On k-equitable and k-balanced labelings of graphs

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Abstract

In this paper, we consider labelings of graphs in which the label on an edge is the absolute value of the difference of its vertex labels. Such a labeling using $\{0,1,2,...,k-1\}$ is called *k-equitable* if the number of vertices (resp. edges) labeled i and the number of vertices (resp. edges) labeled j differ by at most one and is called *k-balanced* if the number of vertices labeled i and the number of edges labeled j differ by at most one. We determine which graphs in certain families are k-equitable or k-balanced and we give also some necessary conditions on these two labelings.

Keywords: Cordial labeling, k-equitable and k-balanced labelings.

1. Introduction

All graphs in this paper are finite, simple and undirected. We fellow the basic notations and terminology of graph theory as in [3].

Let G be a (p,q) graph with vertex set $V(G)$ and edge set $E(G)$ and let us denote the set $\{0,1,2,...,k\}$ by $[0,k]$. A vertex labeling $f:V(G)\to[0,k-1]$ induces an edge labeling $f^*: E(G)\to[0,k-1]$, defined by $f^*(xy)=|f(x)-f(y)|$, for each edge $xy\in E(G)$. For $i\in[0,k-1]$, let $n_i(f)=|\{v\in V(G):\ f(v)=i\}|$ and $m_i(f)=|\{e\in E(G):\ f^*(e)=i\}|$. A labeling f of a graph G is *k-equitable* (resp. *k-balanced*) if $|n_i(f)-n_j(f)|\le 1$ and $|m_i(f)-m_j(f)|\le 1$ (resp. $|n_i(f)-m_j(f)|\le 1$) for all $i,j\in[0,k-1]$. A graph G is called *k-equitable* (resp. *k-balanced*) if it admits a k-equitable (resp. k-balanced) labeling.

The notion of k-equitable labeling of graphs was introduced by Cahit [2] in 1990, who introduced first the notion of 2-equitable labeling under the name of cordial labeling in 1987 [1], while the notion of k-balanced labeling was introduced by Seoud and Abdel Maqsoud [7] in 1999.

Cahit [1] proved the following : every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel $W_n = C_n + K_1$ is cordial if and only if $n \not\equiv 3 \pmod 4$; C_n is cordial if and only if $n \not\equiv 2 \pmod 4$ and an Eulerian graph is not cordial if its size is congruent to $2 \pmod 4$. Shee and Ho [9] determined the cordiality of $C_m^{(n)}$, the one-point union of n copies of C_m. Cahit [2] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if n is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable and all caterpillars are 3-equitable. Kuo, Chang and Kwong [6] determined all m and n for which mK_n is cordial. Szaniszlo [11] showed that K_n is not k-equitable for $3 \leq k < n$, and C_n is k-equitable if and only if k meets all of the following conditions: $n \neq k$; if $k \equiv 2,3 \pmod 4$, then $n \neq k-1$; if $k \equiv 2,3 \pmod 4$, then $n \not\equiv k \pmod{2k}$.

Youssef [12] proved that if G is a k-equitable Eulerian graph of q edges and $k \equiv 2 \, or \, 3 \pmod 4$, then $q \not\equiv k \pmod{2k}$, we call this necessary condition the k-equitable parity condition. As a corollary of the k-equitable parity condition, he also proved that if $k \equiv 2 \, or \, 3 \pmod 4$ and G is odd (p,q) graph with $p \equiv 0 \pmod k$ and $p + q \equiv k \pmod{2k}$, then G is not k-equitable, we call this necessary condition the k-equitable odd parity condition. In [13] Youssef gave some variations on the definition of cordial graph and defined what he called semi-cordial graph and establish some relations between semi- cordial and graceful graphs.

Seoud and Abdel Maqsoud [8] proved that $K_{m,n}$, $3 \leq m \leq n$ is 3-equitable if and only if $(m,n) = (4,4)$, $K_{1,2,n}$, $n \geq 2$ is 3-equitable if and only if $n \equiv 2 \pmod 3$ and $K_{1,m,n}$, $3 \leq m \leq n$ is 3-equitable if and only if $(m,n) = (3,4)$. Seoud and Abdel Maqsoud [7] proved that P_n^2 is k-balanced if and only if $n = 2,3,4 \text{ or } 6$ and $K_{1,m,n}$, $m \leq n$ is k-balanced if and only if (i) $m = 1, n = 1 \text{ or } 2$ and $k = 3$; (ii) $m = 1$ and

$k = n+1$ or $n+2$; or (iii) $k \geq (m+1)(n+1)$. For more details of known results of graph labelings see Gallian [5].

In the next section of this paper we deal with k-equitable graphs. We determine the maximal number of edges in a 3-equitable graph of order n, we show that $C_n + \overline{K}_2$ is 3-equitable if and only if n is even and $n \geq 6$. Finally we show that C_n^2 is 3-equitable if and only if $n \geq 8$. Section 3 deals with k-balanced graphs. We give some necessary conditions for a graph to be k-balanced. Some relations between k-equitable and k-balanced labelings are given.

2. k-equitable graphs

In this section, We give an exact formula for the maximal number of edges in a 3-equitable graph of order n. We also determine the 3-equitablity of the graphs $C_n + \overline{K}_2$ and C_n^2.

Note that if G is (p,q) graph having k-equitable labeling f, then $\left\lfloor \frac{p}{k} \right\rfloor \leq n_i(f) \leq \left\lceil \frac{p}{k} \right\rceil$ and $\left\lfloor \frac{q}{k} \right\rfloor \leq m_i(f) \leq \left\lceil \frac{q}{k} \right\rceil$ for all $i \in [0, k-1]$, and $k-1-f$ is also a k-equitable labeling of G.

Du [4] determined the maximal number of edges in a cordial (that is, 2-equitable) graph of order n. We extend this result for 3-equitable graphs. Let $\delta_k(n)$ be the maximal number of edges in a k-equitable graph of order n. For example, we can easily show that: $\delta_3(n) = n-1$ for $n \leq 3$, $\delta_3(4) = 5$ and $\delta_3(5) = 8$.

Theorem 1. If $n \geq 12$, then

$$\delta_3(n) = \begin{cases} \frac{n^2}{3} + 2, & n \equiv 0 \pmod 3 \\ \frac{(n-1)(n+2)}{3} + 2, & n \equiv 1 \pmod 3 \\ \frac{(n+1)^2}{3} + 2, & n \equiv 2 \pmod 3 \end{cases}$$

Proof. Let G be 3-equitable graph of order n with k-equitable labeling f of maximal number of edges. We note that the edges labeled 2 induced only from the vertices labeled 0 that are adjacent to the vertices labeled 2 and the maximal number of edges labeled 2 in a 3-equitable graph of order n is $\left\lceil \frac{n}{3} \right\rceil \left\lfloor \frac{n+1}{3} \right\rfloor$. The maximal number of edges labeled 0 in G is $\sum_{i=0}^{2} \binom{n_i(f)}{2}$. If $n \geq 12$, one can show that $M_2(f) < M_0(f) < M_1(f)$. So $\delta_3(n) = 3M_2(f) + 2$, where $M_i(f)$ is the maximal number of edges labeled i in G. By considering different cases of n modulo 3, we complete the proof.□

Remark 1. (i) One can easily calculate $\delta_3(n)$, for $6 \leq n \leq 11$, using the same argument as above and obtain: $\delta_3(6) = 11, \delta_3(7) = 17, \delta_3(8) = 23, \delta_3(9) = 28$, $\delta_3(10) = 37$ and $\delta_3(11) = 47$. Also, we can unify the cases in the head of the above theorem in one case as $\delta_3(n) = 3\left\lceil \frac{n}{3} \right\rceil \left\lfloor \frac{n+1}{3} \right\rfloor + 2$, $n \geq 12$. (ii) In light of the proof of Theorem 1, one can extend the result for $k \geq 4$ in the same argument and get $\delta_k(n) = k\left\lceil \frac{n}{k} \right\rceil \left\lfloor \frac{n+1}{k} \right\rfloor + k - 1$, $n \geq 3k$.

Lemma 1. If G is a (p,q) k-equitable graph with $p \equiv 0 \pmod{k}$, then

(i) $G + K_1$ is k-equitable, and

(ii) $G + \overline{K}_2$ is k-equitable.

Proof. Let G be k-equitable (p,q) graph with $p \equiv 0 \pmod{k}$. Now,

(i) Label the vertex of K_1 by the label 0

(ii) Label the two vertices of $\overline{K}_2$ by the labels 0 and $k-1$.□

Cahit [2] showed that W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$, while Seoud and Abdel Maqsoud [8] proved that the fan $F_n = P_n + K_1$ is 3-equitable if and only if $n \geq 1$ except $n = 2$, all double fans $P_n + \overline{K}_2$ are 3-equitable except for $n = 4$ and they conjectured that W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$, but Youssef [12] showed that one direction of the result of Cahit about the 3-equitablity of W_n is not correct by proving that W_n is 3-equitable for all $n \geq 4$.

There are three known reasons at which a graph G fails to be k-equitable:

(1) G has too many edges,

(2) G has the wrong k-equitable parity condition , and

(3) G has the odd parity condition.

In the following theorem we show that the graph in certain case is not 3-equitable for other reason not from the above list.

Theorem 2. For $n \geq 6$, the graph $C_n + \overline{K}_2$ is 3-equitable If and only if n is even.

Proof. If $n \equiv 0 \pmod 6$, then C_n is 3-equitableby by [2] and then $C_n + \overline{K}_2$ is 3-equitable by Lemma 1. If $n \equiv 2 \pmod 6$, let $n = 6m + 2$, $m \geq 1$. We label the vertices of the cycle C_n successively by the labels: $(110)(\prod_{i=1}^{m-1}(200211))(20021)$ and we label the vertices of $\overline{K}_2$ by the labels 0 and 2. If $n \equiv 4 \pmod 6$, let $n = 6m + 4, m \geq 1$. We label the vertices of the cycle successively by the labels: $(111200202)(\prod_{i=1}^{m-1}(110220))(1)$ and we label the vertices of $\overline{K}_2$ by the labels 0 and 2. One can easily verify that the labelings in the above two cases are 3-equitable.

Conversely, suppose $n \geq 3$ is odd and $C_n + \overline{K}_2$ is 3-equitable with 3-equitable labeling f and assume that n'_0, n'_1 and n'_2 are the number of the vertices labeled $0,1$ and 2 respectively which lie on the cycle C_n, m_0', m_1' and m_2' are the number of edges labeled $0,1$ and 2 respectively restricted to the cycle C_n. As C_n is Eulerian graph, then m_1' is even [2].

If n is odd and $n \geq 7$, then $m_1' > 0$, since otherwise all vertex labels on the cycle are all 1 or all either 0 or 2 which implies that $n_1(f) \geq n$ or $n_1(f) \leq 2$, which is absurd in both cases, since $3 \leq n_1(f) < n$. We have the following three cases for the possibilities of the vertex labels of $\bar{K}_2$:

Case 1: $f(V(\bar{K}_2)) = \{0,0\}$ or $\{0,2\}$ or $\{2,2\}$

In this case, $m_1(f) = 2n_1' + m_1'$, that is, $m_1(f)$ is even, which contradicts the fact that $m_1(f)$ must be equal to n which is odd.

Case 2: $f(V(\bar{K}_2)) = \{0,1\}$ or $\{1,2\}$

$m_1(f) = n_0' + n_1' + n_2' + m_1' = n + m_1' > n$, which is absurd.

Case 3: $f(V(\bar{K}_2)) = \{1,1\}$

$m_1(f) = 2n_0' + 2n_2' + m_1'$, which is even, a contradiction as in Case 1.

Hence $C_n + \bar{K}_2$ is not 3-equitable if n is odd and $n \geq 7$

If $n = 5$, if $m_1' > 0$, then $C_5 + \bar{K}_2$ is not 3-equitable as above. If $m_1' = 0$, then $m_1(f) = 10$ a contradiction.

If $n = 3$, then $C_3 + \bar{K}_2$ is not 3-equitable since this graph has 5 vertices and 9 edges while $\delta_3(5) = 8$.

If $n = 4$, then $C_n + \bar{K}_2$ is not 3-equitable by Remark 1 since this graph is of size 12 while $\delta_3(6) = 11$. □

The following lemma is similar to a result in the proof of P_n^2 is 3-equitable in [8].

Lemma 2. If C_n^2 has 3-equitable labeling such that there exists two adjacent vertices of C_n labeled 1, then C_{n+18}^2 is 3-equitable also.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ where $v_i v_j \in E(C_n)$ if and only if $i - j \equiv \pm 1 \pmod{n}$ and let C_n^2 has 3-equitable labeling f such that $f(v_{n-1}) = f(v_n) = 1$. Cut the cycle C_n around the vertex v_n by removing the edges $v_1 v_n$, $v_2 v_n$ and $v_1 v_{n-1}$, and let $V(P_{18}) = \{u_1, u_2, \ldots, u_{18}\}$ where $u_i u_j \in E(P_{18})$ if and only if $|i - j| = 1$. Now, connect v_1 to u_1 and u_{18} to v_n and add all edges between vertices whose the distance between them is 2 to obtain C_{n+18}^2. Now label the vertices of P_{18} successively by the labels: $1,1,2,2,0,0,2,2,0,0,1,1,0,0,2,2,1,1$. We added to C_n^2 6 vertices labeled zero, 6 labeled one and 6 labeled two. For the edge labels, we added 12 edges labeled zero, 12 edges labeled one and 12 edges labeled two. Hence C_{n+18}^2 is 3-equitable. □

Seoud and Abdel Maqsoud [8] sowed that P_n^2 is 3-equitable for all $n \geq 1$ but 3 and conjectured that C_n^2 is not 3-equitable for all $n \geq 3$, however we show the following.

Theorem 3. C_n^2 is 3-equitable if and only if $n \geq 8$.

Proof. $C_3^2 = C_3$ is not 3-equitable by [2]. $C_4^2 = K_4$ and $C_5^2 = K_5$ are not 3-equitable by [11]. $C_6^2 = C_4 + \overline{K}_2$ is not 3-equitable by Theorem 2. If $n = 7$, suppose C_7^2 has 3-equitable labeling f. As C_7^2 is Eulerian graph, then $m_1(f) = 4$, but since $2 \leq n_1(f) \leq 3$, then $m_1(f) \geq 6$ a contradiction. Conversely, we will give a 3-equitable labeling f of $C_n^2, 8 \leq n \leq 25$, with $f(v_{n-1}) = f(v_n) = 1$, then applying Lemma 2 completes the proof by induction. We label the vertices of C_n^2, $8 \leq n \leq 25$ successively as pattern: $(f(v_1), f(v_2), \ldots, f(v_n))$

$n = 8$: $(0,2,0,2,0,2,1,1)$.
$n = 9$: $(0,0,2,2,0,2,1,1,1)$.

$n = 10$: $(0,0,2,2,0,2,2,1,1,1)$.
$n = 11$: $(0,1,2,0,2,0,0,2,1,1,1)$.
$n = 12$: $(0,1,2,0,0,2,2,,0,2,1,1,1)$.
$n = 13$: $(0,1,2,0,2,0,2,0,0,2,1,1,1)$.

$n = 14$: $(0,0,1,2,2,0,2,0,2,0,2,1,1,1)$.
$n = 15$: $(0,1,1,2,0,2,0,0,2,2,0,2,1,1,1)$.

$n = 16$: $(0,0,1,2,2,0,0,2,2,0,2,0,1,1,1,1)$.

$n = 17$: $(0,0,2,2,0,2,2,1,1,0,0,2,2,1,1,1,1)$.

$n = 18$: $(0,0,1,1,1,2,2,0,0,2,2,0,0,2,2,1,1,1)$.
$n = 19$: $(0,0,2,2,0,0,0,2,2,1,1,0,0,2,2,1,1,1,1)$.

$n = 20$: $(0,0,2,2,1,0,0,2,2,1,0,0,2,2,0,2,1,1,1,1)$.

$n = 21$: $(0,0,2,2,0,0,2,2,1,0,2,0,2,1,2,0,1,1,1,1,1)$.

$n = 22$: $(0,0,2,2,0,0,2,2,1,0,2,0,2,1,2,0,0,1,1,1,1,1)$.

$n = 23$: $(0,0,2,2,1,1,0,0,2,2,1,0,2,0,2,0,0,2,2,1,1,1,1)$.

$n = 24$: $(0,0,2,2,0,0,2,2,1,0,0,2,2,1,1,1,0,0,2,2,1,1,1,1)$.

$n = 25$: $(0,0,2,2,0,0,2,2,1,0,0,2,2,0,1,1,1,1,2,2,0,1,1,1,1)$. □

3. k-balanced graphs

In this section, we give some necessary conditions for a graph to be k-balanced and also present some relations between k-equitable and k-balanced labelings.

Lemma 3. If G is a (p,q) k-balanced graph, then $|\,p-q\,| \le k$.

Proof. Let G be k-balanced graph with k-balanced labeling f , then $|\,p-q\,| = \left| \sum_{i=0}^{k-1} n_i(f) - \sum_{i=0}^{k-1} m_i(f) \right| \le \sum_{i=0}^{k-1} |n_i(f) - m_i(f)| \le k$. □

The following lemma is stronger than the result above.

Lemma 4. If G is a (p,q) k-balanced graph, then $\left\lceil \frac{q}{k} \right\rceil - \left\lfloor \frac{p}{k} \right\rfloor \le 1$ if $q \ge p$ or $\left\lceil \frac{p}{k} \right\rceil - \left\lfloor \frac{q}{k} \right\rfloor \le 1$ if $p \ge q$.

Proof. Let $q \ge p$ and let G be k-balanced with k-balanced labeling f , then using the pigeonhole principle, there exists $i \in [0, k-1]$ such that $m_i(f) \ge \left\lceil \frac{q}{k} \right\rceil$, since otherwise $m_i(f) < \left\lceil \frac{q}{k} \right\rceil$ for all $i \in [0, k-1]$ and hence $\sum_{i=0}^{k-1} m_i(f) < k \left\lceil \frac{q}{k} \right\rceil < q$, a contradiction. Again, there exists $j \in [0, k-1]$ such that $n_j(f) \le \left\lfloor \frac{p}{k} \right\rfloor$. Now, $\left\lceil \frac{q}{k} \right\rceil - \left\lfloor \frac{p}{k} \right\rfloor \le m_i(f) - n_j(f) \le 1$. Similarly the case when $p \ge q$. □

Proposition 1. If G is a (p,q) graph having k-balanced labeling f and there exist $i, j \in [0, k-1]$ such that $|n_i(f) - n_j(f)| = 2$ (resp. $|m_i(f) - m_j(f)| = 2$), then $m_i(f) = m_j(f)$ for all

$i, j \in [0, k-1]$ and hence $q \equiv 0 \pmod{k}$ (resp. $n_i(f) = n_j(f)$ for all $i, j \in [0, k-1]$ and hence $p \equiv 0 \pmod{k}$).

Proof. Immediate. □

Corollary 1. If G is a (p,q) graph having k-balanced labeling f, then

(i) If $p \equiv 0 \pmod{k}$ and $q \not\equiv 0 \pmod{k}$, then $n_i(f) = \frac{p}{k}$ for all $i \in [0, k-1]$.

(ii) If $q \equiv 0 \pmod{k}$ and $p \not\equiv 0 \pmod{k}$, then $m_i(f) = \frac{q}{k}$ for all $i \in [0, k-1]$.

Proof. (i) If $n_i(f) \neq \frac{p}{k}$ for some $i \in [0, k-1]$, then there exists $j \in [0, k-1]$ such that $|n_i(f) - n_j(f)| = 2$ and by Proposition 1, $m_i(f) = \frac{q}{k}$ for all $i \in [0, k-1]$ which is absurd since $q \not\equiv 0 \pmod{k}$. Case (ii) is obtained in a similar manner. □

Theorem 4. Let $k \equiv 2 \text{ or } 3 \pmod{4}$. If G is k-balanced Eulerian (p,q) graph and $q \equiv k \pmod{2k}$, then $p \equiv 0 \pmod{k}$.

Proof. Let $k \equiv 2 \text{ or } 3 \pmod{4}$ and let G be k-balanced graph with k-balanced labeling f and $q \equiv k \pmod{2k}$. As G is Eulerian graph, then $\sum_{uv \in E(G)} |f(u) - f(v)| \equiv 0 \pmod{2}$. On the other hand,

$\sum_{uv \in E(G)} |f(u) - f(v)| = \sum_{i=0}^{k-1} i\, m_i(f)$ and as $\frac{q}{k}$ is odd, then there exists $i \in [0, k-1]$ such that $m_i(f)$ is even, since otherwise $\sum_{i=0}^{k-1} i\ m_i(f)$ is odd which is absurd, and hence there exists $j \in [0, k-1]$ such that $|m_i(f) - m_j(f)| = 2$ and from Proposition 1, $p \equiv 0 \pmod k$. □

Combining the above result and the k-equitable parity condition of Youssef [12], we get that if G is Eulerian (p, q) graph with $q \equiv k \pmod{2k}$, $p \not\equiv 0 \pmod k$ and $k \equiv 2\, or\, 3 \pmod 4$, then G is not k-equitable and not k-balanced.

The following result gives a necessary condition for k-equitable graph to be k-balanced graph in certain case.

Proposition 2. Let G be a (p, q) k-equitable graph with $p\ or\ q \equiv 0 \pmod k$.
If $|\ p - q\ | \leq k$, then G is k-balanced.

Proof. Let f be k-equitable labeling of G. If $p \equiv 0 \pmod k$, then $n_i(f) = \frac{p}{k}$ for all $i \in [0, k-1]$ and as $|\ p - q\ | \leq k$, then $\frac{p}{k} - 1 \leq m_i(f) \leq \frac{p}{k} + 1$ for all $i \in [0, k-1]$, that is $|\ n_i(f) - m_j(f)\ | \leq 1$ and hence G is k-balanced. Similarly, if $q \equiv 0 \pmod k$ with same argument as above, G is k-balanced. □

The following result is a gives another relation between k-equitable and k-balanced graphs which generalizes Proposition 2.

Theorem 5. Let G be a (p, q) k-equitable graph. If $-p \pmod k \leq q - p \leq k - p \pmod k$ or $-q \pmod k \leq p - q \leq k - q \pmod k$, then G is k-balanced.

Proof. For fixed p, assume that G has k-equitable labeling f. If $-p \pmod k \leq q - p \leq k - p \pmod k$, as

$\left\lfloor \frac{p}{k} \right\rfloor = \frac{p - p(\mathrm{mod}\, k)}{k}$, we get $k\left\lfloor \frac{p}{k} \right\rfloor \le q \le k(\left\lfloor \frac{p}{k} \right\rfloor + 1)$, and then $\left\lfloor \frac{p}{k} \right\rfloor \le m_i(f) \le \left\lfloor \frac{p}{k} \right\rfloor + 1$, for $i \in [0, k-1]$ and hence $|m_i(f) - n_j(f)| \le 1$, for all $i, j \in [0, k-1]$, then G is k-balanced. The proof is similar for the second inequality. □

The following result shows that every k-equitable tree (resp. unicyclic graph) is k-balanced. Its proof is easy and we omit it.

Proposition 3. If G is (p,q) k-equitable graph with $|p - q| \le 1$, then G is k-balanced.

Cahit [2] conjectured that all trees are k-equitable and proved his conjecture for $k = 2$ in[1]. Speyer and Szaniszlo [10] proved Cahit's conjecture for $k = 3$. Applying Proposition 3 on these two results, we get the following.

Corollary 2. All trees are 2-balanced and 3-balanced.

Again, Proposition 3 and the conjecture of Cahit [2], motivate us to conjecture that.

Conjecture 1. All trees are k-balanced, $k \ge 4$.

Theorem 6. C_n is 3-balanced if and only if $n \ge 3$.

Proof. If $n \not\equiv 3 \,(\mathrm{mod}\, 6)$, then C_n is 3-equitable by [2] and then 3-balanced by Proposition 3. If $n \equiv 3 \,(\mathrm{mod}\, 6)$, let $n = 6m + 3$, $m \ge 0$. Let $f : V(C_n) \to \{0,1,2\}$ be described as follows: We label the vertices of the cycle by the successive labels $(201)\prod_{i=1}^{m}(102201)$. It is straightforward to verify that $n_i(f) = 2m + 1$ for each $i \in [0,2]$ and $m_0(f) = 2m - 1$, $m_1(f) = 2m + 2$, $m_2(f) = 2m + 1$, then f is a 3-balanced labeling of C_n. □

The above result along with the results of Cahit[2] about the 3-equitablity of C_n and of Speyer and Szaniszlo[10] about the 3-equitablity of trees, make us settle for the following conjecture.

Conjecture 2. All connected unicyclic graphs are 3-balanced. And all except C_{6t+3} are 3-equitable.

Analogous to $\delta_k(n)$, let $\sigma_k(n)$ be the maximal number of edges in a k-balanced graph of order n. The following theorem gives the number $\sigma_3(n)$.

Theorem 7. $$\sigma_3(n)=\begin{cases} 3(\left\lfloor \frac{n}{3} \right\rfloor+1), & n \geq 5 \\ 5, & n=4 \\ \frac{n}{2}(n-1), & n=1,2,3 \end{cases}.$$

Proof. If $n \geq 6$, then by Lemma 3, $\sigma_3(n) \leq n+3$ and if $n \equiv 0 \pmod 3$, then $\sigma_3(n)=n+3$ and if $n \equiv 1$ or $2 \pmod 3$, then by Corollary 1 the maximal number of each of the edges labeled $0,1$ and 2 is $\left\lfloor \frac{n}{3} \right\rfloor+1$, so $\sigma_3(n)=3(\left\lfloor \frac{n}{3} \right\rfloor+1)$. For $n \leq 5$, it easy to obtain $\sigma_3(n)$. □

Corollary 3. K_n is 3-balanced if and only if $n \leq 3$.

Remark 2. The reader may extend Theorem 7 for $k \geq 4$ to get $\sigma_k(n)=k(\left\lfloor \frac{n}{k} \right\rfloor+1), \quad n \geq 2k+1.$

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