A CONVOLUTION FORMULA FOR BERNOULLI POLYNOMIALS

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ABSTRACT. In this note, we establish a convolution formula for Bernoulli polynomials in a new and brief way, and some known results are derived as a special case.

1. INTRODUCTION

The classic Bernoulli polynomials $B_n(x), n = 0, 1, 2, \cdots$, are usually defined by the following exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi). \tag{1}$$

In particular, $B_n = B_n(0)$ are called Bernoulli numbers.

In the present note, we will be concerned with the convolution formulae for the Bernoulli polynomials and numbers. The best known such convolution relation is Euler's formula

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \geq 1), \tag{2}$$

which can be rewritten as

$$(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n \quad (n \geq 1), \tag{3}$$

by using the umbral calculas. Very recently, by making use of some connections between the Bernoulli numbers and the Stirling numbers of the second kind, Agoh and Dilcher [1] obtained an explicit formula for $(B_k+B_m)^n$ with arbitrary fixed integers $k, m, n \geq 0$, by virtue of which they deduced some surprising and unusual recurrence relations for Bernoulli numbers. Also, see [4] for a different proof of the Agoh-Dilcher's identity by applying the extended Zeilberger's algorithm.

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Motivated by the work of Agoh and Dilcher, in this note we shall give the expression of $(B_k(x)+B_m(y))^n$ for any non-negative integers k,m,n in a new and brief way. For convenience, in this following we always denote $B_{-n}(x)=0$ for any positive integer n.

Theorem 1.1. *Let k,m,n be any non-negative integers. Then*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}B_{k+i}(x)B_{m+n-i}(y)\\
&\quad=-\frac{k!m!}{(k+m+1)!}(n+\delta(k,m)(k+m+1))B_{k+m+n}(x+y)\\
&\qquad+\sum_{i=0}^{k+m}(-1)^{k+i}\left\{n\binom{k}{i}-m\binom{k}{i-1}\right\}B_{n-1+i}(x+y)\frac{B_{k+m+1-i}(y)}{k+m+1-i}\\
&\qquad+\sum_{i=0}^{k+m}(-1)^{m+i}\left\{n\binom{m}{i}-k\binom{m}{i-1}\right\}B_{n-1+i}(x+y)\frac{B_{k+m+1-i}(x)}{k+m+1-i},
\end{aligned}$$

where $\delta(k,m)=-1$ when $k=m=0$, $\delta(k,m)=0$ when $k=0,m\geq 1$ or $m=0,k\geq 1$, and $\delta(k,m)=1$ otherwise.

Theorem 1.1, as well as the identity of Agoh and Dilcher, leads to several important arithmetic properties of Bernoulli polynomials. For example, setting $n=0, y=1-x$ in Theorem 1.1 and then using $B_n(1-x)=(-1)^nB_n(x)$ $(n\geq 0)$ and $B_{2n+1}=0$ $(n\geq 1)$, we derive

$$\begin{aligned}
B_k(x)B_m(x)=\sum_i\left\{m\binom{k}{2i}+k\binom{m}{2i}\right\}B_{2i}\frac{B_{k+m-2i}(x)}{k+m-2i}&\\
+(-1)^{k+1}\frac{k!m!B_{k+m}}{(k+m)!}&\quad(k,m\geq 1),\qquad(4)
\end{aligned}$$

which appeared in [3, 6] in different ways, and was used by Carlitz [2] to give the proof of a reciprocity formula of the generalized Dedekind sums. Similarly, setting $n=0, x+y=1-z$ in Theorem 1.1 we get

$$\begin{aligned}
&\frac{(-1)^k}{k}\sum_{i=0}^{k}\binom{k}{i}B_{k-i}(z)\frac{B_{m+i}(y)}{m+i}+\frac{(-1)^m}{m}\sum_{i=0}^{m}\binom{m}{i}B_{m-i}(z)\frac{B_{k+i}(x)}{k+i}\\
&\qquad=(-1)^{k+m}\frac{(k-1)!(m-1)!B_{k+m}(z)}{(k+m)!}+\frac{B_k(x)}{k}\frac{B_m(y)}{m}\qquad(5)
\end{aligned}$$

with k,m being both positive integers, which was discovered and used to deduce an extension of the Woodcock's identity [9] by Pan and Sun [7] who made use of an symmetric relation for Bernoulli polynomials due to Sun (see (1.14) in [8] or Corollary 1.4 in [5]).

2. An auxiliary theorem

We deduce Theorem 1.1 from our following auxiliary result.

Theorem 2.1. *Let k, m, n be any non-negative integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}\left(B_i(x)+B_{m+n-i}(y)\right)^k$$
$$=\frac{k+n}{m+1}\sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+i}B_{m+1-i}(x)B_{k+n-1+i}(x+y)$$
$$+\frac{k+n}{m+1}B_{m+1}(y)B_{k+n-1}(x+y)+B_m(y)B_{k+n}(x+y).$$

Proof. Observe that

$$\frac{ve^{xv}}{e^v-1}\frac{(u+v)e^{y(u+v)}}{e^{u+v}-1}=(u+v)\frac{e^{yu}}{e^u-1}\frac{ve^{(x+y)v}}{e^v-1}$$
$$-v\frac{e^{(1-x)u}}{e^u-1}\frac{(u+v)e^{(x+y)(u+v)}}{e^{u+v}-1}. \tag{6}$$

Now, making k-times derivative for the above identity (6) with respect to v, with the help of the Leibniz rule we derive

$$\sum_{j=0}^{k}\binom{k}{j}\frac{\mathrm{d}^j}{\mathrm{d}v^j}\left(\frac{ve^{xv}}{e^v-1}\right)\frac{\mathrm{d}^{k-j}}{\mathrm{d}v^{k-j}}\left(\frac{(u+v)e^{y(u+v)}}{e^{u+v}-1}\right)$$
$$=\frac{ue^{yu}}{e^u-1}\frac{\mathrm{d}^k}{\mathrm{d}v^k}\left(\frac{ve^{(x+y)v}}{e^v-1}\right)+k\frac{e^{yu}}{e^u-1}\frac{\mathrm{d}^{k-1}}{\mathrm{d}v^{k-1}}\left(\frac{ve^{(x+y)v}}{e^v-1}\right)$$
$$+v\frac{e^{yu}}{e^u-1}\frac{\mathrm{d}^k}{\mathrm{d}v^k}\left(\frac{ve^{(x+y)v}}{e^v-1}\right)-v\frac{e^{(1-x)u}}{e^u-1}\frac{\mathrm{d}^k}{\mathrm{d}v^k}\left(\frac{(u+v)e^{(x+y)(u+v)}}{e^{u+v}-1}\right)$$
$$-k\frac{e^{(1-x)u}}{e^u-1}\frac{\mathrm{d}^{k-1}}{\mathrm{d}v^{k-1}}\left(\frac{(u+v)e^{(x+y)(u+v)}}{e^{u+v}-1}\right). \tag{7}$$

Note that

$$\frac{e^{xu}}{e^u-1}-\frac{1}{u}=\sum_{m=0}^{\infty}\frac{B_{m+1}(x)}{m+1}\frac{u^m}{m!}, \tag{8}$$

and

$$\frac{(u+v)e^{x(u+v)}}{e^{u+v}-1}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{m+n}(x)\frac{u^m}{m!}\frac{v^n}{n!}. \tag{9}$$

By putting (8) and (9) into (7), in view of the Cauchy product and the fact $B_n(1-x) = (-1)^n B_n(x)$ $(n \geq 0)$, we derive

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k}\binom{k}{j}B_{i+j}(x)B_{k+m+n-i-j}(y)\right]\frac{u^m}{m!}\frac{v^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[B_m(y)B_{k+n}(x+y) + \frac{kB_{m+1}(y)B_{k+n-1}(x+y)}{m+1}\right]\frac{u^m}{m!}\frac{v^n}{n!} \\
&+ \frac{k}{u}\sum_{n=0}^{\infty}B_{k+n-1}(x+y)\frac{v^n}{n!} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B_{m+1}(y)B_{k+n}(x+y)}{m+1}\frac{u^m}{m!}\frac{v^{n+1}}{n!} \\
&+ \frac{1}{u}\sum_{n=0}^{\infty}B_{k+n}(x+y)\frac{v^{n+1}}{n!} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{k+m+n}(x+y)\frac{u^{m-1}}{m!}\frac{v^{n+1}}{n!} \\
&+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{i=0}^{m}\binom{m}{i}(-1)^{m+i}\frac{B_{m+1-i}(x)}{m+1-i}B_{k+n+i}(x+y)\right]\frac{u^m}{m!}\frac{v^{n+1}}{n!} \\
&+ k\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{i=0}^{m}\binom{m}{i}(-1)^{m+i}\frac{B_{m+1-i}(x)}{m+1-i}B_{k+n-1+i}(x+y)\right]\frac{u^m}{m!}\frac{v^n}{n!} \\
&- k\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{k+m+n-1}(x+y)\frac{u^{m-1}}{m!}\frac{v^n}{n!}. \qquad (10)
\end{aligned}$$

Thus, Theorem 2.1 follows immediately by comparing the coefficients of $u^m v^{n+1}/m!n!$ and $u^m/m!$ in (10), respectively. □

3. The proof of Theorem 1.1

Proof of Theorem 1.1 via Theorem 2.1. By setting $k = 0$ in Theorem 2.1, we have

$$\begin{aligned}
\big(B_0(x) + B_m(y)\big)^n &= n\sum_{i=0}^{m}\binom{m}{i}(-1)^{m+i}B_{n-1+i}(x+y)\frac{B_{m+1-i}(x)}{m+1-i} \\
&- \frac{nB_{m+n}(x+y)}{m+1} + \frac{nB_{m+1}(y)B_{n-1}(x+y)}{m+1} \\
&+ B_m(y)B_n(x+y), \qquad (11)
\end{aligned}$$

which means the case $\delta(k,m) = -1$ or 0 in Theorem 1.1 holds.

Next, consider the case $\delta(k,m) = 1$. We shall use induction on k in Theorem 2.1 to prove Theorem 1.1. Taking $k = 1$ in Theorem 2.1, in light of (11) we obtain

$$
\begin{aligned}
&\left(B_1(x)+B_m(y)\right)^n\\
&=-\frac{(m+n+2)B_{m+n+1}(x+y)}{(m+1)(m+2)}+B_m(y)B_{n+1}(x+y)\\
&\quad+\frac{(n-m)B_{m+1}(y)B_n(x+y)}{m+1}-\frac{nB_{m+2}(y)B_{n-1}(x+y)}{m+2}\\
&\quad+\sum_{i=0}^{m+1}(-1)^{m+i}\left(n\binom{m}{i}-\binom{m}{i-1}\right)B_{n-1+i}(x+y)\frac{B_{m+2-i}(x)}{m+2-i},
\end{aligned}
$$

which implies the case $k=1$ of Theorem 1.1. Now, assume that Theorem 1.1 holds for all positive integers being less than k. By Theorem 2.1 we have

$$
\begin{aligned}
&\left(B_k(x)+B_m(y)\right)^n\\
&=B_m(y)B_{k+n}(x+y)+\frac{(k+n)B_{m+1}(y)B_{k+n-1}(x+y)}{m+1}\\
&\quad+\frac{k+n}{m+1}\sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+i}B_{m+1-i}(x)B_{k+n-1+i}(x+y)\\
&\quad-\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k-1}\binom{k}{j}B_{i+j}(x)B_{k+m+n-i-j}(y)\\
&=B_m(y)B_{k+n}(x+y)+\frac{(k+n)B_{m+1}(y)B_{k+n-1}(x+y)}{m+1}\\
&\quad-\frac{(k+n)B_{k+m+n}(x+y)}{m+1}-\left(B_0(x)+B_{k+m}(y)\right)^n\\
&\quad+\frac{k+n}{m+1}\sum_{i=0}^{m}\binom{m+1}{i}(-1)^{m+i}B_{m+1-i}(x)B_{k+n-1+i}(x+y)\\
&\quad-\sum_{j=1}^{k-1}\binom{k}{j}\left(B_j(x)+B_{k+m-j}(y)\right)^n. \qquad (12)
\end{aligned}
$$

Applying (11) and Theorem 1.1 to (12) we derive

$$\begin{aligned}
&\left(B_k(x)+B_m(y)\right)^n \\
&= B_m(y)B_{k+n}(x+y)+\frac{(k+n)B_{m+1}(y)B_{k+n-1}(x+y)}{m+1} \\
&\quad -\frac{(k+n)B_{k+m+n}(x+y)}{m+1}-B_{k+m}(y)B_n(x+y) \\
&\quad -\frac{nB_{k+m+1}(y)B_{n-1}(x+y)}{k+m+1}+\frac{nB_{k+m+n}(x+y)}{k+m+1} \\
&\quad +\sum_{i=0}^{k+m}(-1)^{k+m+i}\left\{(k+n)\binom{m}{i-k}\right. \\
&\qquad\qquad \left.-n\binom{k+m}{i}\right\}B_{n-1+i}(x+y)\frac{B_{k+m+1-i}(x)}{k+m+1-i} \\
&\quad +\frac{(k+m+n+1)B_{k+m+n}(x+y)}{k+m+1}\sum_{j=1}^{k-1}\binom{k}{j}\Big/\binom{k+m}{j} \\
&\quad -\sum_{i=0}^{k+m}(-1)^{k+i}\sum_{j=1}^{k-1}\binom{k}{j}(-1)^{k-j}\left\{n\binom{j}{i}\right. \\
&\qquad\qquad \left.-(k+m-j)\binom{j}{i-1}\right\}B_{n-1+i}(x+y)\frac{B_{k+m+1-i}(y)}{k+m+1-i} \\
&\quad -\sum_{i=0}^{k+m}(-1)^{m+i}\sum_{j=1}^{k-1}\binom{k}{j}(-1)^{j}\left\{n\binom{m+j}{i}\right. \\
&\qquad\qquad \left.-(k-j)\binom{m+j}{i-1}\right\}B_{n-1+i}(x+y)\frac{B_{k+m+1-i}(x)}{k+m+1-i}. \qquad (13)
\end{aligned}$$

Note that for any non-negative integers i, k, m,

$$\sum_{j=0}^{k}\frac{\binom{k}{j}}{\binom{k+m}{j}}=\frac{k+m+1}{m+1},\quad \sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{m+j}{i}=(-1)^k\binom{m}{i-k}. \qquad (14)$$

It follows from (14) that

$$\sum_{j=1}^{k-1}\binom{k}{j}\Big/\binom{k+m}{j}=\frac{k}{m+1}-\frac{k!m!}{(k+m)!},$$

$$\sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\binom{m+j}{i}=(-1)^k\left(\binom{m}{i-k}-\binom{k+m}{i}\right)-\binom{m}{i},$$

$$\sum_{j=1}^{k-1}(-1)^j(k-j)\binom{k}{j}\binom{m+j}{i-1}=(-1)^{k-1}k\binom{m}{i-k}-k\binom{m}{i-1},$$

$$\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}\binom{j}{i}=\binom{0}{i-k}-(-1)^k\binom{0}{i}-\binom{k}{i},$$

$$\sum_{j=1}^{k-1}(-1)^{k-j}(k+m-j)\binom{k}{j}\binom{j}{i-1}$$
$$=-m\binom{k}{i-1}-(-1)^k(k+m)\binom{0}{i-1}+m\binom{0}{i-k-1}+k\binom{0}{i-k}.$$

Thus, by applying the above five identities to (13) we conclude the induction step after simple calculation. We are done. □

4. Concluding remarks

To conclude this note, we remark that the convolution formulae involving the Euler polynomials given by

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad (|t|<\pi) \tag{15}$$

can also be easily derived by using the above methods, as follows,

$$\sum_{i=0}^{n}\binom{n}{i}E_{k+i}(x)B_{m+n-i}(y)$$
$$=-\frac{1}{2}\sum_{i=0}^{k+m}(-1)^{k+i}\left\{n\binom{k}{i}-m\binom{k}{i-1}\right\}E_{n-1+i}(x+y)E_{k+m-i}(y)$$
$$+\sum_{i=0}^{k+m}\binom{m}{i}(-1)^{m+i}B_{n+i}(x+y)E_{k+m-i}(x)\quad (k,m,n\geq 0), \tag{16}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}E_{k+i}(x)E_{m+n-i}(y)$$
$$=-2\sum_{i=0}^{k+m}\binom{k}{i}(-1)^{k+i}E_{n+i}(x+y)\frac{B_{k+m+1-i}(y)}{k+m+1-i}$$
$$-2\sum_{i=0}^{k+m}\binom{m}{i}(-1)^{m+i}E_{n+i}(x+y)\frac{B_{k+m+1-i}(x)}{k+m+1-i}$$
$$+2\frac{k!m!}{(k+m+1)!}E_{k+m+n+1}(x+y)\quad (k,m,n\geq 0), \tag{17}$$

which can be regarded as the further generalization of the formulae for (12) and (16) in [6] and (2.12) and (2.16) in [7]. We leave them to the interested readers for an exercise.

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