Domino Tiling Graphs

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February 6, 2008

Abstract

This paper investigates tilings of a $2 \times n$ rectangle using vertical and horizontal dominos. It is well-known that these tilings are counted by the Fibonacci numbers. We associate a graph to each tiling by converting the corners and borders of the dominos to vertices and edges. We study the combinatorial, probabilistic, and graph-theoretic properties of the resulting "domino tiling graphs." In particular, we prove central limit theorems for naturally occurring statistics on these graphs. Some of these results are then extended to more general tiling graphs.

2000 Mathematics Subject Classification: 05A15, 05A16, 05C15, 05C45.

1 Introduction

A *domino* is a 1×2 rectangle in the x, y-plane whose corners are located at integer coordinates. A *horizontal domino* has width 2 and height 1, while a *vertical domino* has width 1 and height 2. Given a region S in the plane that is a union of unit squares, a *domino tiling* of S is a covering of the squares of S by non-overlapping dominos. Such tiling problems have been extensively studied in the combinatorial literature [1, 4, 7, 9]. For example, Kasteleyn [10] and Fisher and Temperley [6] proved the following celebrated formula for the number of domino tilings of an $m \times n$ rectangle:

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left[\cos^2\left(\frac{j\pi}{2m+1}\right) + \cos^2\left(\frac{k\pi}{2n+1}\right) \right].$$

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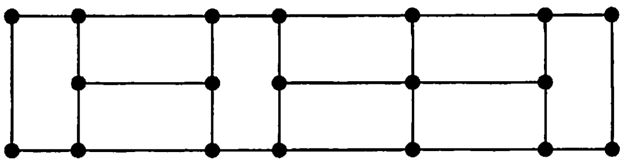

Figure 1: A graph built from a domino tiling.

In the case where $m = 2$, this formula reduces to the Fibonacci number F_{n+1}. To prove this directly, let a_n be the number of domino tilings of a $2 \times n$ rectangle. For $n \geq 2$, we can build such a tiling either by appending one new vertical domino to any of the a_{n-1} tilings of a $2 \times (n-1)$ rectangle, or by appending two horizontal dominos to any of the a_{n-2} tilings of a $2 \times (n-2)$ rectangle. Thus, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. The initial conditions are $a_0 = 1 = a_1$, and hence $a_n = F_{n+1}$.

This paper considers a graph-theoretic variant of the domino tiling problem, which was originally proposed by Anant Godbole [8]. We convert a domino tiling into a graph by putting a vertex at the corner of each domino and converting the borders of the dominos to edges in the natural way. For example, Figure 1 displays a graph obtained from a domino tiling of a 2×9 rectangle.

Suppose we randomly select a domino tiling of a $2 \times n$ rectangle. We can then ask for information about the distribution of various combinatorial statistics on these graphs, such as the number of vertical dominos used, the number of vertices, the number of edges, the diameter, etc. This article derives combinatorial generating functions for these quantities, which lead to formulas for the mean, variance, and moment generating functions of the associated random variables. We use a result of E. Bender to prove central limit theorems that establish the asymptotic normality of these distributions as n tends to infinity. We also study other graph-theoretic properties of domino tiling graphs such as the chromatic number, the existence of Hamiltonian cycles, etc. Some of these results are extended to tilings of an $m \times n$ rectangle by $m \times 1$ and $1 \times m$ subrectangles.

2 Domino Graphs: Combinatorial Analysis

For each integer $n \geq 0$, let Ω_n be the set of all graphs arising from domino tilings of a $2 \times n$ rectangle, and let Ω be the union of all the sets Ω_n. We regard each Ω_n as a probability space where each graph $G \in \Omega_n$ has equal probability $|\Omega_n|^{-1} = 1/F_{n+1}$. We will be studying random variables $X_n : \Omega_n \to \mathbb{N}$ that measure graph-theoretic quantities. To facilitate this discussion, we first derive ordinary generating functions (OGF's) for various statistics defined on all of Ω. These will be used later to find the moment generating functions (MGF's) of the random variables X_n. Recall that the OGF of a weight function $X : \Omega \to \mathbb{N}$ is the power series

$$H_X(z) = \sum_{G\in\Omega} z^{X(G)}.$$

Since we want to keep track of the width of the graphs involved, we usually calculate the two-variable OGF

$$H_X(w,z) = \sum_{G\in\Omega} w^{\text{width}(G)} z^{X(G)}.$$

For a detailed discussion of generating functions, see the texts [3, 13].

In addition to the width of the graph, we will study the following statistics on graphs $G \in \Omega$:

$$\begin{aligned}
N_v(G) &= \text{number of vertical dominos in } G;\\
N_h(G) &= \text{number of horizontal dominos in } G;\\
V(G) &= \text{the number of vertices in } G;\\
E(G) &= \text{the number of edges in } G;\\
\text{Diam}(G) &= \text{the graph-theoretic diameter of } G.
\end{aligned}$$

(Recall that, if we write $d(v,w)$ for the shortest path between vertices v and w of G, we have $\text{Diam}(G) = \max_{v,w} d(v,w)$.) For example, the graph G shown in Figure 1 has $\text{width}(G) = 9$, $N_v(G) = 3$, $N_h(G) = 6$, $V(G) = 19$, $E(G) = 27$, and $\text{Diam}(G) = 7$.

2.1 Word of a Graph

We can encode graphs $G \in \Omega$ as words in the alphabet $\{a, b\}$, where a represents a vertical domino and b represents two horizontal dominos stacked atop each other. For example, the graph shown in Figure 1 is encoded by the word *ababba*. This encoding defines a bijection between Ω and the set $W = \{a, b\}^*$ of all strings of a's and b's.

This encoding leads to an easy derivation of the generating functions for width, N_h, and N_v. Thinking of the letters a and b as non-commuting indeterminates, the binomial theorem and geometric series formulas lead to the formal power series expansion

$$\sum_{x\in W} x = 1 + a + b + aa + ab + ba + bb + \cdots = \sum_{n\geq 0}(a+b)^n = \frac{1}{1-(a+b)}.$$

When we pass from a word x to the corresponding graph G, every letter a contributes 1 to the width and every letter b contributes 2 to the width. Therefore, we can obtain the width generating function by replacing a by w^1 and b by w^2, where w is a new indeterminate:

$$\sum_{G\in\Omega} w^{\mathrm{width}(G)} = \left(\sum_{x\in W} x\right)\Bigg|_{a\to w,\, b\to w^2} = \frac{1}{1-w-w^2}.$$

Of course, this is just a shift of the generating function for the Fibonacci numbers. If we want to keep track of N_v and N_h as well as the width, we need only replace a by vw and b by h^2w^2, where v, w, h are commuting indeterminates. Thus,

$$\sum_{G\in\Omega} w^{\mathrm{width}(G)} v^{N_v(G)} h^{N_h(G)} = \frac{1}{1-vw-h^2w^2}.$$

Evidently, $N_v(G) + N_h(G) = \mathrm{width}(G)$ for any G, so it suffices to keep track of just one of the variables N_v or N_h.

2.2 Vertices and Edges

Our next task is the derivation of the generating functions

$$H_V(z,w) = \sum_{G\in\Omega} z^{V(G)} w^{\mathrm{width}(G)}; \qquad H_E(y,w) = \sum_{G\in\Omega} y^{E(G)} w^{\mathrm{width}(G)}.$$

Recall that a planar graph with v vertices, e edges, and f faces satisfies Euler's relation $v - e + f = 2$. Since the number of faces in any tiling of a $2 \times n$ rectangle is $n + 1$ (n dominos plus the outside region), we have

$$E(G) = V(G) + F(G) - 2 = V(G) + \mathrm{width}(G) - 1$$

for any domino tiling graph G. Therefore, $H_E(y,w) = y^{-1}H_V(y,wy)$, so it suffices to find the OGF $H_V(z,w)$.

Let $H_1 = \sum z^{V(G)}w^{\mathrm{width}(G)}$ where we sum over all graphs G whose rightmost tile is a vertical domino. Let $H_2 = \sum z^{V(G)}w^{\mathrm{width}(G)}$ summed over all G with two horizontal dominos at the right end. Finally, write H for $H_V(z,w)$. We have

$$H = z^2w^0 + H_1 + H_2, \tag{1}$$

where the z^2 term comes from the unique domino tiling graph of width zero, which (by convention) has two vertices and one edge.

We can build the graphs counted by H_1 by either taking a single vertical domino, or appending a vertical domino to a graph ending in a vertical domino, or appending a vertical domino to a graph ending in two horizontal dominos. Therefore,

$$H_1 = z^4w^1 + H_1z^2w^1 + H_2z^2w^1. \tag{2}$$

The first term arises since the graph for a single vertical domino has 4 vertices and width 1. The extra factor z^2w in the second and third terms comes from the addition of the new vertical domino, which always adds two new vertices and increases the width by 1.

Similarly, we can build the graphs counted by H_2 by either taking two stacked horizontal dominos, or appending two such horizontal dominos to a graph ending in a vertical domino, or appending two horizontal dominos to a graph ending in two horizontal dominos. Reasoning as above leads to the formula

$$H_2 = z^6w^2 + H_1z^4w^2 + H_2z^3w^2. \tag{3}$$

Note that adding two horizontal dominos after a vertical domino creates a new vertex in the middle of the old right boundary, plus three new vertices on the new right boundary, explaining the z^4 factor in the previous equation.

We can solve the linear equations (2) and (3) over the field $\mathbb{Q}(z, w)$, which yields the following formulas for H_1 and H_2:

$$H_1 = \frac{z^4w - z^7w^3 + z^8w^3}{1 - z^2w - z^3w^2 - z^6w^3 + z^5w^3},$$

$$H_2 = \frac{z^6w^2}{1 - z^2w - z^3w^2 - z^6w^3 + z^5w^3}.$$

Putting these expressions into (1) yields

$$H_V(z, w) = \frac{z^2 + z^6w^2 - z^5w^2}{1 - z^2w - z^3w^2 - z^6w^3 + z^5w^3}. \tag{4}$$

2.3 Extension to Long Dominos

For each integer $m \geq 2$, define a *long horizontal domino* to be a rectangle of width m and height 1, and define a *long vertical domino* to be a rectangle of width 1 and height m. A natural extension of the preceding discussion is the tiling of $m \times n$ rectangles (for m fixed) by long dominos of area m. Such tilings can still be encoded by words in $W = \{a, b\}^*$, where a represents a single long vertical domino and b now represents m long horizontal dominos stacked atop each other. Reasoning just as before, we have the generating function

$$\sum_{G\in\Omega^{(m)}} w^{\text{width}(G)} v^{N_v(G)} h^{N_h(G)} = \left(\sum_{x\in W} x\right)\Bigg|_{a\to vw,\, b\to h^m w^m}$$
$$= \frac{1}{1 - vw - h^m w^m},$$

where $\Omega^{(m)}$ is the set of all tiling graphs of an $m \times n$ rectangle using long dominos of area m.

The generating function for width and number of vertices can be found by solving equations similar to (1), (2), and (3). More precisely, equations (1) and (2) still hold, while (3) becomes

$$H_2 = z^{2m+2} w^m + H_1 z^{2m} w^m + H_2 z^{m+1} w^m.$$

Solving, we obtain the generating function

$$H_V^{(m)}(z, w) = \sum_{G\in\Omega^{(m)}} w^{\text{width}(G)} z^{V(G)}$$
$$= \frac{z^2 + z^{2m+2} w^m - z^{m+3} w^m}{1 - z^2 w - z^{m+1} w^m - z^{2m+2} w^{m+1} + z^{m+3} w^{m+1}}.$$

Using Euler's formula as before, the edge generating function is

$$H_E^{(m)}(y, w) = y^{-1} H_V^{(m)}(y, wy).$$

3 Domino Graphs: Probabilistic Analysis

Now that we have OGF's for the various combinatorial statistics under discussion, we are ready to analyze the distributions of the associated random variables on the probability spaces Ω_n. The exact distributions of these random variables are uniquely determined by their moment generating functions (MGF's), which are easily derivable from the OGF's. In particular, the exact mean, variance, and higher moments of the random variables can immediately be found by differentiating the MGF's. We give a sample of these calculations in the first subsection below. The next subsection considers the asymptotic behavior of these distributions as n (the width of the graph) tends to infinity. A central limit theorem due to Bender [2] will allow us to prove the asymptotic normality of many of the statistics being studied with a minimum of calculation. A third subsection calculates the expected diameter of a domino tiling graph. Finally, we consider some symmetry properties of these graphs.

3.1 Moment Generating Functions

Suppose we are given a sequence of random variables $X_n : \Omega_n \to \mathbb{N}$ on the finite sample spaces Ω_n. The OGF

$$H(w,x) = \sum_{n\geq 0} w^n \sum_{G\in\Omega_n} x^{X_n(G)}$$

records information about all of the random variables X_n. To compute a quantity involving a particular X_n or Ω_n, we must eventually extract the coefficient of w^n. For example, note that the probability of selecting a particular graph in Ω_n is $1/|\Omega_n| = 1/H(w,1)|_{w^n} = 1/F_{n+1}$. Using this, we can compute the expectation of X_n in terms of H:

$$E[X_n] = \sum_{G\in\Omega_n} X_n(G)P(\{G\}) = F_{n+1}^{-1} \sum_{G\in\Omega_n} X_n(G) = \frac{\frac{\partial H}{\partial x}(w,1)|_{w^n}}{H(w,1)|_{w^n}}.$$

For example, if $X_n(G) = V(G)$ for all n, we can use the OGF (4) to obtain the following exact expression for the expected number of vertices on Ω_n:

$$\frac{(\sqrt{5}(7n+15) - 15n - 3)(-\alpha)^{-n} + (\sqrt{5}(7n+15) + 15n + 3)(-\beta)^{-n}}{10(\alpha^{n+1} - \beta^{n+1})}.$$

(Here and below, we set $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, so that $F_n = (\alpha^n - \beta^n)/\sqrt{5}$.) It is just as easy to use H to find the moment generating functions (MGF's) of the X_n's. Recall that the MGF of X_n is defined as $m_{X_n}(t) = E[e^{tX_n}]$ for real t in some neighborhood of 0. Noting that $H(w,e^t) = \sum_{n\geq 0} w^n \sum_{G\in\Omega_n} e^{tX_n(G)}$, we therefore obtain the formula

$$m_{X_n}(t) = \frac{H(w,e^t)|_{w^n}}{H(w,1)|_{w^n}}. \tag{5}$$

All the moments of X_n can now be found by differentiation, using the well-known formula $E[X_n^k] = m_{X_n}^{(k)}(0)$. Finally, we can compute the variance $\mathrm{Var}(X_n) = m''_{X_n}(0) - m'_{X_n}(0)^2$ and the standard deviation $\sigma_{X_n} = \sqrt{\mathrm{Var}(X_n)}$.

We illustrate these MGF calculations with the random variables $N_v(G)$. From 2.1, we have the OGF $H(x,w) = (1 - xw - w^2)^{-1}$ for the random variables X_n. Application of (5) produces

$$m_{X_n}(t) = (1 - we^t - w^2)^{-1}|_{w^n}/F_{n+1}.$$

Factoring the quadratic polynomial in w, decomposing into partial fractions, and using the geometric series expansion, we obtain the following exact expression for this MGF:

$$m_{X_n}(t) = \frac{(e^t + \sqrt{e^{2t}+4})^{n+1} - (e^t - \sqrt{e^{2t}+4})^{n+1}}{2^{n+1}\sqrt{e^{2t}+4} \cdot F_{n+1}}.$$

Taking derivatives, we compute the expected number of vertical dominos on Ω_n to be

$$E[X_n] = E[N_v] = \frac{2(\alpha^n - \beta^n) + n\sqrt{5}(\alpha^{n+1} + \beta^{n+1})}{5\sqrt{5} \cdot F_{n+1}}.$$

The second moment $E[N_v^2]$ is

$$\frac{2^n(8\alpha^n - 8\beta^n + 4n(\beta^n(6\alpha - 8) + \alpha^n(6\alpha + 2)) + 5n^2(2\alpha^{n+1} - 2\beta^{n+1}))}{25\sqrt{5} \cdot F_{n+1}}.$$

The variance is then given by $\mathrm{Var}(N_v) = E[N_v^2] - E[N_v]^2$.

3.2 Asymptotic Analysis

Given a sequence of random variables $X_n : \Omega_n \to \mathrm{N}$ as above, how can we determine the asymptotic behavior of the X_n's as n tends to infinity? It is clear intuitively that the means and variances grow with n, so we must first normalize the X_n's. Consider the standardized random variables $Z_n = (X_n - E[X_n])/\sqrt{\mathrm{Var}(X_n)}$. One might hope that the cdf's of the Z_n's converge to the cdf of a standard normal random variable Z when n goes to infinity. By the Continuity Theorem from probability, this will hold provided

$$\lim_{n\to\infty} m_{Z_n}(t) = m_Z(t) = e^{t^2/2}$$

for all t near zero. Furthermore, writing $Z_n = aX_n + b$ with $a = \sigma_{X_n}^{-1}$ and $b = -E[X_n]/\sigma_{X_n}$, we have $m_{Z_n}(t) = e^{tb} m_{X_n}(at)$. Thus, in principle, the calculations in the previous subsection furnish all the data we need to find the limiting behavior of N_v, V, etc. In practice, however, the exact expressions we derived above are far too messy to allow the calculation of the required limit, even with the aid of a computer algebra system.

Fortunately, we can invoke the following theorem of Bender [2] to obtain the desired result using a much simpler computation. Suppose

$$f(w,z) = \frac{g(w,z)}{h(w,z)} = \sum_{n,k\geq 0} c_n(k) w^n z^k,$$

where: (a) $h(w,z)$ is a polynomial in w whose coefficients are continuous functions of z; (b) for some r, $h(r,1) = 0$ and all other roots of $h(w,1)$

have larger absolute value; (c) $g(w,z)$ is analytic for z close to 1 and $w < |r|+\epsilon$; and (d) $g(r,1) \neq 0$. Define random variables X_n by setting $P(X_n = k_0) = c_n(k_0)/\sum_k c_n(k) = g(w,z)|_{w^n z^{k_0}}/g(w,1)|_{w^n}$. Then the X_n's are asymptotically normal with mean $n\mu$ and variance $n\sigma^2$, where

$$\mu = \frac{h_z}{rh_w}, \qquad \sigma^2 = \mu^2 + \frac{(h_z/h_w)^2 h_{ww} - 2(h_z/h_w)h_{wz} + h_z + h_{zz}}{rh_w},$$

with all partial derivatives being evaluated at $(w,z) = (r,1)$. We abbreviate the conclusion by writing $X_n \sim N(n\mu, n\sigma^2)$.

To illustrate the use of this theorem, consider again the random variables $X_n(G) = N_v(G)$ for $G \in \Omega_n$. We apply the theorem to the OGF $f(w,z) = 1/(1-zw-w^2)$, taking $g(w,z) = 1$ and $h(w,z) = 1-zw-w^2$. Hypotheses (a) through (d) clearly hold if we choose $r = -\beta$ (the root of $1-w-w^2$ with smallest absolute value). The required partial derivatives are

$$h_w = -z-2w, \; h_z = -w, \; h_{zw} = -1, \; h_{ww} = -2, \; h_{zz} = 0.$$

Evaluating at $(r,1)$ and using the formulas above, we obtain

$$\mu = \frac{\beta}{-\beta(2\beta-1)} = 1/\sqrt{5} \approx 0.4472;$$

$$\sigma^2 = 1/5 + \frac{(\beta/(2\beta-1))^2(-2) - 2(\beta/(2\beta-1))(-1) + \beta + 0}{-\beta(2\beta-1)} = \frac{4\sqrt{5}}{25}.$$

So $N_v \sim N(0.4472n, 0.3578n)$ where n is the width of the graph. We can apply the same sort of analysis to $f(w,z) = 1/(1-w-z^2w^2)$ to find the asymptotic behavior of N_h. Alternatively, since $N_v + N_h = n$ for all graphs of width n, we see immediately that $N_h \sim N(0.5528n, 0.3578n)$.

Now consider $X_n = V$, the number of vertices. Given the OGF (4), we must take $g(w,z) = z^2+z^6w^2-z^5w^2$ and $h(w,z) = 1-z^2w-z^3w^2-z^6w^3+z^5w^3$. Conditions (a) through (d) hold, taking $r = -\beta$ again. After some calculation with partial derivatives, we obtain $\mu = 1+(2/\sqrt{5}) \approx 1.8944$ and $\sigma^2 = 46\sqrt{5}/25 - 4 \approx 0.1143$. Thus, $V \sim N(1.8944n, 0.1143n)$. Since $E = V+n-1$ on Ω_n, we also conclude that $E \sim N(2.8944n, 0.1143n)$.

3.3 Diameter

Suppose $G \in \Omega_n$, where $n \geq 1$. We will show by induction on n that the diameter of G is given by the formula

$$\mathrm{Diam}(G) = N_v(G) + N_h(G)/2 + \begin{cases} 2 & \text{if the word of } G \text{ begins and ends with } b; \\ 1 & \text{otherwise.} \end{cases}$$

One easily verifies the formula when $n = 1$ or $n = 2$. For the induction step, consider $n \geq 3$. Assume that the graph of G is drawn in the first quadrant of the plane, in the $n \times 2$ rectangle with coordinates $(0,0)$, $(0,2)$, $(n,0)$, and $(n,2)$. We first compute $\max d(v,w)$ where v is a vertex of G on the line $x = 0$ and w is a vertex of G on the line $x = n$. A shortest path from v to w will clearly traverse exactly $N_v(G) + N_h(G)/2$ horizontal edges. Furthermore, it is easy to see that such a path will use zero, one, or two vertical edges. By choosing v and w to have different y-coordinates, we can force the path to use at least one vertical edge.

Under what circumstances can we force the path to use *two* vertical edges? If the tiling happens to consist entirely of horizontal dominos, and v and w are opposite corners of the $n \times 2$ rectangle, then clearly we will need to use two vertical edges. If the tiling uses at least one vertical domino, but begins and ends with stacked horizontal dominos, we can again force two vertical edges to be used by choosing $v = (0,1)$ and $w = (n,1)$. For, a direct horizontal path from v to w along the line $y = 1$ must be blocked somewhere in the middle by a vertical domino, which can only be bypassed by taking two extra vertical steps. In all other cases, only one vertical edge need be used. Consider, for example, the case where the word of G begins with a and $v = (0,0)$. If $w = (n,0)$, no vertical edges are needed; if $w = (n,1)$, we go across to $(n,0)$ and then take one vertical edge up to w; if $w = (n,2)$, we take one vertical edge up to $(0,2)$ and then go across to w. The other cases are handled similarly.

So far, we have shown that the maximum value of $d(v,w)$ for v on $x = 0$ and w on $x = n$ is given by the right side of (6). If we consider a pair of vertices v, w not lying on the extreme edges of the rectangle, then v and w belong to a subgraph G' of G that arises from a domino tiling of a rectangle of width smaller than n. So $d(v,w) \leq \mathrm{Diam}(G')$. By induction, $\mathrm{Diam}(G') = N_v(G') + N_h(G')/2 +$ one or two, and this quantity is no greater than the right side of (6). So, in the defining formula $\mathrm{Diam}(G) = \max_{v,w} d(v,w)$, the maximum is attained for suitably chosen vertices v, w on the extreme edges. This completes the proof of (6).

Assuming $n \geq 4$, what is the probability that the word of G begins and ends with b? Erasing the initial and final b's leaves us with the word of a domino graph of width $n - 4$. This process is reversible, so the desired probability is F_{n-3}/F_{n+1}. Taking expectations in (6), we find that the expected diameter for domino graphs of width n is

$$E[\mathrm{Diam}] = E[N_v] + E[N_h]/2 + 1 + F_{n-3}/F_{n+1} \qquad (n \geq 4).$$

3.4 Symmetric Domino Graphs

Recall that $\Omega_n^{(m)}$ is the set of all graphs obtained from tilings of an $n \times m$ rectangle by long dominos of area m. Define $a_n^{(m)} = |\Omega_n^{(m)}|$; these numbers satisfy the generalized Fibonacci recurrence

$$a_n^{(m)} = a_{n-1}^{(m)} + a_{n-m}^{(m)}$$

subject to the obvious initial conditions. We will compute the probability that a graph $G \in \Omega_n^{(m)}$ is symmetric about the line $x = n/2$. This event occurs iff the word of G is a palindrome.

There are three general methods for building symmetric graphs. First, we can concatenate an arbitrary graph in $\Omega_{n/2}^{(m)}$ with its mirror image in the line $x = n/2$; clearly, this can only be done when n is even. Second, we can place one vertical domino centered on the line $x = n/2$, then add a domino tiling graph of width $(n-1)/2$ to its left, and put the mirror-image graph to its right. This is only possible when n is odd. Third, we can place a stack of m horizontal dominos centered on the line $x = n/2$, then add a domino tiling graph of width $(n-m)/2$ to the left of the stack, and put the mirror image of this graph to the right of the stack. Evidently, the third method is possible iff $(n-m)/2$ is an integer, so that n and m must have the same parity. Considering the four possible cases, we conclude that $P(G \in \Omega_n^{(m)} \text{ is symmetric}) =$

$$\begin{cases} (a_{n/2}^{(m)} + a_{(n-m)/2}^{(m)})/a_n^{(m)} & \text{if } m \text{ is even and } n \text{ is even;} \\ a_{(n-1)/2}^{(m)}/a_n^{(m)} & \text{if } m \text{ is even and } n \text{ is odd;} \\ a_{n/2}^{(m)}/a_n^{(m)} & \text{if } m \text{ is odd and } n \text{ is even;} \\ (a_{(n-1)/2}^{(m)} + a_{(n-m)/2}^{(m)})/a_n^{(m)} & \text{if } m \text{ is odd and } n \text{ is odd.} \end{cases}$$

4 Hamiltonian Cycles

Recall that a *Hamiltonian cycle* in a graph is a cycle that visits each vertex exactly once. Hamiltonian paths are defined similarly. It is easy to see that every graph $G \in \Omega_n^{(m)}$ has a Hamiltonian path. Starting at the northwest corner, we greedily take vertical edges whenever we can, moving to the right only when we hit the upper or lower boundary of the rectangle. On the other hand, the question of whether Hamiltonian cycles exist in these graphs is more subtle. Our goal in this section is to prove the following result.

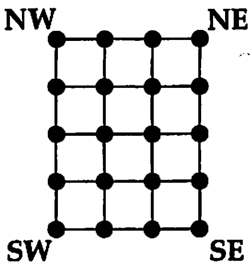

Figure 2: The grid graph $Gr(4,5)$.

Theorem 1. *Let $G \in \Omega_n^{(m)}$ be a domino tiling graph with word w. G has a Hamiltonian cycle iff m is odd or every maximal string of consecutive b's in the word bwb has odd length.*

4.1 Grid Graphs

The first step towards proving Theorem 1 is to analyze *grid graphs*. For integers $c, d \geq 1$, define $Gr(c,d)$ to be the graph with vertex set $\{1,2,\ldots,c\} \times \{1,2,\ldots,d\}$ and edge set

$$\{\{(x,y),(x+1,y)\} : 1 \leq x < c, 1 \leq y \leq d\} \cup$$
$$\{\{(x,y),(x,y+1)\} : 1 \leq x \leq c, 1 \leq y < d\}.$$

For example, Figure 2 displays the graph $Gr(4,5)$. It will be convenient to label the four corner vertices as follows:

$$SW = (1,1), \quad NW = (1,d), \quad SE = (c,1), \quad NE = (c,d).$$

One easily confirms that $Gr(c,d)$ is *bipartite* with vertex partition $V_0 = \{(x,y) : x+y \equiv 0 \pmod 2\}$, $V_1 = \{(x,y) : x+y \equiv 1 \pmod 2\}$. The following observation will be the key to understanding Hamiltonian paths and cycles in grid graphs and domino tiling graphs:

$$\text{Any path from } v \in V_i \text{ to } w \in V_j \text{ must have length } \ell \equiv i-j \pmod 2. \tag{6}$$

Lemma 2. *Fix integers $c, d \geq 1$.*

(1) $Gr(c,d)$ has a Hamiltonian cycle iff c or d is even.

(2) $Gr(c,d)$ has a Hamiltonian path from NW to SW (resp. from SE to NE) iff c is odd or d is even.

(3) *There exist paths P_1 from NW to NE and P_2 from SE to SW in $Gr(c,d)$ such that each vertex of $Gr(c,d)$ belongs to exactly one of these paths iff c or d is even.*

Proof. (1) It is easy to find Hamiltonian cycles in G if c or d is even, by following the patterns indicated in Figure 3. For the converse, assume c and d are both odd. A Hamiltonian cycle in $Gr(c,d)$ would have cd vertices and cd edges. This cycle would be a closed path from $NW \in V_0$ to $NW \in V_0$ of odd length, in violation of observation (6).

(2) By symmetry, it suffices to consider the case of paths from NW to SW. Figure 4 illustrates how such paths may be constructed when c is odd or d is even. Conversely, assume c is even and d is odd. The vertices NW and SW both belong to V_0 in this case, so every path from NW to SW must have even length. But, for this path to visit every vertex once, the length must be $cd-1$, which is odd. So no Hamiltonian path from NW to SW exists.

(3) Figure 5 shows how to construct P_1 and P_2 in the cases where c is even or d is even. Conversely, assume c and d are both odd. Let ℓ_1, ℓ_2 be the number of edges in paths P_1 and P_2. Since all four corners lie in V_0, ℓ_1 and ℓ_2 are both even, by (6). On the other hand, the requirement that each vertex appear in exactly one of the paths implies that $\ell_1+\ell_2 = cd-2$ is odd, which is a contradiction. □

4.2 Simplifying Domino Graphs

Henceforth, we consider a fixed domino graph $G \in \Omega_n^{(m)}$ with word w. Suppose that w contains two consecutive a's. The corresponding portion of the graph G is shown in Figure 6. Since the removal of edges uv and xy disconnects the graph, it is clear that a Hamiltonian cycle in G (if one exists) must use both of these edges. Similarly, the cycle must use vw and yz. Therefore, the edge vy cannot be used in the cycle. If we delete this edge and erase the vertices v and y (replacing the edges uv, vw with uw and replacing xy, yz with xz), then the new graph has a Hamiltonian cycle iff the old graph does. The new graph is the domino tiling graph whose word w' is obtained from w by deleting one of the consecutive a's. Note also that the condition in Theorem 1 holds for w iff it holds for w'. By iterating, we see that it suffices to prove the theorem in the case where w never has two consecutive a's.

Next, consider a domino tiling graph (using m-dominos) whose word is b^k. This graph consists of a $k \times m$ array of $m \times 1$ dominos. Shrinking the horizontal dimension of each domino from m to 1, we see that this graph is isomorphic to $Gr(k+1, m+1)$. In particular, by part (1) of the lemma, this graph has a Hamiltonian cycle iff m is odd or k is odd. Thus, the theorem

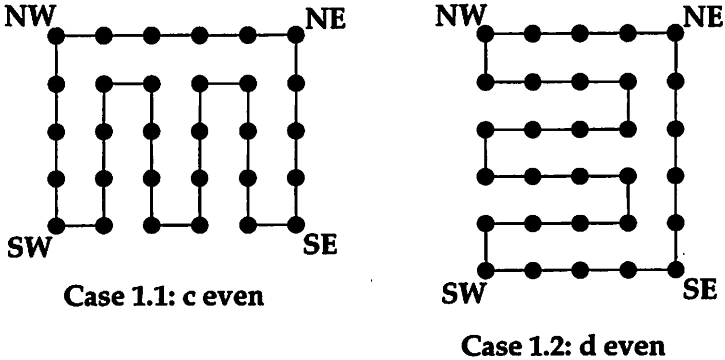

Figure 3: Building Hamiltonian cycles in a grid.

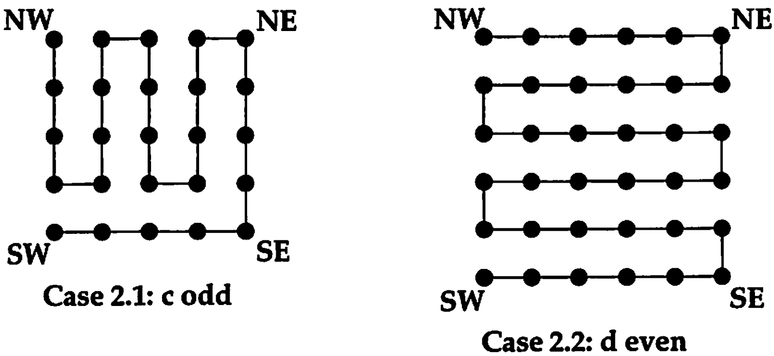

Figure 4: Building Hamiltonian paths in a grid.

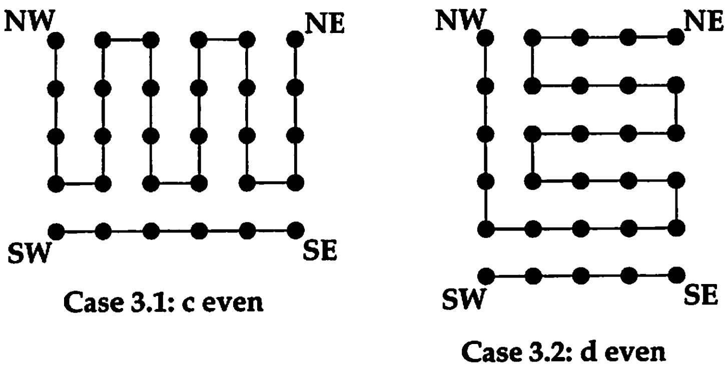

Figure 5: Building P_1 and P_2.

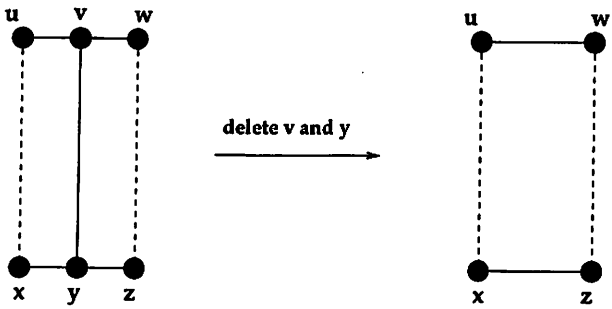

Figure 6: Two consecutive vertical dominos.

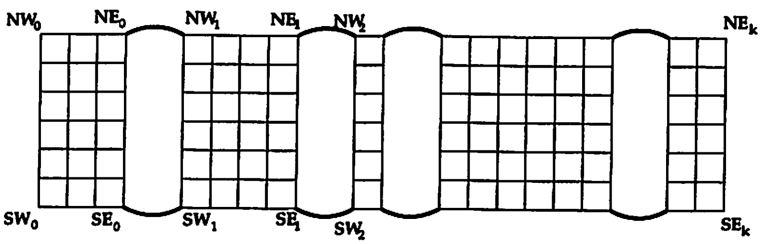

Figure 7: Viewing a domino graph as a series of linked grids.

is true when the word of G contains no a.

In the remaining cases, by shrinking horizontal dominos as above, we can think of G as a concatenation of grid graphs linked by "bridge edges" coming from the vertical dominos, as shown in Figure 7. If the word of G is $w = b^{n_0}ab^{n_1}ab^{n_2}\cdots ab^{n_k}$, then the grid graphs in the figure (from left to right) are copies of

$$Gr(n_0+1, m+1), Gr(n_1+1, m+1), \ldots, Gr(n_k+1, m+1).$$

Here the leftmost grid graph is absent if $n_0 = 0$, and the rightmost grid graph is absent if $n_k = 0$. Thus there are four cases to consider. We prove the theorem in each of these cases in the next subsection.

4.3 Case Analysis

In all four cases, notice that a Hamiltonian cycle (if it exists) must use all of the "bridge edges" shown as arcs in Figure 7. More precisely, suppose we traverse the cycle starting with the bridge edge from NE_0 to NW_1. What is the next bridge edge we visit? It cannot be the edge from SW_1 to SE_0, lest we never use the bridge edges leading to the right side of the graph. It cannot be the edge from SE_1 to SW_2, since there would be no way to get back from NE_1 to SW_1 on the "return trip" without crossing the path from NW_1 to SE_1. Thus, the next bridge edge must be NE_1 to NW_2. Proceeding similarly, we see that the bridge edges *must* be visited in the following order:

$$NE_0, NW_1; \; NE_1, NW_2; \; \ldots; \; NE_{k-1}, NW_k;$$

$$SW_k, SE_{k-1}; \; \ldots; \; SW_2, SE_1; \; SW_1, SE_0.$$

For each grid graph between two adjacent vertical dominos, we must therefore find two paths P_1 and P_2 as in part (3) of the lemma. For the leftmost grid (if it is present), we must find a Hamiltonian path from SE_0 to NE_0 to complete the Hamiltonian cycle. For the rightmost grid (if it is present), we must find a Hamiltonian path from NW_k to SW_k. Part (2) of the lemma tells us when such paths exist. It is now easy to analyze the four cases. Write the word of G as $w = b^{n_0}ab^{n_1}ab^{n_2}\cdots ab^{n_k}$ as above, where $k > 0$, $n_0, n_k \geq 0$ and $n_1, \ldots, n_{k-1} > 0$.

(1) Suppose $n_0 = n_k = 0$. Part (3) of the lemma shows that the required Hamiltonian cycle exists iff m is odd or n_i is odd for all $1 \leq i \leq k-1$. The second alternative holds iff every run of b's in bwb has odd length.

(2) Suppose $n_0 > 0$ and $n_k = 0$. By parts (2) and (3) of the lemma, the required cycle exists iff m is odd, or n_0 is even and n_i is odd for $1 \leq i \leq k-1$. The second alternative holds iff every run of b's in bwb has odd length.

(3) Suppose $n_0 = 0$ and $n_k > 0$. The cycle exists iff m is odd, or n_k is even and n_i is odd for $1 \leq i \leq k-1$, which holds iff every run of b's in bwb has odd length.

(4) Suppose $n_0, n_k > 0$. The cycle exists iff m is odd, or n_0 is even and n_k is even and n_i is odd for $1 \leq i \leq k-1$, which holds iff every run of b's in bwb has odd length.

5 Coloring Properties

This section determines the chromatic number, the chromatic index, and the chromatic total of domino tiling graphs in $\Omega_n^{(m)}$.

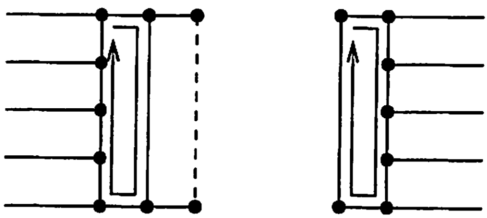

Figure 8: Odd cycles in domino graphs.

5.1 Chromatic Number

A graph G is *k-colorable* iff there exists a labelling of the vertices of G using k available colors such that any two adjacent vertices are assigned different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum k such that G is k-colorable. It is easy to see that $\chi(G) = 2$ iff G is a bipartite graph with at least one edge.

Theorem 3. *Let $G \in \Omega_n^{(m)}$ be a graph with word w. Then G is bipartite (so $\chi(G) = 2$) iff m is odd, or $w = a^k$ for some k, or $w = b^j$ for some j, or the word awa does not contain two adjacent a's. Otherwise, $\chi(G) = 3$.*

Proof. If $w = a^k$ or $w = b^j$, then G is isomorphic to a grid graph, which (as observed earlier) is bipartite. If awa has no two adjacent a's, then G is isomorphic to a subgraph of a grid graph, so is bipartite. (To see this, shrink the horizontal dimensions of all horizontal dominos from length m to length 1, and note that all edges in the resulting graph have length 1.) If m is odd, color the vertices of G by coloring (x, y) red if $x + y$ is even, and coloring (x, y) blue if $x + y$ is odd. Since all edge lengths in G are odd, each edge connects a blue vertex to a red vertex. Thus $\chi(G) = 2$ in all these cases.

Conversely, suppose m is even, w contains both a's and b's, and awa contains two adjacent a's. If w itself contains two adjacent a's, w must have a subword of the form baa or aab. In this case, we can get an odd-length cycle by traversing the edges on the perimeter of the vertical domino associated to the middle a of this subword (see Figure 8). Bipartite graphs have no odd cycles, so $\chi(G) > 2$ in this case. Similarly, if w begins with ab or ends with ba, traversing the perimeter of the initial or final vertical domino will also yield an odd cycle (see Figure 8).

Take a graph G as in the preceding paragraph. We show that G is 3-colorable, thereby proving that $\chi(G) = 3$. We define the color of a vertex of G by induction on the x-coordinate of the vertex. Our coloring will have the property that all vertices on a given vertical line $x = c$ will be colored with one of two colors (which may depend on c). For the base step, label the vertices on the line $x = 0$ with two alternating colors. For the induction step, assume $c > 0$ and all vertices to the left of $x = c$ have already been colored. Now consider various cases.

- The line $x = c$ has only two vertices, located at $(c, 0)$ and (c, m). If $(c-1, 0)$ and $(c-1, m)$ have the same color, use the two other colors to color $(c, 0)$ and (c, m). Otherwise, color $(c, 0)$ the same color as $(c-1, m)$, and color (c, m) the same color as $(c-1, 0)$.
- The line $x = c$ has $m+1$ vertices, all of which are joined by horizontal edges to vertices further left. By induction, these vertices (on the line $x = c - m$) alternate between two colors as we scan from bottom to top. By alternating the same two colors in the other order along the line $x = c$, we can extend the proper coloring to this line.
- The line $x = c$ has $m + 1$ vertices, but only the top and bottom vertex are joined to vertices further left. If the vertices $(c - 1, 0)$ and $(c - 1, m)$ have the same color, use the two other colors in an alternating pattern to color the vertices on $x = c$. Otherwise, if $(c-1, 0)$ and $(c-1, m)$ have two different colors, use the third color for $(c, 0)$, and then alternate between this color and the color of $(c-1, 0)$ as you scan up the line $x = c$.

□

5.2 Chromatic Index

The *chromatic index* of a graph G, denoted $\chi_I(G)$, is the minimum number of colors required to label the edges of the graph so that no vertex is incident to two edges of the same color. Vizing's Theorem [11] states that the chromatic number of a simple graph G with maximum degree Δ is either Δ or $\Delta + 1$. A graph G is called *class 1* iff $\chi_I(G) = \Delta(G)$, and *class 2* otherwise. Erdős and Wilson [5] showed that, asymptotically, almost all graphs on n vertices are class 1.

Theorem 4. *All graphs $G \in \Omega_n^{(m)}$ are class 1.*

Proof. Fix $G \in \Omega_n^{(m)}$, and let Δ be the maximum degree of any vertex in G. Obviously $\Delta \leq \chi_I(G)$, so it suffices to exhibit a proper edge-coloring of G using only Δ colors. We consider three cases.

- Case 1: $\Delta = 4$. Figure 9 indicates how to color the edges of G using 4 colors. For convenience, horizontal dominos have been shrunk to unit squares in the figure. Given this shrinking convention, we color horizontal edges between the lines $x = 2k$ and $x = 2k + 1$ red, and color horizontal edges between the lines $x = 2k + 1$ and $x = 2k + 2$ blue, for all k. Use a similar coloring rule for vertical edges of unit length, using two new colors (say yellow and green). Any "long" vertical edge can be colored green. It is clear from the figure that this edge-coloring is proper.

- Case 2: $\Delta = 3$ and m is odd. Figure 10 shows how to find a proper edge-coloring of G using 3 colors. Color all horizontal edges red and blue, as in Case 1. We must use the third color (say green) for all long vertical edges. The remaining vertical edges all form paths of length m from $y = 0$ to $y = m$. Since the maximum degree is not 4, each such path has a vertical domino to its immediate left or right. Therefore, the horizontal edges touching this path in the region $1 \leq y \leq m-1$ (if any) all have the same color. See Figure 10. If this color is red, say, then we can label the edges of the path green, blue, green, blue, ..., green from bottom to top. If this color is blue, we alternate between green and red instead. In any case, every such path can be colored without needing a fourth color.

- Case 3: $\Delta = 3$ and m is even. Figure 11 shows how to find a proper edge-coloring of G using 3 colors. The pattern is similar to the one in Case 2, except we modify the colors for horizontal and vertical edges in the region $1 \leq y \leq 2$ as shown in the figure, so that the vertical paths can still begin and end with green edges.

□

5.3 Chromatic Total

The *chromatic total* of a graph G, denoted $\chi_T(G)$, is the minimum number of colors required to label the vertices and edges of G so that: (a) no two adjacent vertices have the same color; (b) no two edges touching a given vertex have the same color; and (c) an edge does not have the same color as either of its endpoints. One sees immediately that $\chi_T(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. *Vizing's total coloring conjecture* states that $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$ for every graph G.

Theorem 5. *For all $m, n \geq 1$ and all graphs $G \in \Omega_n^{(m)}$, $\chi_T(G) = \Delta(G)+1$.*

The proof of this result is rather long, so we divide it into steps.

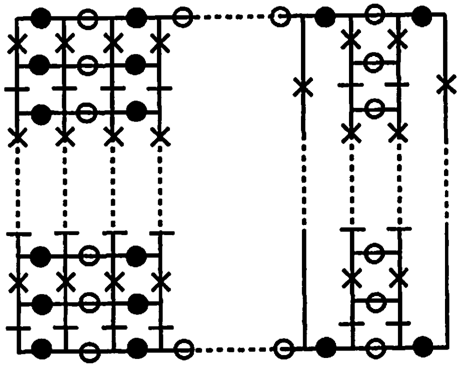

Figure 9: Edge-coloring a graph when $\Delta = 4$.

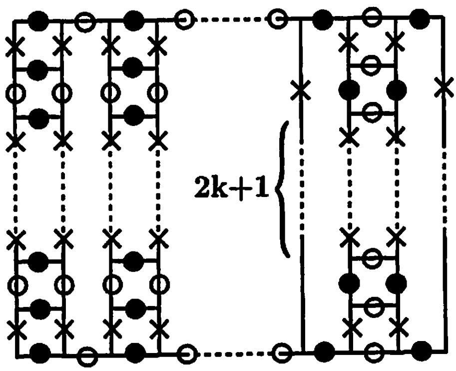

Figure 10: Edge-coloring a graph when $\Delta = 3$ and m is odd.

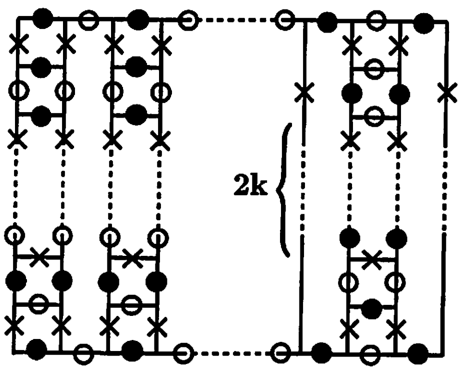

Figure 11: Edge-coloring a graph when $\Delta = 3$ and m is even.

Step 1: Coloring Grids

We first prove that for all grid graphs G with $\Delta(G) \neq 1$, $\chi_T(G) = \Delta(G)+1$. (The exceptional case consists of a graph with one edge, in which $\Delta(G) = 1$ and $\chi_T(G) = 3$.) This result follows by inspection of the coloring pattern illustrated in Figure 12, which shows how to totally color grid graphs of maximum degree 4 using 5 colors. By restricting to the subgraph consisting of the top two rows of vertices (resp. the top row), one obtains total colorings of grid graphs of maximum degree 3 (resp. 2) using 4 (resp. 3) colors.

Now take a domino graph $G \in \Omega_n^{(m)}$. We will produce a total coloring of G using five colors. Modify G by shrinking horizontal edges of length m to length 1 and deleting any vertical edges of length m (but not their endpoints). The resulting graph G' is a subgraph of a grid graph, and can therefore be totally colored with 5 colors. Observe that the two colors used for vertical edges in Figure 12 (denoted by $\times$'s and $-$'s) are never used to color vertices or horizontal edges. Therefore, we can safely use either of these colors to color the long vertical edges in G that are not present in G'. This produces a total coloring of G using five colors.

Since $\chi_T(G) \geq \Delta(G) + 1$, the proof is complete for domino graphs of maximum degree 4. We are also done when $m = 1$ or when the graph uses only horizontal or only vertical dominos, since domino graphs are grid graphs in those cases. For the rest of the proof, assume $m > 1$, $\Delta(G) = 3$, and both horizontal and vertical dominos are used in the graph. We need to find a total coloring of G using only four colors. This coloring will be built from special colorings of certain subgraphs of G, which we now analyze.

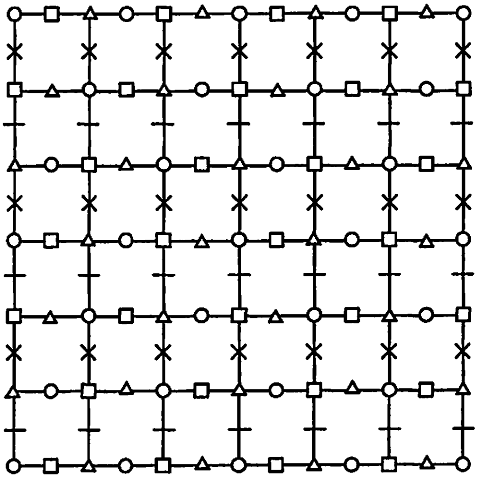

Figure 12: A total coloring of a grid graph.

Step 2: Coloring Ladders

A grid graph $Gr(2,k)$ will be called a *vertical ladder* with k *rungs* (the horizontal edges). The graphs obtained by omitting the top rung, the bottom rung, or both will also be called ladders. Grid graphs $Gr(k,2)$ will be called *sideways ladders.*

We need to understand how a partial coloring of the top of a vertical ladder can be extended, one rung at a time, further down the ladder. Consider the situation in Figure 13, in which the two vertices and three edges shown in the left picture have already been colored in such a way that the left vertex has the same color as the top-right edge. (Here and below, the letters A, B, C, D are variables standing for four *distinct* colors.) We claim there are exactly two ways to extend this coloring down to the next rung, as shown in the right two pictures of the figure. Furthermore, the two extensions retain the property that the left vertex of the bottom rung has the same color as the edge just above the right vertex of that rung. Thus, the extension process can be iterated. It is clear that the two pictured extensions are valid colorings possessing the indicated property. To see that there are no other extensions, first observe that the colors used for the two new vertical edges are forced by the definition of the total coloring and the fact that only four colors are available. Next, the lower-left vertex can only be colored D or B. If it is colored D, then the remaining edge and vertex must be colored A and B in some order, and either order leads to a valid coloring. If the lower-left vertex is colored B instead, then the lower-right

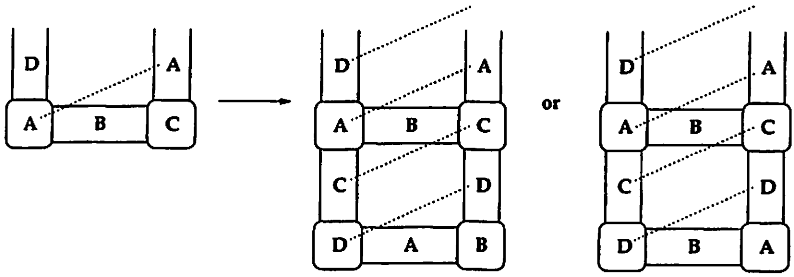

Figure 13: Building a coloring rung by rung.

vertex must be colored A, leaving no color for the edge between these two vertices. This completes the proof of the claim. The same argument shows that there are two ways to color the top rung of the ladder (located just above the part of the ladder shown in the figure), and in both colorings the top-right vertex will be colored D. Thus, the pattern of equal colors indicated by the diagonal dotted lines in the figure persists all the way down the ladder, provided that it is present in the first rung from the top. Of course, we can reflect and rotate this coloring pattern to obtain other valid colorings of vertical and sideways ladders.

Step 3: Coloring Horizontal Domino Stacks

We know that our graph G contains stacks of horizontal dominos, no two of which are consecutive (lest $\Delta(G) = 4$). Here we find some total colorings of each individual stack of horizontal dominos that have certain additional properties. Consider a vertical ladder $Gr(2, m+1)$, which represents a stack of m horizontal dominos. Given a total coloring of this ladder with 4 colors, and given a corner vertex v of the ladder (i.e., one of the vertices NW, SW, NE, or SE), let $c(v)$ be the color assigned to vertex v, and let $S(v)$ be the set of two distinct colors assigned to the edges leading to v. Note that $c(v) \notin S(v)$, and there is a unique color $z(v)$ not in $\{c(v)\} \cup S(v)$. We say that the given total coloring has *property L* iff $c(SW) \in S(NW)$ and $c(NW) \notin S(SW)$. We say that the given total coloring has *property R* iff $c(SE) = z(NE)$ and $c(NE) \in S(SE)$.

We claim that for all $m \geq 2$, there exist total colorings of $Gr(2, m+1)$ with property L and property R. Figure 14 exhibits such colorings using colors 1, 2, 3, and 4 for $m = 5, 6, 7$. (For ease of reading, vertices are indicated by large circles and edges are not drawn.) To obtain such colorings for other choices of $m \geq 2$, start with the graph in the figure whose number of dominos is congruent to m mod 3, and either omit or duplicate the

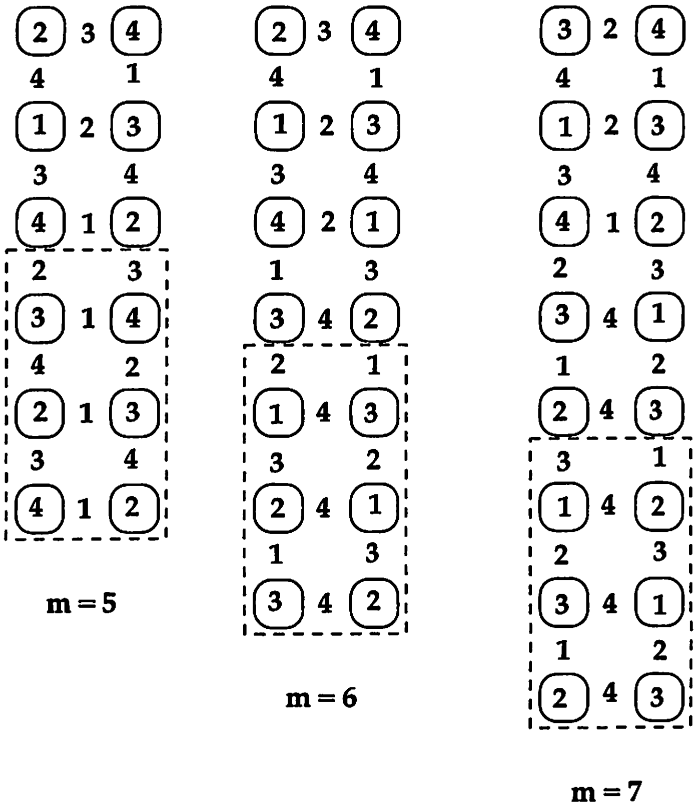

Figure 14: Total colorings with properties L and R.

portion of the graph enclosed in a dotted box to obtain the desired number of rungs. For every m, other valid total colorings with the same properties can be obtained by permuting the four colors in any fashion.

Step 4: Linking Horizontal Stacks

Our given domino tiling graph has maximum degree 3, which implies that there are never two adjacent stacks of m horizontal dominos in the graph. Instead, each such stack is linked to the next stack either by a two-edge "bridge" formed by a single vertical domino, or a sideways ladder formed by multiple vertical dominos. Furthermore, there may be a sideways ladder (with right rung missing) linked on the left side of the leftmost horizontal stack. Similarly, there may be a sideways ladder (with left rung missing)

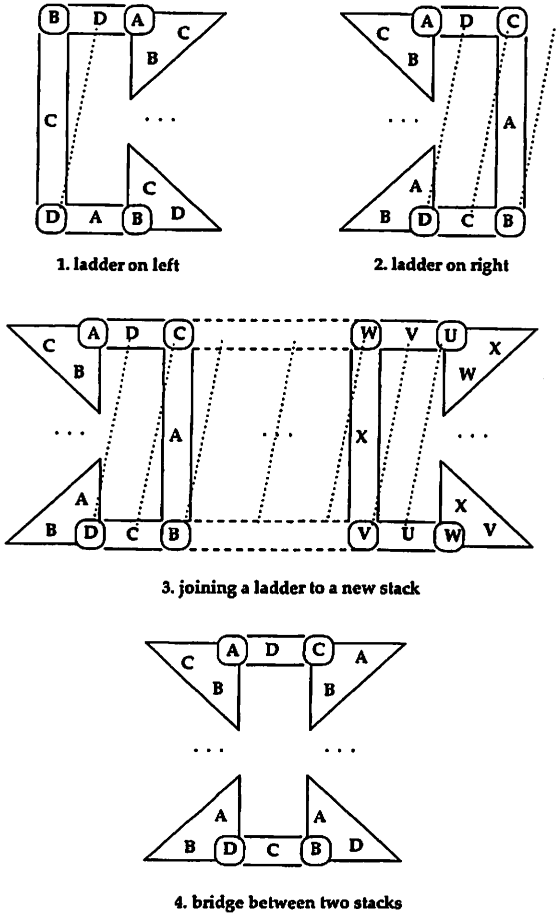

Figure 15: Linking successive horizontal stacks.

linked on the right side of the rightmost stack. We show that all of these possible linkages between and appendages to the horizontal stacks can be colored with four colors without violating the total coloring property. The key is to always color the horizontal stacks using colorings with properties L and R.

We start with the leftmost horizontal stack in G. Arbitrarily color it using a total coloring with 4 colors that has properties L and R. If there is a sideways ladder to the left of this stack, begin to color the ladder following the pattern shown in panel 1 of Figure 15. Here we use the fact that the stack coloring has property L. Then use step 2 (adapted to sideways ladders) to extend this coloring to the left end of G.

Returning to the leftmost horizontal stack, we now extend the total coloring to the right, one step at a time. Suppose we have just colored a given horizontal stack, which is adjoined on the right by a sideways ladder (with left rung missing) formed by a succession of at least two vertical dominos (or one vertical domino at the far right end). We can begin to color this sideways ladder as shown in panel 2 of Figure 15, thanks to property R. Then use step 2 to extend this coloring to the right end of the sideways ladder. Here there are two cases. If the right end of the ladder is the right end of the whole graph, we are done. If the right end of the ladder links to a new horizontal stack, then panel 3 of Figure 15 shows that we can find a total coloring of this stack (with labels suitably permuted) that has properties L and R. Thus, the inductive coloring construction can continue.

The final possibility is that a horizontal stack is linked immediately to another horizontal stack by a two-edge "bridge" arising from a single intervening vertical domino. In this case, panel 4 of Figure 15 shows how the coloring of the first stack (which has property R on the right) can be extended to a coloring of the next stack (with property L on the left and R on the right) with a suitable permutation of the colors. This completes the proof of the theorem.

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