Quasi-almostmedian graphs

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Abstract

We introduce quasi-almostmedian graphs as a natural nonbipartite generalization of almostmedian graphs. They are filling a gap between quasi-median graphs and quasi-semimedian graphs. We generalize some results of almostmedian graphs and deduce some results from a bigger class of quasi-semimedian graphs. The consequence of this is another characterization of almostmedian graphs as well as two new characterizations of quasi-median graphs.

1 Introduction and preliminaries

Median graphs constitute the most important subclass of partial cubes, i.e. isometric subgraphs of hypercubes. They have been intensively studied during past 25 years. There are known more then 50 characterizations of median graphs, see the survey [13].

Several generalizations of median graphs are known. Quasi-median graphs are nonbipartite, see [1] or book [11], while almostmedian and semimedian graphs remains in the class of partial cubes and where introduced in [10], see also [3, 5, 4, 12] for more information. Quasi-semimedian graphs were introduced by Brešar in [2] as natural nonbipartite generalization of semimedian graphs as well as a generalization of quasi-median graphs. This class was then further investigated in [6] where several characterization are presented.

In this paper we define quasi-almostmedian graphs as a natural (nonbipartite) generalization of almostmedian graphs. We will show a similar

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relation between quasi-almostmedian and quasi-semimedian graphs as it is between almostmedian and semimedian graphs, based on equality of two edge relations as shown recently in [3]. Furthermore we define the almost-quadrangle property that is characteristic for quasi-almostmedian graphs (as analogue to the semi-quadrangle property from [6]). As a consequence we obtain a new characterization of almostmedian and quasi-median graphs, as well as a connection between quasi-almostmedian graphs and weak modularity.

We continue with definitions of several basic graph theoretical concepts and refer to standard texts or to [11] for the terms not listed here.

The *distance* $d_G(u, v)$, or briefly $d(u, v)$, between two vertices u and v in a graph G is defined as the number of edges on a shortest u, v-path. A subgraph H of G is called *isometric*, if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, and H is *convex* if for every $u, v \in V(H)$ all shortest u, v-paths belong to H. Convex subgraphs are isometric.

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a, x) is adjacent to (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes.* Q_3 is usual (graphically) cube and with Q_3^- we denote cube Q_3 without one vertex. Graph $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_t}$ is called *Hamming graph.* Isometric subgraphs of Hamming graphs are called *partial Hamming graphs.* Graph $K_2 \Box K_3$ without one vertex is called a *house.*

For partial cubes, the sets W_{ab} and U_{ab} that we shall define below play a crucial role. Let ab be an edge of connected graph $G = (V, E)$. Then

$$W_{ab} = \{w \in V \mid d_G(a, w) < d_G(b, w)\}, \text{ and}$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}.$$

We will use the notation U_{ab} also for subgraphs induced by the vertices of set U_{ab}.

A vertex x is a *median* for triple of vertices u, v, and w of G if x lies on a shortest u, v-path, on a shortest u, w-path, and on a shortest v, w-path. A graph G is a *median graph* if there exists a unique median to every triple $u, v, w \in V(G)$.

It follows from results in [1] that median graphs are precisely the bipartite graphs in that all U_{ab}'s are convex. By this result, the following definitions from [10] make sense. A bipartite graph is a *semimedian* graph if it is a partial cube in that all U_{ab}'s are connected. Similarly, a bipartite graph is *almostmedian* if it is a partial cube for which every U_{ab} is an isometric subgraph of G. It is clear that median graphs are almostmedian graphs, that almostmedian graphs are semimedian graphs, and that semimedian graphs are partial cubes. If we do not restrict to a bipartite case we obtain the following definitions: a graph G is *quasi-semimedian* if it is a partial Hamming graph and all U_{ab}'s are connected, and G is *quasi-almostmedian* if it is a partial Hamming graph and every U_{ab} is isometric.

One of the most useful relations for the investigation of metric properties of graphs in general and partial cubes and Cartesian products in particular is the Djoković relation $\sim$ (cf. [9]). Two edges $e = xy$ and $f = uv$ of G are in the relation $\sim$ if $x \in W_{uv}$ and $y \in W_{vu}$. Clearly, $\sim$ is reflexive and symmetric, however not transitive in general. This cannot happen in the class of partial Hamming graphs where $\sim$ is transitive (see [15]). (Note that in the case of partial cubes we often use the notation Θ for $\sim$.)

Edges e and f are in relation $\approx$ if $e \sim f$ or there exist edges e' and f' that belong to the same clique, such that $e \sim e'$ and $f' \sim f$. The relation $\approx$ was first introduced in [2] (denoted there by Δ) and is reflexive, symmetric and is transitive for partial Hamming graphs [2]. Also $\sim \subseteq \approx$.

Another relevant relation defined on the edge set of a graph is δ. We say an edge e is in relation δ to an edge f if $e = f$ or if e and f are opposite edges of an induced 4-cycle. Clearly δ is reflexive and symmetric. Moreover, it is contained in $\sim$. Thus its transitive closure δ^* is contained in $\sim$ and in $\approx$ in the class of partial Hamming graphs. In [6] it is shown that a bipartite graph is quasi-semimedian if and only if $\sim = \delta^*$.

Suppose that $e = e_1 \delta e_2 \delta \ldots \delta e_k = f$ is a sequence of edges by virtue of which e and f are in relation δ^*. The union of squares that contain e_i and e_{i+1}, where $i = 1, 2, \ldots, k-1$ forms a ladder. In such a case we shall say that e and f are connected by a "ladder". Clearly a ladder does not necessarily provide a shortest path between e and f.

We will frequently use the following results of Brešar [2], Chepoi [7], and Wilkeit [14], respectively.

Theorem 1 *A connected graph G is partial Hamming graph if and only if*
(i) the relation $\approx$ is transitive,
(ii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and
(iii) If P is a path connecting the endpoints of an edge xy, then P contains an edge ab with $xy \approx ab$.

Lemma 2 *Let G be a partial Hamming graph and K a clique in G. Then for any vertex $u \in V(G)$ the distances from u to vertices of K are either equal or there exists a unique $x \in K$ that is closer to u than other vertices of K.*

Lemma 3 *If G is a partial Hamming graph then: if a vertex $w \in V(G)$ has the same distance to adjacent vertices x and y of G, then any two neighbors of $a \in W_{xy}$ and $b \in W_{yx}$ of w are adjacent.*

2 The main result

Suppose that vertices u, w, x, y of G have the following properties: $d(u, x) = d(u, y) = k = d(u, w) - 1$ and w is adjacent to x and y. *The quadrangle property* for these vertices is fulfilled if there exists a common neighbor v of x and y with $d(u, v) = k - 1$. In [6] *the semi-quadrangle property* was introduced. The difference is that there exists an edge ab with the property $ab\delta^* xw$ and $d(u, a) = k - 1$. Note that if $ab\delta xw$ and $b = y$ we have the quadrangle property. *The almost-quadrangle property* is fulfilled if there exists an edge ab with $ab\delta xw$ and $d(u, a) = k - 1$, see Figure 1. Again if $b = y$ we obtain the quadrangle property. The graph G satisfies *the almost-quadrangle property* if the almost-quadrangle property is fulfilled for all vertices u, w, x, y with the above properties.

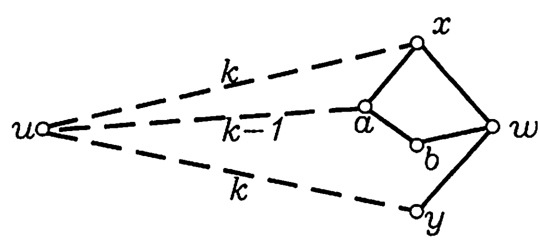

Figure 1: Almost-quadrangle property

In this section we prove that the almost-quadrangle property is characteristic for quasi-almostmedian graphs in the class of partial Hamming graphs as analogue to the result for quasi-semimedian graphs and semi-quadrangle property from [6], as well as analogue from [3] where convex cycle C_{2n}, $n \geq 3$, is forbidden and $\sim = \delta^*$ must hold. But first a lemma.

Lemma 4 *Let P be a shortest path in a partial Hamming graph G. Then no two edges of P are in relation $\approx$.*

Proof Suppose that edges uv and ab of the shortest path P are in relation $\approx$. We may assume that u is the first vertex of P and b the last one. Thus $u, v \in W_{ab}$ and $a, b \in W_{vu}$. Since P is a shortest path, $uv \not\sim ab$. Thus there exist two edges xy and wz of the same clique in G with $uv \sim xy$ and $wz \sim ab$. We can chose the notation so that $y \in W_{vu}$ and $z \in W_{ab}$. If xy and wz are not incident, y is closer to v than both w and z by Lemma 2. This is a contradiction to $v \in W_{ab}$, since by Theorem 1 *(ii)* v should be closer to z. Thus xy and wz must be incident and suppose that $y = z$.

Again by Theorem 1 (*ii*), $W_{vu} = W_{yx}$. A contradiction with $b \in W_{yx}$ since $d(b,y) = d(b,x)$. We have similarities in all the other cases when $y = w$, $x = z$, or $x = w$ and the proof is complete. □

Theorem 5 *Let G be a partial Hamming graph. The following assertions are equivalent:*

(*i*) *G is quasi-almostmedian,*

(*ii*) *G satisfies the almost-quadrangle property,*

(*iii*) *$\delta^* =\sim$ and G does not contain convex cycles C_{2r}, $r \geq 3$.*

Proof (*i*) $\Rightarrow$ (*ii*) Let G be a quasi-almostmedian graph. Then G is a quasi-semimedian and semi-quadrangle property holds. Let u, x, y, w be vertices with properties $d(u,x) = d(u,y) = k = d(u,w) - 1$ and w be a common neighbor of x and y. By semi-quadrangle property there exists (at least) an edge ab that is in relation δ with xw. Thus, also $ab \sim xw$. Let P be a shortest u,x-path and let Q be a shortest u,y-path. Consider the path that is a subwalk of the walk $P \cup Q \cup \{yw\}$ and starts in x and ends in w. On this path exists an edge zv for which $zv \approx xw$ holds by (*iii*) of Theorem 1. Suppose that there exists a clique with edges e and f such that $xw \sim e$ and $f \sim zv$. Since zv is on a shortest path from u, one of the vertices z or v is closer to u. Hence by Theorem 1 (*ii*) one of the endvertices of f is closer to u than the other. If e and f have no vertex in common we deduce by Lemma 2 that this endvertex of f is closer to u than both endvertices of e. This contradicts $u \in W_{xw}$, since by Theorem 1 (*ii*), u should be closer to one endvertex of e. Otherwise, if e and f have a vertex in common it must be by Theorem 1 (*ii*) and Lemma 2 closer to u than the remaining vertices s and t of e and f, respectively. Consider now the path starting in t and proceeding by the f, zv-ladder to v continuing on Q to y, then to w and down the wx, e-ladder to s. By Theorem 1 (*iii*) there exists an edge on this path that is in relation $\approx$ with st. This edge must be yw, since all the other edges on this path lie on some shortest path already containing an edge in relation $\approx$ with st (otherwise we get a contradiction with Lemma 4). By the transitivity of $\approx$ we have $xw \approx wy$, which is impossible.

Thus we have $zv \sim xw$. Suppose that z is closer to u than v. The ladder between ab and zv is isometric in W_{wx} and has the same length as a shortest v,w-path. But then

$$\begin{aligned} d(a,u) &\leq d(a,z) + d(z,u) \\ &= d(b,v) + d(v,u) - 1 \\ &= d(w,v) - 1 + d(v,u) - 1 \\ &= k - 1. \end{aligned}$$

Moreover, a is a neighbor of x and $d(u,x) = k$. Thus $d(u,a) \geq k-1$ which yields the desired equality.

(ii) $\Rightarrow$ *(iii)* Suppose that G contains a convex cycle C of length $2k$ for $k > 2$. Pick two antipodal vertices u and w of C and let x, y be the neighbors of w on C. By almost-quadrangle condition, there exists an edge ab as in Fig. 1. Since C is convex and a lies on a shortest path between u and x, we conclude that $a \in C$. Since b belongs to a shortest path between the vertices a and w of C, we obtain a contradiction with convexity of C.

Thus $\delta^* \neq \sim$. Since $\delta^* \subseteq \sim$ in partial Hamming graphs, we have $\sim \not\subseteq \delta^*$. Thus $uv \sim ab$ and there is no ladder between uv and ab. Among all such pairs of edges in G let they be chosen such that their distance $n \geq 2$ is as small as possible. The assumption that G fulfills the almost-quadrangle property will lead us to a contradiction. Indeed, let w be a neighbor of v on a shortest v, b-path. Then by the almost-quadrangle property for a, u, w, v there exists an edge xy with $xy\delta uv$ and $d(x,a) = n-1$. However $xy\delta uv$ implies $xy \sim uv$ and by transitivity of $\sim$ in partial Hamming graphs we have $xy \sim ab$. Hence a contradiction with minimality of n, because the ladder between xy and ab would imply the ladder between uv and ab.

(iii) $\Rightarrow$ *(i)* In this part of the proof we closely follow the proof of the Theorem 4 from [3]. The main difference is that here we must deal with relation $\approx$ not with $\sim$ (Θ).

Since for G $\delta^* = \sim$ holds, G is quasi-semimedian. Let G be without convex cycles C_{2k}, $k > 2$. The assumption that G is not quasi-almostmedian will lead us to a contradiction. If G is not quasi-almostmedian then there are two edges ab and xy which are in relation $\sim$ but no ladder between them is isometric. Among all such pairs of edges in G let they be chosen so that their distance $n \geq 2$ is as small as possible. Let C be a cycle formed by a shortest path $a = u_0, u_1 u_2, \ldots, u_n = x$, edge xy, a shortest path $y = v_n, v_{n-1}, \ldots, v_1, v_0 = b$ and edge ba.

The assumption that there exists a path P that violates the convexity of C will lead us to a contradiction. We may assume that P connects two vertices of C so that internal vertices of P are not on C, and that P is a shortest path between its endvertices. We distinguish three cases.

Case 1. Suppose P is a shortest path between u_k and v_m that is shorter than at least one of the u_k, v_m-paths on C. Note that in this case $0 < k < n$ and $0 < m < n$. Note that any path from a vertex in W_{ab} to a vertex in W_{ba} contains at least one edge that is in relation $\approx$ with ab. If there are two such edges this path is not the shortest path and by Lemma 3 there exists an edge $e = uv$ on P that is in relation $\sim$ with ab. Thus P contains an edge $e = uv \sim ab$ since $u_k \in W_{ab}$ and $v_m \in W_{ba}$.

Notice that P together with C defines two cycles, and at least one of them is shorter than C. Without loss of generality we may assume that the cycle shorter than C contains the edge ab. Since C is a shortest cycle in G with respect to having two edges in relation $\sim$ with no isometric ladder between them, we infer that there is an isometric ladder between ab and $e = uv$, to prevent a contradiction with minimality of C. Let wz be the edge on this ladder, forming a square with ab. We claim that $w \neq u_1$. Indeed, if $w = u_1$, then $d(w,x) = d(a,x) - 1$, and since $wz \sim xy \sim ab$ we have $d(w,x) = d(a,x) - 1 = d(z,y) = d(b,y) - 1$. Thus the cycle formed by wz, xy, the shortest w,x-path, and the shortest z,y-path would be shorter than C. However, wz and xy are also not connected by an isometric ladder (because such a ladder would imply the existence of isometric ladder between ab and xy), so we get a contradiction with minimality of C. This proves that $w \neq u_1$ and by symmetry $z \neq v_1$.

If aw is not in relation $\approx$ with any $u_t u_{t+1}$ consider a path between a and w that first traverses the path $u_1, \ldots, u_k$, then goes along P between u_k and u, and finally traverses one side (in W_{ab}) of the isometric uv, wz-ladder. By Theorem 1 (*iii*) there exists an edge on this path that is in relation $\approx$ with aw. This edge is not on the uv, wz-ladder since there would be two edges in relation $\approx$ on a shortest a,u-path contrary to Lemma 4 and must thus lie on the P.

Suppose now that aw is in relation $\approx$ with some edge on C between u_0 and u_k. Suppose $aw \approx u_t u_{t+1}$. If $aw \sim u_t u_{t+1}$, then we infer that $d(w, u_{t+1}) = d(a, u_{t+1}) - 1$, hence

$$d(w,x) = d(a,x) - 1.$$

Since $wz \sim ab \sim xy$ we get

$$d(w,x) = d(a,x) - 1 = d(z,y) = d(b,y) - 1$$

which leads to the same contradiction with minimality as before (namely, $wz \sim xy$ and the distance between wz and xy is smaller than n, yet there is no isometric ladder between them, because the ladder obtained from such an isometric ladder by adding ab would be an isometric ladder between ab and xy). Thus $aw \not\sim u_t u_{t+1}$ and suppose that there exists a clique with edges e and f such that $aw \sim e$ and $f \sim u_t u_{t+1}$. By Theorem 1 (*ii*) one of the endvertices of f is closer to a than the other, since u_t is closer to a than u_{t+1}. If e and f have no vertex in common we deduce by Lemma 2 that this endvertex of f is closer to a than both endvertices of e. This contradicts that $a \in W_{u_t u_{t+1}}$, since by Theorem 1 (*ii*) a should be closer to one endvertex of f. Otherwise, if e and f have a vertex in common it must be by Theorem 1 (*ii*) and Lemma 2 closer to a than the remaining vertices s and t of e and f, respectively. Now consider a s,t-path that is a subwalk of the walk that starts in s and traverses one side (in W_{wa}) of the e, wa-ladder to w, proceeds on the side of W_{wz} in wz, uv-ladder to u, returns by P to u_k, follows to vertices $u_{k-1}, u_{k-2}, \ldots, u_{t+1}$, and returns to t by the $u_t u_{t+1}, f$-ladder on the side of $W_{u_{t+1}u_t}$. On this path there exists an edge that is in relation $\approx$ with st. This edge can not be on wz, uv-ladder which is a subladder of isometric ab, uv-ladder, since there already is aw. This edge is also clearly not on $u_0 u_k$-path (since there is $u_t u_{t+1}$ and thus must be on the path P.

In each case we have an edge on P in W_{ab} that is in relation $\approx$ with aw. By applying the same reasoning in the subgraph W_{ba} we infer that bz is in relation $\approx$ with some edge on P in W_{ba}. Since $aw \approx bz$ we derive that two different edges of P are in relation $\approx$. By Lemma 4, P is not a shortest path and this case is concluded.

Case 2. P is a shortest path between u_k and v_m that is of the same length as both u_k, v_m-paths on C, otherwise we have Case 1. Let C_1 and C_2 be the cycles that P creates with C; note that they are of the same length as C. Since P is a shortest path, all its edges are pairwise not in relation $\approx$ by Lemma 4. Hence, by Theorem 1 *(iii)* there exists for every edge e on P exactly one edge on $C_1 \backslash P$ (respectively $C_2 \backslash P$) that is in relation $\approx$ with e. Hence all edges of C_1 (respectively C_2) on C between u_k and v_m are pairwise not in relation $\approx$. Thus they form a shortest u_k, v_m-path on C_1 (respectively C_2). We infer that $u_{k-1}u_k \sim v_m w$, where w is a neighbor of v_m on P, and also $u_{k+1}u_k \sim v_m w$ (if $k = 0$ replace u_{k-1} by v_0 and if $k = n$ replace u_{k+1} by v_n). Since $\sim \subseteq \approx$ and $\approx$ is transitive, we have $u_{k-1}u_k \approx u_{k+1}u_k$ which is clearly a contradiction.

Case 3. P is a shortest path connecting two vertices of C in W_{ab} (or W_{ba}). Since the u_0, u_n-path is shortest (as well as v_0, v_n-path), we infer that P is of the same length as the path between the corresponding vertices on C. This observation combined with Case 1 implies that C is isometric. Note that the cycle C' (obtained by replacing in C the u_k, u_m-path with P) is also isometric from the same reason as C. Let w_{k+1} be the neighbor of u_k on P. We infer that edges $u_k u_{k+1}$ and $u_k w_{k+1}$ are in relation $\sim$ with the same antipodal edge on C in W_{ba}. As in Case 2 this yields a contradiction.

We conclude that C is convex, a desired contradiction. □

3 Some consequences

The equivalence of *(i)* and *(ii)* in Theorem 5 immediately implies the following result in bipartite case, which is a new characterization of almostmedian graphs.

Corollary 6 *A graph G is almostmedian if and only if G is a partial cube that satisfy the almost-quadrangle property.*

In [6] the semi-triangle property was introduced as follows. A graph G satisfies the *semi-triangle property* if for any vertices $u, x, y \in V(G)$ where $d(u,x) = d(u,y) = k \geq 2$ such that $xy \in E(G)$, there exists a triangle with vertices a, b, c such that $xy\delta^* ab$ and $d(u,a) - 1 = d(u,b) - 1 = d(u,c) < k$. This is a generalization of the *triangle property* where $x = a$ and $y = b$. A graph G is called *semi-weakly-modular* if it satisfies both semi-quadrangle and semi-triangle property and *weakly modular* if it satisfies both triangle and quadrangle property.

Let $H_n(u,v)$ be a graph obtained from $P_2 \Box P_n$, $n \geq 1$, where additional vertex u is adjacent to both endvertices of one edge of $P_2 \Box P_n$ where both endvertices have degree 2 and similarly vertex v to the other such edge in $P_2 \Box P_n$. Furthermore, with

$$I_G(u,v) = \{w \in V(G) \mid w \text{ is on a shortest } u,v\text{-path}\}$$

we denote the *interval* between u and v. In [6] the following result was shown.

Theorem 7 *A graph G is quasi-semimedian if and only if*

(i) G is semi-weakly-modular,

(ii) for every induced $H_n(u,v)$, $n \geq 1$, we have $I_G(u,v) \cap H_n(u,v) = \{u,v\}$, and

(iii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$.

We can not define the almost-triangle property from the semi-triangle property in the same manner as in the case of quadrangle properties. This can easy be seen if we delete vertex v from $H_n(u,v)$, $n > 2$, since this graph is quasi-almostmedian but the "almost-triangle property" is not satisfied. Thus we define a graph to be *almost-weakly-modular* if it satisfies the almost-quadrangle and semi-triangle property. We can now show the analogue of the above theorem.

Corollary 8 *A graph G is quasi-almostmedian if and only if*

(i) G is almost-weakly-modular,

(ii) for every induced $H_n(u,v)$, $n \geq 1$, we have $I_G(u,v) \cap H_n(u,v) = \{u,v\}$, and

(iii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$.

Proof Let G be quasi-almostmedian. Then G is quasi-semimedian and (*ii*) and (*iii*) follows from Theorem 7, as well as semi-triangle property. The almost-quadrangle property follows from Theorem 5. Let now (*i*), (*ii*), and (*iii*) be fulfilled for a graph G. If G is almost-weakly-modular, then G is semi-weakly-modular. Thus G is quasi-semimedian by Theorem 7, and from (*i*) and Theorem 5 we yield the desired result. □

At the end of this paper we state two new characterizations of quasi-median graphs. The most appropriate definition for quasi-median graphs is from [8]. Graph G is *quasi-median* if it is a weakly-modular graph that does not contain $K_4 - e$ or $K_{2,3}$ as an induced subgraph. We recall also a part of Theorem 4.1 from [6] that a graph G is quasi-median if and only if G is a quasi-semimedian graph without Q_3^- and without a house as a convex subgraphs. With this result the next corollary is clear.

Corollary 9 *A connected graph G is quasi-median if and only if G is quasi-almostmedian and G has neither Q_3^- nor a house as a convex subgraph.*

If we combine the last two corollaries we receive the second characterization of quasi-median graphs.

Corollary 10 *A connected graph G is quasi-median if and only if*
(i) G is almost-weakly-modular,
(ii) for every induced $H_n(u,v)$, $n \geq 1$, we have $I_G(u,v) \cap H_n(u,v) = \{u,v\}$,
(iii) for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and
(iv) G has neither Q_3^- nor a house as a convex subgraph.

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