GENERALIZED F-NOMIAL MATRIX AND FACTORIZATIONS

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ABSTRACT. In this study we define the generalized $k-$order Fibonacci matrix and the $n \times n$ generalized Pascal matrix $\mathcal{F}_n(GF)$ associated with generalized $F-nomial$ coefficients. We find the inverse of generalized Pascal matrix $\mathcal{F}_n(GF)$ associated with generalized $F-nomial$ coefficients. In the last section we factorize this matrix via generalized $k-$order Fibonacci matrix and give illustrative examples for these factorizations.

1. INTRODUCTION

In [3] for a fixed n, the $n \times n$ lower triangular Pascal matrix

$$P_n = [p_{i,j}]_{i,j=1,2,\ldots,n}$$

is defined by

$$p_{i,j} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} .$$

The Pascal matrices has many applications in probability, numerical analysis, surface reconstruction and combinatorics. In [1] the relationships between the Pascal matrix and Vandermonde, Frobenius, Stirling matrices are studied. Also in [1] another applications in stability properties of numerical methods for solving ordinary differential equations are shown. Lee and Kim [13] factorized the Pascal matrix involving the Fibonacci matrix. Pascal matrices, Binomial coefficients, Fibonomial coefficients, $F-nomial$ coefficients, their generalizations and factorizations are studied by many authors. For details see [4, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22, 23].

For integers i, j and n, $1 \leq i, j \leq n$ the $n \times n$ Pascal matrix via Fibonomial coefficients named as Fibo Pascal matrix $P_n = (p_{ij})$ similar to the

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Pascal matrix in [10] as follows

$$p_{i,j} = \begin{cases} \binom{i-1}{j-1}_F & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}.$$

was studied in [20]. The inverse of this matrix was also given in [20].

Silvester [8] obtained a matrix representation for usual Fibonacci sequence. Kalman [7] extended this matrix representation for a generalization of Fibonacci sequence. Kalman supposed that the $(n+k)th$ term of that sequence defined recursively as a linear combination of the preceding k terms,

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are constant coefficients.

Miles [16] defined the generalized k-Fibonacci numbers as shown for $n > k \geq 2$

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k},$$

where $f_1 = f_2 = \ldots = f_{k-2} = 0$ and $f_{k-1} = f_k = 1$.

Er [6] defined k sequences of the generalized $order-k$ Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \text{ for } n > 0 \text{ and } 1 \leq i \leq k,$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence.

Akbulak and Bozkurt [2] defined order-m generalized Fibonacci k numbers by matrix representation. Using this matrix representation they obtained sums, some identities and the generalized Binet formula of generalized order$-m$ Fibonacci k-numbers.

In [14] the $n \times n$ $k-$Fibonacci matrix is defined and the inverse of the $k-$Fibonacci matrix is given as follows:

$$F(k)_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ -1 & \cdots & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \cdots & -1 & 1 \end{bmatrix}. \quad (1.1)$$

In [18] the $n \times n$ $k-$Pell matrix $M(k)_n$ is defined and the factorizations of $k-$Pell matrix are given. The inverse of the $k-$Pell matrix is given.

In this paper we define and study the $n \times n$ Pascal matrix via generalized $F-nomial$ coefficients. We define the generalized $k-$order Fibonacci matrix and factorize this matrix. We also find the inverse of generalized $k-$order Fibonacci matrix. Finally in the last section we factorize the $n \times n$ Pascal matrix via generalized $F-nomial$ coefficients involving generalized $k-$order Fibonacci matrix.

2. Generalized k-order Fibonacci Matrix

For a positive integer $k \geq 2$, we define the generalized $k-$order Fibonacci sequence $\{v_n(k)\}$ as

$$v_1(k) = ... = v_{k-2}(k) = 0, v_{k-1}(k) = 1, v_k(k) = d_1$$

and for $n > k \geq 2$

$$v_n(k) = d_1 v_{n-1}(k) + d_2 v_{n-2}(k) + d_3 v_{n-3}(k) + ... + d_k v_{n-k}(k)$$

We call $v_n(k)$ the nth generalized $k-$Fibonacci number. For example if $k = 4$, then the 4th sequence of the generalized $4-order$ Fibonacci sequence is

$$0, 0, 1, d_1, d_1^2 + d_2, d_1^3 + 2d_1d_2 + d_3, d_1^4 + 3d_1^2d_2 + d_2^2 + 2d_1d_3 + d_4, \ldots.$$

For some special cases;

- If $k = 2$ and $d_1 = d_2 = 1$ then $\{v_n(2)\}$ is the usual Fibonacci sequence $\{F_n\}$.
- If $k = 2$ and $d_1 = 2$, $d_2 = 1$ then $\{v_n(2)\}$ is the usual Pell sequence $\{P_n\}$.
- If $k = 2$ and $d_1 = 1$, $d_2 = 2$ then $\{v_n(2)\}$ is the usual Jacobsthal sequence $\{J_n\}$.
- If $k = 3$ and $d_1 = d_2 = d_3 = 1$ then $\{v_n(3)\}$ is the Tribonacci sequence $\{T_n\}$.
- If $d_1 = d_2 = d_3 = ... = d_k = 1$ then $\{v_n(k)\}$ is the $k-$Fibonacci sequence defined in [14].
- If $d_1 = 2$, and $d_2 = d_3 = d_4 ... = d_k = 1$ then $\{v_n(k)\}$ is the $k-$Pell sequence.[18].
- If $d_1 = 1$, $d_2 = 2$, and $d_3 = d_4 ... = d_k = 1$ then $\{v_n(k)\}$ is the $k-$Jacobsthal sequence [18].

Now we introduce new matrix. The $n \times n$ generalized $k-$order Fibonacci matrix

$$F_n(k) = [f_{i,j}(k)]_n$$

is defined as for a fixed $k \geq 2$,

$$f_{i,j}(k) = \begin{cases} v_{i-j+1} & \text{if } i - j + 1 \geq 0, \\ 0 & \text{if } i - j + 1 < 0, \end{cases} \tag{2.1}$$

where $v_n = v_{n+k-2}(k)$. For $k = 4$ and $n = 4$ the matrix is as follows:

$$F_4(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ d_1^2 + d_2 & d_1 & 1 & 0 \\ d_1^3 + 2d_1d_2 + d_3 & d_1^2 + d_2 & d_1 & 1 \end{bmatrix}$$

G.Y. Lee and J.S. Kim gave the factorizations of the k-Fibonacci matrix in [14]. Now we give factorizations of the generalized $k-$order Fibonacci matrix $F_n(k)$, where the method is similar to the method in [14]. The selection of the matrices S_l and G_l are same with the selection of Lee and Kim. We use this method for finding the inverse of the generalized $k-$order Fibonacci matrix $F_n(k)$.

Let I_n be the identity matrix of order n, and let L_k be a $k \times k$ lower triangular matrix as follows:

$$L_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ d_1 & 1 & 0 & 0 & \cdots & 0 \\ d_2 & 0 & 1 & 0 & \cdots & 0 \\ d_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ d_{k-1} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.2}$$

Set $S_l = L_{k+1} \oplus I_l$, $l = 1, 2, \ldots$. We define $n \times n$ matrices $\overline{F_n(k)} = [1] \oplus F_{n-1}(k)$, $G_1 = I_n$, $G_2 = I_{n-2} \oplus L_2$, $G_3 = I_{n-3} \oplus L_3, \ldots, G_k = I_{n-k} \oplus L_k$, $G_{k+1} = I_{n-k-1} \oplus L_{k+1}$ and for $k + 2 \leq l \leq n$, $G_l = I_{n-l} \oplus S_{l-k-1}$. In particular $S_0 = L_{k+1}$ and $G_n = S_{n-k-1}$.

We have the following theorem:

Theorem 1. *The generalized $k-$order Fibonacci matrix $F_n(k)$ can be factorized by G_l, $1 \leq l \leq n$, as follows:*

$$F_n(k) = G_1 G_2 \ldots G_n$$

Now we give another factorization of $F_n(k)$. An $n \times n$ matrix $D_n(k) = [d_{ij}(k)]$ is defined as

$$d_{ij}(k) = \begin{cases} v_i & \text{if } j = 1, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

then we can give the following theorem:

Theorem 2. *For $n \geq 2$,*

$$F_n(k) = D_n(k)(I_1 \oplus D_{n-1}(k))(I_2 \oplus D_{n-2}(k)) \ldots (I_{n-2} \oplus D_2(k)).$$

It can be computed that the inverse of L_k is

$$L_k^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -d_1 & 1 & 0 & 0 & \cdots & 0 \\ -d_2 & 0 & 1 & 0 & \cdots & 0 \\ -d_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{k-1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$D_n(k)^{-1} = \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ -v_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -v_n & 0 & & 1 \end{bmatrix}$$

Corollary 1. *Let $G_i^{-1} = H_i$ for $i = 1, 2, ..., n$. Then we have*

$$\begin{aligned} F_n(k)^{-1} &= H_n H_{n-1} ... H_2 H_1 \\ &= (I_{n-2} \oplus D_2(k)^{-1} ... (I_1 \oplus D_{n-1}(k)^{-1}) D_n(k)^{-1} \end{aligned}$$

The inverse of the matrix $F_n(k)$ is obtained as the matrix $F_n(k)^{-1} = F'_{ij}$, as

$$F'_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -d_{i-j}, & 0 < i - j \leq k \\ 0 & \text{otherwise} \end{cases}$$

that is

$$F(k)_n^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -d_1 & 1 & \ddots & & & & \vdots \\ \vdots & -d_1 & \ddots & \ddots & & & \vdots \\ -d_{k-1} & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -d_{k-1} & \cdots & -d_1 & 1 \end{bmatrix}. \qquad (2.3)$$

3. The Generalized F-Nomial Matrix

In this section we will define the generalized $F-$ *nomial* coefficients which is generalization of the $F-$ *nomial* coefficients. The $F-$ *nomial* coefficients was defined as follows in [5]. Let F be a natural numbers' sequence $\{n_F\}_{n\geq 0}$, and $n, k \in \mathbb{N}$, such that $n \geq k$, the $F-$*nomial* coefficient is identified with the symbol

$$\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)_F!}$$

where $n_F! = n_F\,(n-1)_F ... 1_F$ with $0_F! = 1$.[5]

We now give a generalization of $F-nomial$ coefficient which we name as generalized $F-nomial$ coefficient.

Definition 1. *Let GF be any sequence $\{n_{GF}\}_{n\geq 0}$, whose any element is different from 0 and $n,k \in \mathbb{N}$. Then generalized $F-nomial$ coefficient ($GF-nomial$ coefficient) is defined and shown with the symbol*

$$\binom{n}{k}_{GF} = \frac{n_{GF}!}{k_{GF}!(n-k)_{GF}!}$$

where $n_{GF}! = n_{GF}\,(n-1)_{GF} ... 1_{GF}$ with $0_{GF}! = 1$.

For some special cases:

• If GF is a natural numbers' sequence $\{n_F\}_{n\geq 0}$, and $n,k \in \mathbb{N}$, such that $n \geq k$ then $GF-nomial$ coefficient reduce to $F-nomial$ coefficient which is defined in [5].

• If GF is sequence of natural numbers that is $n_{GF} = n$ the $GF-nomial$ coefficients reduce to ordinary binomial coefficients

$$\binom{n}{k}_{GF} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

• If we get GF as Fibonacci sequence $\{F_n\}_{n\geq 0}$ one obtain Fibonomial coefficients that is

$$\binom{n}{k}_{GF} = \frac{F_n!}{F_k!F_{n-k}!} = \binom{n}{k}_{Fib}$$

• Finally if the nth element of the sequence GF is $n_{GF} = n_q = \dfrac{(q^n-1)}{q-1}$ one can obtain $q-$binomial (Gaussian) coefficients.

Now we define generalization of the Pascal matrix similar to the matrices in [10] and [20].

Definition 2. *The $n \times n$ generalized Pascal matrix $\mathcal{F}_n(GF)$ associated with $GF-nomial$ coefficients is defined as*

$$\mathcal{F}_n(GF;i,j) = \binom{i-1}{j-1}_{GF}, \quad i,j = 1,2,\ldots,n, \tag{3.1}$$

with

$$\binom{i-1}{j-1}_{GF} = 0 \quad \text{if } j > i.$$

For example 5×5 generalized Pascal matrix associated with $GF-nomial$ coefficients $\mathcal{F}_5(GF)$ is

$$\mathcal{F}_5(GF) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & \binom{2}{1}_{GF} & 1 & 0 & 0 \\ 1 & \binom{3}{1}_{GF} & \binom{3}{2}_{GF} & 1 & 0 \\ 1 & \binom{4}{1}_{GF} & \binom{4}{2}_{GF} & \binom{4}{3}_{GF} & 1 \end{bmatrix} \tag{3.2}$$

If GF is sequence of natural numbers that is $n_{GF} = n$ then we obtain usual Pascal matrix [3].

If we get GF as Fibonacci numbers $\{F_n\}_{n\geq 0}$ we obtain Fibonomial matrix [20].

If we get GF as Pell numbers $\{P_n\}_{n\geq 0}$ we obtain Pell Pascal matrix [20].

For example, if $n=5$ then these matrices are given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 3 & 6 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 \\ 1 & 12 & 30 & 12 & 1 \end{bmatrix}$$

respectively.

We now give the following recurrence

$$a_n = -\sum_{i=1}^{n-1} a_i \binom{n-1}{i-1}_{GF} \quad , \quad a_1 = 1 \tag{3.3}$$

for finding the inversion formula of the $GF-nomial$ coefficients $\binom{n}{k}_{GF}$.

The inverse of the Pascal matrix associated with $GF-nomial$ coefficients is given in the following theorem.

Theorem 3. *Let $\mathcal{F}_n^{-1}(GF)$ be the $n\times n$ matrix defined by*

$$\mathcal{F}_n^{-1}(GF;i,j) = \begin{cases} a_{i-j+1}\binom{i-1}{j-1}_{GF} & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

where a_n is in (3.3). Then $\mathcal{F}_n^{-1}(GF)$ is inverse of $\mathcal{F}_n(GF)$ the Pascal matrix associated with $GF-nomial$ coefficients.

Proof. It's clear that $(\mathcal{F}_n(GF)\mathcal{F}_n^{-1}(GF))_{ij} = 0$ for $i<j$. If $i=j$, then we have

$$\begin{aligned} (\mathcal{F}_n(GF)\mathcal{F}_n^{-1}(GF))_{ii} &= \sum_{s=1}^{n} \mathcal{F}_n(GF;i,s)\mathcal{F}_n^{-1}(GF;s,i) \\ &= \mathcal{F}_n(GF;i,i)\mathcal{F}_n^{-1}(GF;i,i) \\ &= \binom{i-1}{i-1}_{GF} a_{i-i+1}\binom{i-1}{i-1}_{GF} = 1. \end{aligned}$$

We will prove that $(\mathcal{F}_n(GF)\mathcal{F}_n^{-1}(GF))_{ij} = 0$ for $i > j$. Suppose $i > j$ then

$$\begin{aligned}
(\mathcal{F}_n(GF)\mathcal{F}_n^{-1}(GF))_{ij} &= \sum_{s=1}^{n} \mathcal{F}_n(GF;i,s)\mathcal{F}_n^{-1}(GF;s,j) \\
&= \binom{i-1}{j-1}_{GF} a_1 \binom{j-1}{j-1}_{GF} \\
&\quad + \binom{i-1}{j}_{GF} a_2 \binom{j}{j-1}_{GF} \\
&\quad + \cdots + \binom{i-1}{i-1}_{GF} a_{i-j+1} \binom{i-1}{j-1}_{GF} \\
&= \frac{[i-1]_{GF}!}{[j-1]_{GF}!} \left(\frac{a_1}{[i-j]_{GF}!} + \frac{a_2}{[i-j-1]_{GF}!\,[1]_{GF}!} \right. \\
&\quad \left. + \cdots + \frac{a_{i-j+1}}{[i-j]_{GF}!} \right) \\
&= \frac{[i-1]_{GF}!}{[j-1]_{GF}![i-j]_{GF}!} \left(\frac{a_1[i-j]_{GF}!}{[i-j]_{GF}!} \right. \\
&\quad + \frac{a_2[i-j]_{GF}!}{[i-j-1]_{GF}![1]_{GF}!} \\
&\quad \left. + \cdots + \frac{a_{i-j+1}[i-j]_{GF}!}{[i-j]_{GF}!} \right) \\
&= \frac{[i-1]_{GF}!}{[j-1]_{GF}![i-j]_{GF}!} \left(a_1 \binom{i-j}{0}_{GF} \right. \\
&\quad \left. + a_2 \binom{i-j}{1}_{GF} + \cdots + a_{i-j+1} \binom{i-j}{i-j}_{GF} \right) \\
&= \frac{[i-1]_{GF}!}{[j-1]_{GF}![i-j]_{GF}!} \left(\sum_{k=1}^{i-j} a_k \binom{i-j}{k-1}_{GF} \right. \\
&\quad \left. + a_{i-j+1} \binom{i-j}{i-j}_{GF} \right)
\end{aligned}$$

Using (3.3), we obtain

$$\sum_{s=1}^{n} \mathcal{F}_n(GF;i,s)\mathcal{F}_n^{-1}(GF;s,j) = 0$$

for $i > j$. □

4. Factorizations of The F-Nomial Matrix via generalized k–order Fibonacci matrix

In this section, we discuss new factorizations of $\mathcal{F}_n(GF)$ generalized Pascal matrix associated with $GF - nomial$ coefficients. We define new $n \times n$ matrix $\mathcal{L}_n(k, GF)$ as follows

$$\mathcal{L}_n(k, GF; i, j) = \binom{i-1}{j-1}_{GF} - \sum_{s=1}^{k} d_s \binom{i-s-1}{j-1}_{GF}, \tag{4.1}$$

For $k = 2$ and $n = 3$ we have

$$\mathcal{L}_3(2, GF) = \begin{bmatrix} 1 & 0 & 0 \\ 1 - d_1 & 1 & 0 \\ 1 - d_1 - d_2 & \binom{2}{1}_{GF} - d_1 & 1 \end{bmatrix}$$

Theorem 4. *Let $\mathcal{L}_n(k, GF)$ be $n \times n$ matrix as in (4.1) and $F_n(k)$ be $n \times n$ matrix as in (2.1). Then*

$$\mathcal{F}_n(GF) = F_n(k)\ \mathcal{L}_n(k, GF).$$

Proof. Since the inverse of the matrix $F_n(k)$ is given in (1.1) then it is enough to prove $F_n(k)^{-1}\mathcal{F}_n(GF) = \mathcal{L}_n(k, GF)$. Let $F_n(k)^{-1} = [f'(k)_{ij}]$ be the inverse of the matrix $F_n(k)$ then

$$\begin{aligned} l_{11} &= \sum_{s=1}^{n} F'_{1s} p_{s1} \\ &= F'_{11}p_{11} + F'_{12}p_{21} + \dots + F'_{1n}p_{n1} \\ &= F'_{11}p_{11} \\ &= 1_{GF} \end{aligned}$$

since $F'_{ij} = 0$ for $j \geq 2$.

$$\begin{aligned} l_{1j} &= \sum_{s=1}^{n} F'_{1s} p_{sj} \\ &= F'_{11}p_{1j} + F'_{12}p_{2j} + F'_{13}p_{3j} + \dots + F'_{1n}p_{nj} \\ &= F'_{11}p_{1j} \\ &= 0 \end{aligned}$$

since $p_{1j} = 0$ for $j \geq 2$.

$$\begin{aligned} l_{21} &= \sum_{s=1}^{n} F'_{2s} p_{s1} \\ &= F'_{21} p_{11} + F'_{22} p_{21} + F'_{23} p_{31} + \ldots + F'_{2n} p_{n1} \\ &= F'_{21} p_{11} + F'_{22} p_{21} \\ &= -1_F + 1_F \\ &= 0 \end{aligned}$$

since $F'_{ij} = 0$ for $j \geq 2$.

$$\begin{aligned} l_{22} &= \sum_{s=1}^{n} F'_{2s} p_{s2} \\ &= F'_{21} p_{12} + F'_{22} p_{22} + F'_{23} p_{32} + \ldots + F'_{2n} p_{n2} \\ &= F'_{22} p_{22} \\ &= 1_{GF} \end{aligned}$$

since $p_{1j} = 0$ for $j \geq 2$.

$$\begin{aligned} l_{2j} &= \sum_{s=1}^{n} F'_{2s} p_{sj} \\ &= F'_{21} p_{1j} + F'_{22} p_{2j} + F'_{23} p_{3j} + \ldots + F'_{2n} p_{nj} \\ &= F'_{21} p_{1j} + F'_{22} p_{2j} \\ &= 0 \end{aligned}$$

since $p_{1j} = 0$ for $j \geq 2$.
Finally from the definition of $F_n(k)^{-1}$ and the recurrence relation (4.1) for

$i \geq j$ we have

$$\begin{aligned}
\sum_{s=1}^{n} F'(k)_{is} p_{sj} &= F'_{i1} p_{1j} + \dots + F'_{i(i-k-2)} p_{(i-k-2)j} \\
&\quad + F'_{i(i-k-1)} p_{(i-k-1)j} + F'_{i(i-k)} p_{(i-k)j} \\
&\quad + F'_{i(i-k+1)} p_{(i-k+1)j} + F'_{i(i-k+2)} p_{(i-k+2)j} \\
&\quad + \dots + F'_{i(i-1)} p_{(i-1)j} + F'_{ii} p_{ij} \\
&\quad + F'_{i(i+1)} p_{(i+1)j} + \dots + F'_{in} p_{nj} \\
&= -\binom{i-k-1}{j-1}_{GF} - \binom{i-k}{j-1}_{GF} \\
&\quad -\binom{i-k+1}{j-1}_{GF} - \dots - \binom{i-3}{j-1}_{GF} \\
&\quad -\binom{i-2}{j-1}_{GF} + \binom{i-1}{j-1}_{GF} \\
&= l_{ij}
\end{aligned}$$

since $F'_{ij} = 0$ for $i - j > k$.

It is obviously seen that for $i < j$, $l_{ij} = 0$ and for $i = j$, $l_{ij} = 1$. □

Now, we define a new $n \times n$ matrix $R_n(k, GF)$ as follows.

$$R_n(k, GF; i, j) = \binom{i-1}{j-1}_{GF} - \sum_{s=1}^{k} d_s \binom{i-1}{j+s-1}_{GF}. \tag{4.2}$$

For $k = 3$ and $n = 3$ we have

$$R_3(3, GF) = \begin{bmatrix} 1 & 0 & 0 \\ 1 - d_1 & 1 & 0 \\ 1 - d_1\binom{2}{1}_{GF} - d_2 & \binom{2}{1}_{GF} - d_1 & 1 \end{bmatrix}$$

Theorem 5. *Let $R_n(k, GF)$ be the matrix as in (4.2). Then*

$$\mathcal{F}_n(GF) = R_n(k, GF) F_n(k).$$

5. Illustrative Example

Example 1. *For the sequence $GF = (-1)^n \dfrac{n}{n^2+1}$, $k = 3$, and $n = 4$ then*

$$\mathcal{F}_4\left((-1)^n \frac{n}{n^2+1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -\frac{4}{5} & 1 & 0 \\ 1 & \frac{3}{5} & \frac{3}{5} & 1 \end{bmatrix}$$

and

$$F_4(3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ d_1^2 + d_2 & d_1 & 1 & 0 \\ d_1^3 + 2d_1d_2 + d_3 & d_1^2 + d_2 & d_1 & 1 \end{bmatrix}.$$

The matrices

$$L_4\left(3, (-1)^n \frac{n}{n^2+1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - d_1 & 1 & 0 & 0 \\ 1 - d_1 - d_2 & -\frac{4}{5} - d_1 & 1 & 0 \\ 1 - d_1 - d_2 - d_3 & \frac{3}{5} + \frac{4d_1}{5} - d_2 & \frac{3}{5} - d_1 & 1 \end{bmatrix}$$

and

$$R_4\left(3, (-1)^n \frac{n}{n^2+1}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - d_1 & 1 & 0 & 0 \\ 1 + \frac{4d_1}{5} - d_2 & -\frac{4}{5} - d_1 & 1 & 0 \\ 1 - \frac{3d_1}{5} - \frac{3d_2}{5} - d_3 & \frac{3}{5} - \frac{3d_1}{5} - d_2 & \frac{3}{5} - d_1 & 1 \end{bmatrix}$$

For first factorization

$$\begin{aligned} & F_4(3) L_4\left(3, (-1)^n \frac{n}{n^2+1}\right) \\ = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ d_1^2 + d_2 & d_1 & 1 & 0 \\ d_1^3 + 2d_1d_2 + d_3 & d_1^2 + d_2 & d_1 & 1 \end{bmatrix} \\ & \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - d_1 & 1 & 0 & 0 \\ 1 - d_1 - d_2 & -\frac{4}{5} - d_1 & 1 & 0 \\ 1 - d_1 - d_2 - d_3 & \frac{3}{5} + \frac{4d_1}{5} - d_2 & \frac{3}{5} - d_1 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ d_1^2 + d_2 & d_1 & 1 & 0 \\ d_1^3 + 2d_1d_2 + d_3 & d_1^2 + d_2 & d_1 & 1 \end{bmatrix} \\ = & \mathcal{F}_4\left((-1)^n \frac{n}{n^2+1}\right) \end{aligned}$$

and the second is

$$R_4\left(3,(-1)^n\frac{n}{n^2+1}\right)F_4(3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-d_1 & 1 & 0 & 0 \\ 1+\frac{4d_1}{5}-d_2 & -\frac{4}{5}-d_1 & 1 & 0 \\ 1-\frac{3d_1}{5}-\frac{3d_2}{5}-d_3 & \frac{3}{5}-\frac{3d_1}{5}-d_2 & \frac{3}{5}-d_1 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ d_1^2+d_2 & d_1 & 1 & 0 \\ d_1^3+2d_1d_2+d_3 & d_1^2+d_2 & d_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -\frac{4}{5} & 1 & 0 \\ 1 & \frac{3}{5} & \frac{3}{5} & 1 \end{bmatrix}$$

$$= \mathcal{F}_4\left((-1)^n\frac{n}{n^2+1}\right)$$

6. Conclusion

In this study we defined the generalized $k-$order Fibonacci sequence, generalized $k-$order Fibonacci matrix and the $n \times n$ generalized Pascal matrix $\mathcal{F}_n(GF)$ associated with generalized $F-$*nomial* coefficients. We found the inverse of generalized Pascal matrix $\mathcal{F}_n(GF)$ associated with generalized $F-$*nomial* coefficients. In the last section we factorized this matrix via generalized $k-$order Fibonacci matrix. By our factorizations the results in [13], [24], [14], [18], [19], [20] are our special cases.

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