 1 \leq t < s < n-t$).

In [5] Akelbek and Kirkland give the scrambling indices of digraphs $D_{s,n}$ when $D_{s,n}$ are primitive.

Theorem 1.1 *([5]) Let* $n \geq 4$, $1 \leq s \leq n-1$ *and* $\gcd(n, s) = 1$. *Then*

$$k(D_{s,n}) = \begin{cases} n - s + (\frac{s-1}{2})n, & \text{if } s \text{ is odd,} \\ n - s + (\frac{n-1}{2})s, & \text{if } s \text{ is even.} \end{cases}$$

In the following section, we will give the scrambling indices of digraphs $D_{t,s,n}$ when $D_{t,s,n}$ are primitive.

2 Main results

First, we give a useful lemma that appeared in [5].

Lemma 2.1 *([5]) Let p and s be positive integers such that* $\gcd(p,s)=1$ *and $p > s \geq 2$. Then for each l, $1 \leq l \leq \max\{s-1, \lfloor \frac{p}{2} \rfloor\}$, the equation $xp+ys=l$ has a unique integral solution (x,y) with $|x| \leq \lfloor \frac{s}{2} \rfloor$ and $|y| \leq \lfloor \frac{p}{2} \rfloor$.*

Note that the existence of solution (x,y) is guaranteed by Lemma 2.1. Henceforth, we say (x,y) is a solution of equation $xp+ys=l$ with minimum absolute value to mean that $|x| \leq \lfloor \frac{s}{2} \rfloor$, $|y| \leq \lfloor \frac{p}{2} \rfloor$ and $xp+ys=l$.

Lemma 2.2 *Let p, s and l be positive integers such that* $\gcd(p,s)=1$, *$p > s \geq 2$, and $1 \leq l \leq \max\{s-1, \lfloor \frac{p}{2} \rfloor\}$. If (x_0, y_0) is the solution of the equation $xp+ys=l$ with minimum absolute value, then all integral solutions of the equation $xp+ys=l$ are $\{(x_0+ms, y_0-mp) \mid m$ is an integer$\}$.*

Proof Since (x_0,y_0) is a solution of the equation $xp+ys=l$, we have

$$x_0p + y_0s = l.$$

Let (x_1,y_1) be any solution of the equation $xp+ys=l$. Then

$$x_1p + y_1s = l.$$

So $(x_1-x_0)p + (y_1-y_0)s = 0$, that is,

$$(x_1-x_0)p = (y_0-y_1)s.$$

Noting $\gcd(p,s)=1$, then $(x_1-x_0)|s$, and so there is an integer m_1 such that $x_1-x_0=m_1s$, that is, $x_1=x_0+m_1s$. In this case, $y_1=y_0-m_1p$.

On the other hand, for each integer m, it is clear that (x_0+ms, y_0-mp) is a solution of the equation $xp+ys=l$. The lemma now follows. □

For positive integers p and s with $2 \leq s < p$ and $\gcd(p,s)=1$, for the sake of convenience, we denote

$$k(p,s) = \begin{cases} (\frac{s-1}{2})p, & \text{if } s \text{ is odd,} \\ (\frac{p-1}{2})s, & \text{if } s \text{ is even.} \end{cases} \tag{2.1}$$

Theorem 2.3 *Let $n \geq 4$, $1 \leq t < s < n-t$, and* $\gcd(n-t,s)=1$. *Then*

$$k(D_{t,s,n}) = n - s + k(n-t,s).$$

Proof Denote $p = n - t$. Let C_s and C_p be the cycles of lengths s and p, respectively.

First, we will prove that $k(D_{t,s,n}) \leq n - s + k(n-t, s)$. Consider the following two cases.

Case 1.1 s is odd.

For any $u, v \in V(D_{t,s,n})$, take $v_i, v_j \in V(C_s)$ such that $u \xrightarrow{n-s} v_i$ and $v \xrightarrow{n-s} v_j$. If $v_i = v_j$, then $k_{u,v}(D_{t,s,n}) \leq n - s$. We now suppose $v_i \neq v_j$ so that $d(v_i, v_j) = l$, for some $1 \leq l \leq s-1$. Since $\gcd(s,p) = 1$ and $p > s \geq 2$, by Lemma 2.1, the equation $xp + ys = l$ has a unique integral solution (x, y) with $|x| \leq \lfloor \frac{s}{2} \rfloor$ and $|y| \leq \lfloor \frac{p}{2} \rfloor$.

Moreover, since $l < s$ and $l < p$, x and y are both nonzero and must have opposite signs. We consider the following two subcases.

Subcase 1.1.1 $x > 0$ and $y < 0$.

Then $xp = l - ys$. If v_j is in the overlap of C_s and C_p, then

$$u \xrightarrow{n-s} v_i \xrightarrow{l} v_j \xrightarrow{-ys} v_j \quad \text{and} \quad v \xrightarrow{n-s} v_j \xrightarrow{xp} v_j$$

are the walks of length $n - s + xp$ from u to v_j and v to v_j.

If v_j is on C_s but not on C_p, noticing that there exists at least one vertex of the walk $v \xrightarrow{n-s} v_j$ on C_p, then

$$u \xrightarrow{n-s} v_i \xrightarrow{l} v_j \xrightarrow{-ys} v_j \quad \text{and} \quad v \xrightarrow{n-s} v_j + xC_p$$

are the walks of length $n - s + xp$ from u to v_j and v to v_j, where xC_p is the walk around C_p x times. Thus

$$k_{u,v}(D_{t,s,n}) \leq n - s + xp \leq n - s + \left(\frac{s-1}{2}\right)p.$$

Subcase 1.1.2. $x < 0$ and $y > 0$.

Since $v_i \xrightarrow{l} v_j$ and $v_i, v_j \in V(C_s)$, then $v_j \xrightarrow{s-l} v_i$. We have $-xp = (s-l) + (y-1)s$. By the argument from Subcase 1.1.1, there exist walks of length $n - s - xp$ from u to v_i and v to v_i. Thus

$$k_{u,v}(D_{t,s,n}) \leq n - s - xp \leq n - s + \left(\frac{s-1}{2}\right)p.$$

Case 1.2. s is even.

Then $p = n - t$ is odd. For any $u, v \in V(D_{t,s,n})$, take $v_i, v_j \in V(C_p)$ such that $u \xrightarrow{n-s} v_i$ and $v \xrightarrow{n-s} v_j$. If $v_i = v_j$, then $k_{u,v}(D_{t,s,n}) \leq n - s$. Now we suppose that $v_i \neq v_j$. Since $v_i, v_j \in V(C_p)$, we have that $d(v_i, v_j) \leq \frac{p-1}{2}$ or $d(v_j, v_i) \leq \frac{p-1}{2}$. Without loss of generality, let $d(v_i, v_j) \leq \frac{p-1}{2}$, so that there is a positive integer $1 \leq l \leq \frac{p-1}{2}$ such that $v_i \xrightarrow{l} v_j$ and $v_j \xrightarrow{p-l} v_i$.

By Lemma 2.1, the equation $xp + ys = l$ has a unique integral solution (x, y) with $|x| \le \lfloor \frac{s}{2} \rfloor$ and $|y| \le \lfloor \frac{p}{2} \rfloor$.

If $x \ne 0$, then $y \ne 0$, and x and y must have opposite signs. By arguments similar to Case 1.1,

$$k_{u,v}(D_{t,s,n}) \le n - s + |y|s \le n - s + \left(\frac{p-1}{2}\right)s.$$

If $x = 0$, then the equation is $ys = l$, and so $y > 0$. If v_j is in C_s as well as C_p, then

$$u \xrightarrow{n-s} v_i \xrightarrow{l} v_j \quad \text{and} \quad v \xrightarrow{n-s} v_j \xrightarrow{ys} v_j$$

are the walks of length $n - s + ys$ from u to v_j and v to v_j. If v_j is on C_p but not on C_s, noticing that there exists at least one vertex of the walk $v \xrightarrow{n-s} v_j$ on C_s, then

$$u \xrightarrow{n-s} v_i \xrightarrow{l} v_j \quad \text{and} \quad v \xrightarrow{n-s} v_j + yC_s$$

are the walks of length $n - s + ys$ from u to v_j and v to v_j. Thus

$$k_{u,v}(D_{t,s,n}) \le n - s + ys \le n - s + \left(\frac{p-1}{2}\right)s.$$

Combining the above Cases 1.1 and 1.2, we have that

$$k(D_{t,s,n}) \le n - s + k(p, s).$$

Next, for $v_i \in V(D_{t,s,n})$ and a positive integer x, we denote $R_x(v_i)$ the set of all those vertices which can be reached by a walk of length x starting from vertex v_i. Let $h = n - s + k(p, s) - 1$. We find a vertex $v \in V(D_{t,s,n})$ such that $R_h(v_{n-t}) \cap R_h(v) = \phi$.

We consider the following three cases.

Case 2.1. s is odd and p is even.

Then $h = n - s + (\frac{s-1}{2})p - 1$. It is clear that

$$\left(\frac{p}{2}\right)s - \left(\frac{s-1}{2}\right)p = \frac{p}{2},$$

and $(\frac{p}{2}, -\frac{s-1}{2})$ is the integral solution of the equation $xs + yp = \frac{p}{2}$ with minimum absolute value. Let $\frac{p}{2} \equiv r(\text{mod} s)$, so that $\frac{p}{2} = r + t's$ for some nonnegative integer t', where $1 \le r \le s - 1$. Then

$$\left(\frac{p}{2} - t'\right)s - \left(\frac{s-1}{2}\right)p = r, \tag{2.2}$$

and so $(\frac{p}{2} - t', -\frac{s-1}{2})$ is the integral solution of the equation $xs + yp = r$ with minimum absolute value.

If $n-t-r \geq s-t$, then the vertex $v_{n-t-r} \notin V(C_s)$, $d(v_{n-t}, v_{s-t}) = n-s$, and $d(v_{n-t-r}, v_{s-t}) = n-s-r$. Thus $R_h(v_{n-t}) = R_{(\frac{s-1}{2})p-1}(v_{s-t})$, and $R_h(v_{n-t-r}) = R_{(\frac{s-1}{2})p+r-1}(v_{s-t})$. We claim that $R_{(\frac{s-1}{2})p-1}(v_{s-t}) \cap R_{(\frac{s-1}{2})p+r-1}(v_{s-t}) = \phi$. In fact, if $v_{i_0} \in R_{(\frac{s-1}{2})p-1}(v_{s-t}) \cap R_{(\frac{s-1}{2})p+r-1}(v_{s-t})$, then there are nonnegative integers a_1, a_2, b_1, b_2, such that

$$\begin{cases} (\frac{s-1}{2})p - 1 = d(v_{s-t}, v_{i_0}) + a_1 s + b_1 p, \\ (\frac{s-1}{2})p + r - 1 = d(v_{s-t}, v_{i_0}) + a_2 s + b_2 p, \end{cases} \tag{2.3}$$

and so $r = (a_2 - a_1)s + (b_2 - b_1)p$. By (2.2) and Lemma 2.2, there exists an integer m such that $b_1 - b_2 = \frac{s-1}{2} + ms$, and so $b_1 \geq \frac{s-1}{2}$ or $b_2 \geq \frac{s+1}{2}$. This contradicts (2.3). Then $R_{(\frac{s-1}{2})p-1}(v_{s-t}) \cap R_{(\frac{s-1}{2})p+r-1}(v_{s-t}) = \phi$, and $R_h(v_{n-t}) \cap R_h(v_{n-t-r}) = \phi$.

If $n-t-r < s-t$, noticing that $n-t-r = p-r = 2(r+t's) - r = r + 2t's$, then $t' = 0$, $n-t-r = r$, and so $1 \leq r < s-t$. We claim that $R_h(v_{n-t}) \cap R_h(v_r) = \phi$. In fact, if $v_{i_0} \in R_h(v_{n-t}) \cap R_h(v_r)$, then there are nonnegative integers a_1, a_2, b_1, b_2, such that

$$\begin{cases} h = n - t - r + d(v_r, v_{i_0}) + a_1 s + b_1 p, \\ h = d(v_r, v_{i_0}) + a_2 s + b_2 p, \end{cases} \tag{2.4}$$

and so $(a_2 - a_1)s + (b_2 - b_1)p = n - t - r = r$. By (2.2) and Lemma 2.2, there exists an integer m such that $b_1 - b_2 = \frac{s-1}{2} + ms$, and so $b_1 \geq \frac{s-1}{2}$ or $b_2 \geq \frac{s+1}{2}$. Noticing $1 \leq r < s-t$ and $p = n-t > n-s$, by (2.4),

$$d(v_r, v_{i_0}) + a_1 s + b_1 p = h - (n-t-r) = t + r - s + \left(\frac{s-1}{2}\right)p - 1 < \left(\frac{s-1}{2}\right)p - 1,$$

and

$$d(v_r, v_{i_0}) + a_2 s + b_2 p = h = n - s + \left(\frac{s-1}{2}\right)p - 1 < p + \left(\frac{s-1}{2}\right)p - 1 = \left(\frac{s+1}{2}\right)p - 1.$$

These contradict $b_1 \geq \frac{s-1}{2}$ or $b_2 \geq \frac{s+1}{2}$. Then $R_h(v_{n-t}) \cap R_h(v_r) = \phi$.

Case 2.2. Both s and p are odd.

We have that

$$\left(\frac{p-1}{2}\right)s - \left(\frac{s-1}{2}\right)p = \frac{p-s}{2},$$

and $(\frac{p-1}{2}, -\frac{s-1}{2})$ is the integral solution of the equation $xs + yp = \frac{p-s}{2}$ with minimum absolute value. Let $\frac{p-s}{2} \equiv r(\text{mod} s)$, so that $\frac{p-s}{2} = r + t's$ for some nonnegative integer t', where $1 \leq r \leq s-1$. Then

$$\left(\frac{p-1}{2} - t'\right)s - \left(\frac{s-1}{2}\right)p = r,$$

and so $(\frac{p-1}{2}-t', -\frac{s-1}{2})$ is the integral solution of the equation $xs+yp=r$ with minimum absolute value. Clearly, $n-t-r=p-r=2(r+t's)+s-r=r+(2t'+1)s>s-t$.

By argument similar to Case 2.1, we have that $R_h(v_{n-t})\cap R_h(v_{n-t-r})=\phi$.

Case 2.3. s is even.

In this case, p is odd, and $h=n-s+(\frac{p-1}{2})s-1$. It is clear that

$$\Big(\frac{s}{2}\Big)p-\Big(\frac{p-1}{2}\Big)s=\frac{s}{2}, \tag{2.5}$$

and $(\frac{s}{2}, -\frac{p-1}{2})$ is the integral solution of the equation $xp+ys=\frac{s}{2}$ with minimum absolute value. We claim that $R_h(v_{n-t})\cap R_h(v_{n-t-\frac{s}{2}})=\phi$. Otherwise, we suppose that there is a vertex $v_{i_0}\in V(D_{t,s,n})$ such that $v_{i_0}\in R_h(v_{n-t})\cap R_h(v_{n-t-\frac{s}{2}})$.

If $n-t-\frac{s}{2}\le s-t$, then $n\le\frac{3s}{2}$, and there are nonnegative integers a_1,a_2,b_1,b_2, such that

$$\begin{cases} h=\frac{s}{2}+d(v_{n-t-\frac{s}{2}},v_{i_0})+a_1s+b_1p,\\ h=d(v_{n-t-\frac{s}{2}},v_{i_0})+a_2s+b_2p,\end{cases} \tag{2.6}$$

and so $(a_2-a_1)s+(b_2-b_1)p=\frac{s}{2}$. By (2.5) and Lemma 2.2, there exists an integer m such that $a_1-a_2=\frac{p-1}{2}+mp$, and so $a_1\ge\frac{p-1}{2}$ or $a_2\ge\frac{p+1}{2}$. Noticing that $n\le\frac{3s}{2}$, by (2.6),

$$d(v_{n-t-\frac{s}{2}},v_{i_0})+a_1s+b_1p=n-s+\Big(\frac{p-1}{2}\Big)s-1-\frac{s}{2}\le\Big(\frac{p-1}{2}\Big)s-1,$$

and

$$d(v_{n-t-\frac{s}{2}},v_{i_0})+a_2s+b_2p=n-s+\Big(\frac{p-1}{2}\Big)s-1\le\Big(\frac{p+1}{2}\Big)s-1.$$

These contradict $a_1\ge\frac{p-1}{2}$ or $a_2\ge\frac{p+1}{2}$.

If $n-t-\frac{s}{2}>s-t$, then there are nonnegative integers a_1,a_2,b_1,b_2, such that

$$\begin{cases} h=n-s+d(v_{s-t},v_{i_0})+a_1s+b_1p,\\ h=n-s-\frac{s}{2}+d(v_{s-t},v_{i_0})+a_2s+b_2p,\end{cases} \tag{2.7}$$

and so $(a_2-a_1)s+(b_2-b_1)p=\frac{s}{2}$. By (2.5) and Lemma 2.2, there exists an integer m such that, $a_1-a_2=\frac{p-1}{2}+mp$, and so $a_1\ge\frac{p-1}{2}$ or $a_2\ge\frac{p+1}{2}$. By (2.7),

$$d(v_{s-t},v_{i_0})+a_1s+b_1p=n-s+\Big(\frac{p-1}{2}\Big)s-1-(n-s)=\Big(\frac{p-1}{2}\Big)s-1,$$

and

$$d(v_{s-t}, v_{i_0}) + a_2 s + b_2 p = n - s + \left(\frac{p-1}{2}\right)s - 1 - \left(n - s - \frac{s}{2}\right) = \left(\frac{p+1}{2}\right)s - 1.$$

These contradict $a_1 \geq \frac{p-1}{2}$ or $a_2 \geq \frac{p+1}{2}$.

Thus $R_h(v_{n-t}) \cap R_h(v_{n-t-\frac{s}{2}}) = \phi$ for Case 2.3.

Combining the above Cases 2.1, 2.2 and 2.3, we have that there is a vertex $v \in V(D_{t,s,n})$ such that $R_h(v_{n-t}) \cap R_h(v) = \phi$. Thus for any integer h_0 with $1 \leq h_0 \leq h$, $R_{h_0}(v_{n-t}) \cap R_{h_0}(v) = \phi$. It implies that $k(D_{t,s,n}) \geq h + 1 = n - s + k(p, s)$. The theorem now follows. □

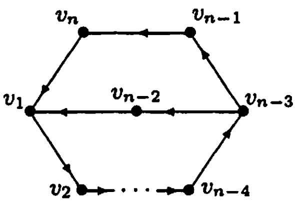
Combining Theorems 1.1 and 2.3, we have the following result on the scrambling indices of all primitive digraphs with exactly two cycles.

Corollary 2.4 *Let D be a primitive digraph of order n ($n \geq 4$) with exactly two cycles (say their lengths are s and p, and $s < p$). Then*

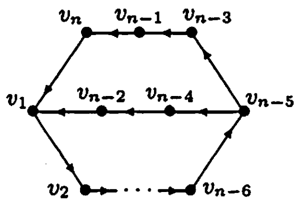
$$k(D) = n - s + k(p, s) = \begin{cases} n - s + (\frac{s-1}{2})p, & \text{if } s \text{ is odd,} \\ n - s + (\frac{p-1}{2})s, & \text{if } s \text{ is even.} \end{cases}$$

3 Remarks

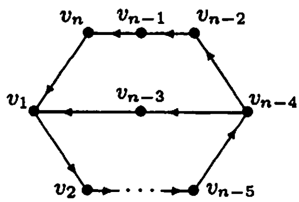
In [8], when authors study the scrambling indices of primitive minimally strong digraphs, they derive the following three digraphs and obtain theirs scrambling indices (see Lemmas 3.10, 3.11 and 3.12 in [8]).



Digraph H_n ($n \geq 4$)



Digraph Q_n ($n \geq 6$)



Digraph W_n ($n \geq 6$)

Note that each of primitive digraphs H_n, Q_n and W_n has exactly two cycles, and that H_n is isomorphic to $D_{1,n-2,n}$, Q_n is isomorphic to $D_{2,n-3,n}$, and W_n is isomorphic to $D_{1,n-3,n}$. By Theorem 2.3 or Corollary 2.4, the scrambling indices of above three digraphs follow directly.

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