On the strongly c-harmoniousness of cycle with some chords *

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Abstract. For $1 \leq s \leq n-3$, let $C_n(i; i_1, i_2, \cdots, i_s)$ denotes an n-cycle with consecutive vertices $x_1, x_2, ..., x_n$ to which the s chords $x_i x_{i_1}, x_i x_{i_2}, ..., x_i x_{i_s}$ have been added. In this paper, we discuss strongly c-harmonious problem of the graph $C_n(i; i_1, i_2, \cdots, i_s)$. A shell of width n is a fan $C_n(1; 3, 4, \cdots, n-1)$ and a vertex with degree $n-1$ is called apex. $MS\{n^m\}$ is a graph consisting of m copies of shell of width n having a common apex. If $m \geq 1$ is odd, then the multiple shell $MS\{n^m\}$ is harmonious.

Key words: harmonious graph; strongly c-harmonious graph; labelling; cycle; chord; multiple shell

Mathematics Subject Classifications: 05C78,05C90,05B30

1 Introduction

Graphs labelling, Where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as Coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Harmonious graphs naturally arose in the study

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by Graham and Sloane [1] of modular versions of additive base problems stemming from error-correcting codes. They also proved that some graphs are harmonious. Only graphs without loops, isolated vertices and multiple edges will be considered in this paper. The symbol Z_n denotes a ring of integers modulo n. Graph G=(V, E) is said to be a (p, q) graph if it has p vertices and q edges. If there exists an injection f: $V \to Z_q$, such that the induced mapping $f^*(uv) \equiv f(u) + f(v) \pmod q$ is a bijection from E onto Z_q, then f is said to be a harmonious labelling of G. A graph which admits such a labelling is called a harmonious graph. Chang, Hsu, and Rogers (see [3]) and Grace (see [4], [5]) have investigated subclasses of harmonious graphs. Chang et al. defined an injective labelling f of a graph G with q edges to be strongly c-harmonious labelling if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f^*(xy)$=$f(x) + f(y)$ for each edge xy are $c, c+1, \ldots, c+q-1$. Grace called such a labelling sequential labelling. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. S. Xu in [6] proved that all cycles with a chord are harmonious except that C_6 and the distance in C_6 between the endpoints of the chord is 2. In [8] Deb and Limaye showed that a variety of multiple shells are harmonious and they conjectured that all multiple shells are harmonious. Gallian in [7] surveyed the results on harmonious labelling of graphs and opened the problem whether a cycle with some chords are harmonious or not.

For $1 \le s \le n-3$, let $C_n(i; i_1, i_2, \cdots, i_s)$ denotes an n-cycle with consecutive vertices $x_1, x_2, \ldots, x_n$ to which the s chords $x_i x_{i_1}, x_i x_{i_2}, \ldots, x_i x_{i_s}$ have been added. In this paper, we shall discuss the strongly c-harmonious problem of the graph $C_n(i; i_1, i_2, \cdots, i_s)$, and obtain the following graphs are strongly c-harmonious. Let $n \equiv k(\bmod 4)$

k	graph	range of t	range of s
all	$C_n(1;3,4,\ldots,n\text{-}1)$		
1,3	$C_n(1;4,5,\ldots,n\text{-}2)$		
all	$C_n(1;5,6,\ldots,n\text{-}3)$		
2	$C_n(1;3,4,\ldots,t+2)$	4s-2,4s-1	$[1,\lfloor\frac{n-1}{4}\rfloor]$
0	$C_n(1;3,4,\ldots,t+2)$	4s-3,4s-4	$[1,\lfloor\frac{n+1}{4}\rfloor]$
1	$C_n(n;2,3,\ldots,2s,2s+1)$		$[1,\frac{n-1}{4}]$
1	$C_n(n;2,3,\ldots,2s)$		$[1,\frac{n-1}{4}]$
3	$C_n(n;3,4,\ldots,2s+1,2s+2)$		$[1,\frac{n-3}{4}]$
3	$C_n(n;3,4,\ldots,2s+1)$		$[1,\frac{n+1}{4}]$
1	$C_n(n;4t+2,4t+3,\ldots,4t+s+1)$	≥ 0	$\geq 1, 2t+s \leq \frac{n-3}{2}$
3	$C_n(n;4t+3,4t+4,\ldots,4t+s+2)$	≥ 1	$\geq 1, 2t+s \leq \frac{n-3}{2}$
1,3	$C_n(n;3,5,\ldots,2s+1)$		$[1,\frac{n-3}{2}]$
1,3	$C_n(\frac{n+1}{2};\frac{n+5}{2},\frac{n+9}{2},\ldots,\frac{n+1}{2}+2s)$		$[1,\frac{n-1}{4}]$
0	$C_n(1;n\text{-}1,n\text{-}3,\ldots,n\text{-}2s+1)$		$[1,\frac{n-2}{2}]$
3	$C_n(n;4,6,\ldots,2+2s)$		$[1,\frac{n-3}{4}]$
1	$C_n(1;n\text{-}1,n\text{-}2,n\text{-}4,\ldots,n\text{-}2s)$		$[1,\frac{n-3}{2}]$
2	$C_n(3;n,n\text{-}1,n\text{-}3,\ldots,n\text{-}2s+1)$		$[2,\frac{n-4}{2}]$
1	$C_n(\frac{n+1}{2};\frac{n+1}{2}\text{-}2,\frac{n+1}{2}\text{-}4,\ldots,\frac{n+1}{2}\text{-}2s,$ $\frac{n+1}{2}+2,\frac{n+1}{2}+4,\ldots,\frac{n+1}{2}+2t)$	$[0,\frac{n-1}{4}]$	$[1,\frac{n-1}{4}]$

By above definition, we obtain the following results.

Theorem 1.1 *If G is a (p,q) graph, then*

(1) the graph G is not harmonious when $p > q+1$;

(2) a strongly c-harmonious graph is also a harmonious graph. □

Theorem 1.2 (Graham and Sloane [1]) *The n-cycle is harmonious if and only if $n \equiv 1$ or 3 (mod 4).* □

Theorem 1.3 *(1) If (p,q) graph $G=(V,E)$ is strongly c-harmonious, then $\sum_{x\in V} d(x)f(x)= q(q-1)/2+cq$, where $d(x)$ is the degree of vertex x.*

If k-regular (p,q) graph G is strongly c-harmonious, then $q(q-1)+2cq \equiv 0 \pmod{2k}$.

(2) If (p,q) graph $G=(V,E)$ is harmonious, then $\sum_{e\in E} f^*(e)= q(q-1)/2+ kq$ for some k.

Proof For part (1), there is

$$\sum_{x\in V} d(x)f(x)= \sum_{xy\in E}(f(x)+f(y))= \sum_{e\in E} f^*(e)=q(q+2c-1)/2$$

$=q(q-1)/2+cq$.

Therefore, there is $q(q-1)+2cq \equiv 0 \pmod{2k}$ when G is a k-regular (p,q) graph. Part (2) is a corollary of part (1). □

Theorem 1.4 (M.Z.Youssef [2]) *If G is a harmonious graph, then $G^{(m)}$ (the graph consisting of m copies of G with one fixed vertex in common) is harmonious for any odd $m \geq 1$.* □

A shell of width n is a fan $C_n(1;3,4,\cdots,n-1)$ and a vertex with degree $n-1$ is called apex. $MS\{n^m\}$ is a graph consisting of m copies of shell of width n having a common apex. Deb and Limaye in [8] obtained that "all $MS\{n^3\}$ are harmonious". The following theorem extends this result.

Theorem 1.5 *If $m \geq 1$ is odd, then $MS\{n^m\}$ is harmonious.*

Proof By Theorem 2.5, we have the fan $C_n(1;3,4,\cdots,n-1)$ is strongly c-harmonious and apex is labelled 0. Hence it is also harmonious. This result immediately follows by Theorem 1.4. □

Let Z be the set of all integers. The symbol $[a,b]$ is defined by $\{x|\ x \in Z, a \leq x \leq b\}$, $[a,b]_k$ is defined by $\{x|\ x \in Z, a \leq x \leq b, x \equiv a (mod\ k)\}$, and the symbol $\lfloor x \rfloor$ denotes the greatest integer y such that $y \leq x$. When f is a function defined on the set S, let $f(S)$ denotes the set $\{f(x)|\ x \in S\}$.

Example Figure 1 shows a $C_8(1;4,5,6)$. Figure 2 is a strongly 4-harmonious labelling of $C_9(1;5,6)$.

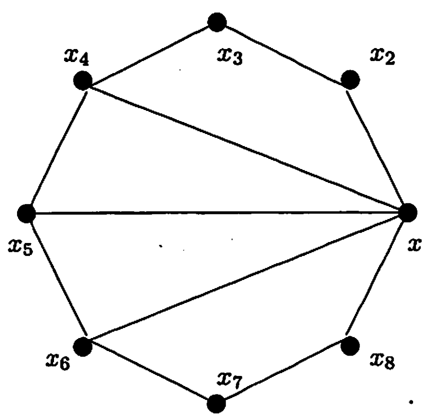

Figure 1

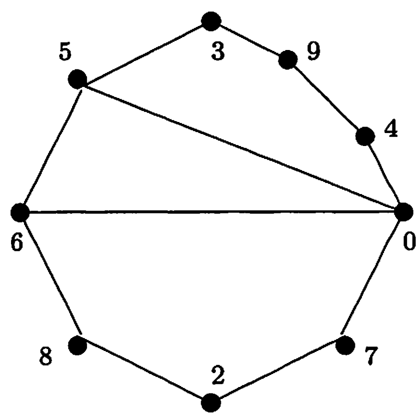

Figure 2

2 C_n with consecutive chords

In the following, we use V and E to denote the vertex set and the edge set of $C_n(i; i_1, i_2, \cdots, i_s)$, respectively.

Theorem 2.1 *When $n \equiv 1 \pmod 4$ and $n \geq 9$, the graph $C_n(n; 4t+2, 4t+3, ..., 4t+s+1)$ is strongly $(n-1)/2$-harmonious for $t \geq 0$, $s \geq 1$ and $2t+s \leq (n-3)/2$.*

Proof This graph has n vertices and $n+s$ edges. We construct the function f: $V \to Z_{n+s}$ as follows: $f(x_{2i-1})=i-1$ if $i \in [1, (n+1)/2]$,
$f(x_{2i})=(n-1)/2+i$ if $i \in [1, t]$,
$f(x_{2i})=\lfloor s/2 \rfloor + (n-1)/2 + i$ if $i \in [t+1, t+(n-1)/4]$,
$f(x_{2i})=s+(n-1)/2+i$ if $i \in [t+(n+3)/4, (n-1)/2]$.

It is not difficult to check that the f is an injection from V to Z_{n+s}. Let

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) |\ i \in [1, t]\}$
$=\{(n-3)/2+2i, (n-1)/2+2i |\ i \in [1, t]\}=[(n+1)/2, 2t+(n-1)/2]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) |\ i \in [t+1, t+(n-1)/4]\}$
$=\{\lfloor s/2 \rfloor+(n-3)/2+2i, \lfloor s/2 \rfloor+(n-1)/2+2i |\ i \in [t+1, t+(n-1)/4]\}$
$=[\lfloor s/2 \rfloor+(n+1)/2+2t, \lfloor s/2 \rfloor+2t+n-1]$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) |\ i \in [t+(n+3)/4, (n-1)/2]\}$
$=\{s+(n-3)/2+2i, s+(n-1)/2+2i |\ i \in [t+(n+3)/4, (n-1)/2]\}$
$=[s+n+2t, s+(3n-3)/2]$,
$D=\{f^*(x_n x_1)\} \cup \{f^*(x_n x_{4t+1+j}) |\ j \in [1, s]\}=\{(n-1)/2\} \cup \{(n-3)/2+i |\ i \in [2t+2, 2t+\lfloor (s+2)/2 \rfloor]\} \cup \{\lfloor s/2 \rfloor+n-1+i |\ i \in [2t+1, 2t+\lfloor (s+1)/2 \rfloor]\}$
$=\{(n-1)/2\} \cup [(n+1)/2+2t, 2t+\lfloor (s+2)/2 \rfloor+(n-3)/2] \cup [\lfloor s/2 \rfloor+n+2t, 2t+s+n-1]$.

Therefore, $f^*(E)=A \cup B \cup C \cup D=[(n-1)/2, s+(3n-3)/2]$. This implies that the f^* is a bijection from E to $[(n-1)/2, s+(3n-3)/2]$. □

Theorem 2.2 *When $n \equiv 3$ (mod 4) and $n \geq 11$, the graph $C_n(n; 4t+3, 4t+4, ..., 4t+s+2)$ is strongly $(n-1)/2$-harmonious for $t \geq 1$, $s \geq 1$ and $2t+s \leq (n-3)/2$.*

Proof This graph has n vertices and $n+s$ edges. We give the labelling f: $V \rightarrow Z_{n+s}$ as follows: $f(x_{2i-1})=i-1$ if $i \in [1,(n+1)/2]$,
$f(x_{2i})=(n-1)/2+i$ if $i \in [1,t]$,
$f(x_{2i})=\lfloor(s-1)/2\rfloor+(n+1)/2+i$ if $i \in [t+1, t+\frac{n+1}{4}]$,
$f(x_{2i})=(n-1)/2+s+i$ if $i \in [\frac{n+5}{4}+t, \frac{n-1}{2}]$.

It is not difficult to check that the f is an injection from V to Z_{n+s}. Let
$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1,t]\}$
$=\{(n-3)/2+2i, (n-1)/2+2i| \ i \in [1,t]\}=[(n+1)/2, 2t+(n-1)/2]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [t+1, t+(n+1)/4]\}$
$=\{\lfloor(s-1)/2\rfloor+(n-1)/2+2i, \lfloor(s-1)/2\rfloor+(n+1)/2+2i| \ i \in [t+1, t+(n+1)/4]\}$
$=[\lfloor(s-1)/2\rfloor+(n+3)/2+2t, \lfloor(s-1)/2\rfloor+2t+n+1]$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [t+(n+5)/4, (n-1)/2]\}$
$=\{s+(n-3)/2+2i, s+(n-1)/2+2i| \ i \in [t+(n+5)/4, (n-1)/2]\}=[s+n+2t+1, s+(3n-3)/2]$,
$D=\{f^*(x_nx_1)\}\cup\{f^*(x_nx_{4t+2+j})| \ j \in [1,s]\}=\{(n-1)/2\}\cup\{(n-3)/2+i| \ i \in [2t+2, 2t+\lfloor(s+3)/2\rfloor]\}\cup\{\lfloor(s-1)/2\rfloor+n+i| \ i \in [2t+2, 2t+\lfloor(s+2)/2\rfloor]\}$
$=\{(n-1)/2\}\cup[(n+1)/2+2t, 2t+\lfloor(s-1)/2\rfloor+(n+1)/2]\cup[\lfloor(s-1)/2\rfloor+n+2t+2, 2t+s+n]$.

Therefore, $f^*(E)=A\cup B\cup C\cup D=[(n-1)/2, s+(3n-3)/2]$. □

Theorem 2.3 *Let integer $n \geq 6$. (1) Let $n \equiv 2$ (mod 4), and t=$4s-2$ or $4s-1$ where $1 \leq s \leq \lfloor(n-1)/4\rfloor$. Then the graph $C_n(1; 3, 4, ..., t+2)$ is strongly $(n+2)/2$-harmonious if t=$4s-2$, or strongly $n/2$-harmonious if t=$4s-1$.*

(2) Let $n \equiv 0$ (mod 4), t=$4s-3$ or $4s-4$ where $1 \leq s \leq \lfloor(n+1)/4\rfloor$. Then the graph $C_n(1; 3, 4, ..., t+2)$ is strongly $(n+2)/2$-harmonious if t=$4s-4$,

or strongly $n/2$-harmonious if $t{=}4s-3$.

Proof Each graph has n vertices and $n+t$ edges. We construct the function f: $V \to Z_{n+t}$ as follows.

For part (1), $f(x_{2i}){=}n+2s-i$ if $i \in [1, n/2]$, $f(x_1){=}0$,
$f(x_{2i-1}){=}(n+2)/2+2s-i$ if $i \in [2, s+(n+2)/4]$,
$f(x_{2i-1}){=}n/2+1-i$ if $i \in [\frac{n+6}{4}+s, \frac{n}{2}]$.

It is not difficult to check that the f is an injection from V to Z_{n+t}. Let

$A{=}\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i-2}x_{2i-1}) | \ i \in [2, s+(n+2)/4]\}$
$\quad{=}\{3n/2+4s+1-2i, 3n/2+4s+2-2i | \ i \in [2, s+(n+2)/4]\}$
$\quad{=}[n+2s, 3n/2+4s-2]$,
$B{=}\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i-1}x_{2i-2}) | \ i \in [s+(n+6)/4, n/2]\}$
$\quad{=}\{2s+3n/2+1-2i, 2s+3n/2+2-2i | \ i \in [(n+6)/4+s, n/2]\}$
$\quad{=}[2s+n/2+1, n-1]$,
$C{=}\{f^*(x_nx_1), f^*(x_2x_1)\}{=}\{2s+n/2, 2s+n-1\}$,
when $t{=}4s-2$,
$D{=}\{f^*(x_1x_j) | \ j \in [3, t+2]\}{=}\{n/2+2s+1-i, n+2s-i | \ i \in [2, 2s]\}$
$\quad{=}[(n+2)/2, (n-2)/2+2s] \cup [n, 2s+n-2]$.
When $t{=}4s-1$,
$D{=}\{f^*(x_1x_j) | \ j \in [3, t+2]\}{=}\{n/2+2s+1-i, n+2s-j | \ i \in [2, 2s+1], j \in [2, 2s]\}{=}[n/2, (n-2)/2+2s] \cup [n, 2s+n-2]$.

Therefore, $f^*(E){=}A \cup B \cup C \cup D{=}[a, 4s+(3n-4)/2]$, where $a{=}n/2$ if $t{=}4s-1$, or $a{=}(n+2)/2$ if $t{=}4s-2$.

For part (2), $f(x_{2i}){=}n+2s-1-i$ if $i \in [1, n/2]$, $f(x_1){=}0$,
$f(x_{2i-1}){=}n/2+2s-i$ if $i \in [2, s+n/4]$,
$f(x_{2i-1}){=}n/2+1-i$ if $i \in [\frac{n+4}{4}+s, \frac{n}{2}]$.

It is not difficult to check that the f is an injection from V to Z_{n+t}. Let

$A{=}\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i-2}x_{2i-1}) | \ i \in [2, s+n/4]\}$
$\quad{=}\{3n/2+4s-1-2i, 3n/2+4s-2i | \ i \in [2, s+n/4]\}$

$=[n+2s-1, 3n/2+4s-4]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i-1}x_{2i-2})|\ i\in[s+(n+4)/4, n/2]\}$

$=\{2s+3n/2-2i, 2s+3n/2+1-2i|\ i\in[(n+4)/4+s, n/2]\}$

$=[2s+n/2, n-1]$,

$C=\{f^*(x_nx_1), f^*(x_2x_1)\}=\{2s+n/2-1, 2s+n-2\}$,

when $t=4s-4$,

$D=\{f^*(x_1x_j)|\ j\in[3,t+2]\}=\{n/2+2s-i, n+2s-1-i|\ i\in[2,2s-1]\}$

$=[(n+2)/2,(n-4)/2+2s]\cup[n,2s+n-3]$.

When $t=4s-3$,

$D=\{f^*(x_1x_j)|\ j\in[3,t+2]\}$

$=\{n/2+2s-i, n+2s-1-j|\ i\in[2,2s], j\in[2,2s-1]\}$

$=[n/2,(n-4)/2+2s]\cup[n,2s+n-3]$.

Therefore, $f^*(E)=A\cup B\cup C\cup D=[a, 4s+(3n-8)/2]$, where $a=n/2$ if $t=4s-3$, or $a=(n+2)/2$ if $t=4s-4$. □

Theorem 2.4 *Let integer $n\geq 9$. Then the graph $C_n(1;5,6,...,n-3)$ is strongly $(n-1)/2$-harmonious if n is odd; the graph $C_n(1;5,6,...,n-3)$ is strongly $(n-4)/2$-harmonious if n is even.*

Proof This graph has n vertices and $2n-7$ edges.

Case 1: When n is odd, if $n=9$ see Figure 2.

If $n\geq 11$, we construct the function f: $V\to Z_{2n-7}$ as follows: $f(x_1)=0$, $f(x_2)=n-3$, $f(x_4)=2n-9$, $f(x_{2i})=n+i-5$ if $i\in[3,(n-3)/2]$, $f(x_{2i-1})=(n-7)/2+i$ if $i\in[2,(n-3)/2]$, $f(x_{n-2})=1$, $f(x_{n-1})=(3n-9)/2$, $f(x_n)=n-4$.

Since $f(V)=\{0,n-3,2n-9,1,(3n-9)/2,n-4\}\cup[n-2,(3n-13)/2]\cup[(n-3)/2,n-5]$, the f is an injection from V to Z_{2n-7}. In the following we show that the f^* is a bijection. Let

$A=\{f^*(x_{2i}x_{2i+1})=3(n-5)/2+2i|\ i\in[3,(n-5)/2]\}$

$=[(3n-3)/2,(5n-25)/2]_2$,

$B=\{f^*(x_{2i}x_{2i-1})=(3n-17)/2+2i|\ i\in[3,(n-3)/2]\}$

$=[(3n-5)/2,(5n-23)/2]_2$,

$C=\{f^*(x_{2i-1}x_1)=(n-7)/2+i, f^*(x_{2i}x_1)=n-5+i \mid i\in[3,(n-3)/2]\}$
$=[(n-1)/2, n-5]\cup[n-2,(3n-13)/2]$,
$D=\{f^*(x_1x_2), f^*(x_3x_2), f^*(x_3x_4), f^*(x_4x_5), f^*(x_{n-3}x_{n-2}), f^*(x_{n-1}x_{n-2}), f^*(x_{n-1}x_n), f^*(x_nx_1)\}=\{n-3,(3n-9)/2,(5n-21)/2,(5n-19)/2,(3n-11)/2,(3n-7)/2,(5n-17)/2,n-4\}$.

Therefore, $f^*(E)=A\cup B\cup C\cup D=[(n-1)/2,(5n-17)/2]$.

Case 2: When n is even, we construct the function the f: $V\to Z_{2n-7}$ as follows:
$f(x_1)=0$, $f(x_2)=n-3$, $f(x_4)=2n-8$, $f(x_{2i})=n+i-5$ if $i\in[3,(n-2)/2]$, $f(x_{2i+1})=(n-8)/2+i$ if $i\in[1,(n-4)/2]$, $f(x_{n-1})=1$, $f(x_n)=n-5$.

Since $f(V)=\{0,n-3,2n-8,1,n-5\}\cup[n-2,(3n-12)/2]\cup[(n-6)/2,n-6]$, the f is an injection from V to Z_{2n-7}. In the following we show that the f^* is a bijection. Let
$A=\{f^*(x_{2i}x_{2i+1}), f^*(x_{2i}x_{2i-1}) \mid i\in[3,(n-4)/2]\}$
$=\{(3n-18)/2+2i,(3n-20)/2+2i \mid i\in[3,(n-4)/2]\}$
$=[(3n-8)/2,(5n-26)/2]$,
$B=\{f^*(x_{2i}x_1)=n-5+i \mid i\in[3,(n-4)/2]\}=[n-2,(3n-14)/2]$,
$C=\{f^*(x_{2i+1}x_1)=(n-8)/2+i \mid i\in[2,(n-4)/2]\}=[(n-4)/2,n-6]$,
$D=\{f^*(x_1x_2), f^*(x_3x_2), f^*(x_3x_4), f^*(x_4x_5), f^*(x_{n-3}x_{n-2}), f^*(x_{n-1}x_{n-2}), f^*(x_{n-1}x_n), f^*(x_nx_1)\}=\{n-3,(3n-12)/2,(5n-22)/2,(5n-20)/2,(5n-24)/2,(3n-10)/2,n-4,n-5\}$.

Therefore, $f^*(E)=A\cup B\cup C\cup D=[(n-4)/2,(5n-20)/2]$. □

Theorem 2.5 *When integer $n\geq 4$, the graph $C_n(1;3,4,\cdots,n-1)$ is strongly c-harmonious.*

Proof The graph $C_n(1;3,4,\cdots,n-1)$ has n vertices and $2n-3$ edges. We construct the function f: $V\to Z_{2n-3}$ as follows: $f(x_1)=0$, and label the other vertices by distinguishing 4 cases.

Case 1 When n is odd and $n\geq 5$, let $f(x_{2i+1})=n-2+i$ if $i\in[1,\frac{n-1}{2}]$, $f(x_{2i})=\frac{n-3}{2}+i$ if $i\in[1,\frac{n-1}{2}]$. Then
$f(V)=[(n-1)/2,n-2]\cup[n-1,(3n-5)/2]\cup\{0\}=[(n-1)/2,(3n-5)/2]\cup\{0\}$.

Since $|f(V)|=n$, the f is an injection from V to Z_{2n-3}. By the definition of f we have

$\{f^*(x_1x_{2i})|\ i\in[2,\frac{n-1}{2}]\}=\{\frac{n-3}{2}+i|\ i\in[2,\frac{n-1}{2}]\}=[(n+1)/2,n-2]$,

$\{f^*(x_1x_{2i+1})|\ i\in[1,\frac{n-3}{2}]\}=\{n-2+i|\ i\in[1,\frac{n-3}{2}]\}=[n-1,(3n-7)/2]$,

$\{f^*(x_{2i-1}x_{2i})|\ i\in[2,\frac{n-1}{2}]\}=\{3(n-3)/2+2i|\ i\in[2,\frac{n-1}{2}]\}$
$=[(3n-1)/2,(5n-11)/2]_2$,

$\{f^*(x_{2i+1}x_{2i})|\ i\in[1,\frac{n-1}{2}]\}=\{(3n-7)/2+2i|\ i\in[1,\frac{n-1}{2}]\}$
$=[(3n-3)/2,(5n-9)/2]_2$,

$f^*(x_1x_n)=(3n-5)/2$, $f^*(x_1x_2)=(n-1)/2$.

Since $f^*(E)=[(n-1)/2,(5n-9)/2]$, the f is a strongly $(n-1)/2$-harmonious labelling of the graph $C_n(1;3,4,\cdots,n-1)$.

Case 2 When $n\equiv 2 \pmod 4$ and $n\geq 6$, let

$f(x_{2i+1})=n+i-2$ if $i\in[1,\frac{n-2}{2}]$, $f(x_{2i})=\frac{n}{2}+i-2$ if $i\in[1,\frac{n}{2}]$.

Then $f(V)=[\frac{n}{2}-1,\frac{3n-6}{2}]\cup\{0\}$. Since $|f(V)|=n$, the f is an injection from V to Z_{2n-3}. By the definition of f, we have

$\{f^*(x_1x_{2i})|\ i\in[2,\frac{n-2}{2}]\}=\{\frac{n}{2}+i-2|\ i\in[2,\frac{n-2}{2}]\}=[\frac{n}{2},n-3]$,

$\{f^*(x_1x_{2i+1})|\ i\in[1,\frac{n-2}{2}]\}=\{n+i-2|\ i\in[1,\frac{n-2}{2}]\}=[n-1,\frac{3n-6}{2}]$,

$\{f^*(x_{2i-1}x_{2i})|\ i\in[2,\frac{n}{2}]\}=\{\frac{3n}{2}+2i-5|\ i\in[2,\frac{n}{2}]\}=[\frac{3n}{2}-1,\frac{5n}{2}-5]_2$,

$\{f^*(x_{2i+1}x_{2i})|\ i\in[1,\frac{n-2}{2}]\}=\{\frac{3n}{2}+2i-4|\ i\in[1,\frac{n-2}{2}]\}=[\frac{3n}{2}-2,\frac{5n}{2}-6]_2$,

$\{f^*(x_1x_n),f^*(x_1x_2)\}=\{n-2,\frac{n}{2}-1\}$.

Since $f^*(E(G))=[n/2-1,5n/2-5]$, the f is a strongly $(n-2)/2$-harmonious labelling of graph $C_n(1;3,4,\cdots,n-1)$.

Case 3 When $n\equiv 0 \pmod 4$ and $n\geq 8$, let

$f(x_{2i})=n+i-2$ if $i\in[1,\frac{n}{2}]$, $f(x_{2i+1})=\frac{n}{2}+i-1$ if $i\in[1,\frac{n}{2}-1]$. Then $f(V)=[\frac{n}{2},\frac{3n-4}{2}]\cup\{0\}$. Since $|f(V)|=n$, the f is an injection from V to Z_{2n-3}. By the definition of f we have

$\{f^*(x_1x_{2i+1})|\ i\in[1,\frac{n-2}{2}]\}=\{\frac{n}{2}+i-1|\ i\in[1,\frac{n-2}{2}]\}=[\frac{n}{2},n-2]$,

$\{f^*(x_1x_{2i})|\ i\in[2,\frac{n-2}{2}]\}=\{n+i-2|\ i\in[2,\frac{n-2}{2}]\}=[n,\frac{3n-6}{2}]$,

$\{f^*(x_{2i-1}x_{2i})|\ i\in[2,\frac{n}{2}]\}=\{\frac{3n}{2}+2i-4|\ i\in[2,\frac{n}{2}]\}=[\frac{3n}{2},\frac{5n}{2}-4]_2$,

$\{f^*(x_{2i+1}x_{2i})|\ i\in[1,\frac{n-2}{2}]\}=\{\frac{3n}{2}+2i-3|\ i\in[1,\frac{n-2}{2}]\}=[\frac{3n}{2}-1,\frac{5n}{2}-5]_2$,

$\{f^*(x_1x_n), f^*(x_1x_2)\}=\{n-1, \frac{3n}{2}-2\}$.

Since $f^*(E(G))=[n/2, 5n/2-4]$, the f is a strongly $n/2$-harmonious labelling of the graph $C_n(1;3,4,\cdots,n-1)$.

Case 4 When $n=4$, we directly construct the f as follows: $f(x_2)=2$, $f(x_3)=1$, $f(x_4)=4$. □

Theorem 2.6 *The graph $C_n(1;4,5,\cdots,n-2)$ is strongly $(n-1)/2$-harmonious when n is odd and $n \geq 7$, or strongly $(n-2)/2$-harmonious when $n \equiv 0$ (mod 4) and $n \geq 8$, or strongly $(n-4)/2$-harmonious when $n \equiv 2$ (mod 4) and $n > 6$, or strongly 3-harmonious when $n = 6$.*

Proof The graph $C_n(1;4,5,\cdots,n-2)$ has n vertices and $2n-5$ edges. When $n=6$, let $f(x_1)=3$, $f(x_2)=1$, $f(x_3)=2$, $f(x_4)=5$, $f(x_5)=0$, $f(x_6)=6$.

When $n > 6$, we construct the function f: $V \rightarrow Z_{2n-5}$ as follows: $f(x_1)=0$ and we assign labels to the other vertices by distinguishing three cases.

Case 1 When n is odd and $n \geq 7$, $f(x_{2i+1})=(n-5)/2+i$ if $i \in [1, \frac{n-1}{2}]$, $f(x_{2i})=n-3+i$ if $i \in [1, \frac{n-1}{2}]$.

It is not difficult to check that the f is an injection from V to Z_{2n-5}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i+1}) = (3n-11)/2+2i \mid i \in [1,(n-1)/2]\}$
$\quad=[(3n-7)/2, (5n-13)/2]_2$,

$B=\{f^*(x_{2i}x_{2i-1}) = (3n-13)/2+2i \mid i \in [2,(n-1)/2]\}$
$\quad=[(3n-5)/2, (5n-15)/2]_2$,

$C=\{f^*(x_1x_2), f^*(x_nx_1)\}=\{n-2, n-3\}$,

$D=\{f^*(x_1x_i) \mid i \in [4, n-2]\}=\{(n-5)/2+i, n-3+i \mid i \in [2,(n-3)/2]\}$
$\quad=[(n-1)/2, n-4] \cup [n-1, (3n-9)/2]$.

Therefore, $f^*(E)=A \cup B \cup C \cup D=[(n-1)/2, (5n-13)/2]$.

Case 2 When $n \equiv 0$ (mod 4) and $n \geq 8$, $f(x_{2i+1})=(n-2)/2+i$ if $i \in [1, \frac{n-4}{2}]$,

$f(x_{2i})=n-3+i$ if $i \in [1, \frac{n-2}{2}]$, $f(x_{n-1})=1$, $f(x_n)=(n-2)/2$.

It is not difficult to check that the f is an injection from V to Z_{2n-5}.

By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i+1}) = 3n/2-4+2i|\ i \in [1,(n-4)/2]\}=[3n/2-2, 5n/2-8]_2$,

$B=\{f^*(x_{2i}x_{2i-1}) = 3n/2-5+2i|\ i \in [2,(n-2)/2]\}=[3n/2-1, 5n/2-7]_2$,

$C=\{f^*(x_1x_2), f^*(x_nx_1), f^*(x_nx_{n-1}), f^*(x_{n-1}x_{n-2})\}$

$=\{n-2, n/2-1, n/2, 3n/2-3\}$,

$D=\{f^*(x_1x_i)|\ i \in [4, n-2]\}=[(n+1)/2, n-3] \cup [n-1, (3n-8)/2]$.

Therefore, $f^*(E)=A \cup B \cup C \cup D=[(n-2)/2, (5n-14)/2]$.

Case 3 When $n \equiv 2 \pmod 4$ and $n \geq 10$,

$f(x_{2i})=(n-6)/2+i$ if $i \in [1, \frac{n}{2}]$,

$f(x_{2i+1})=n-2+i$ if $i \in [1, \frac{n-4}{2}]$, $f(x_{n-1})=2$.

It is not difficult to check that the f is an injection from V to Z_{2n-5}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i+1}) = 3n/2-5+2i|\ i \in [1,(n-4)/2]\}=[3n/2-3, 5n/2-9]_2$,

$B=\{f^*(x_{2i}x_{2i-1}) = 3n/2-6+2i|\ i \in [2,(n-2)/2]\}=[3n/2-2, 5n/2-8]_2$,

$C=\{f^*(x_1x_2), f^*(x_nx_1), f^*(x_nx_{n-1}), f^*(x_{n-1}x_{n-2})\}$

$=\{n/2-2, n-3, n-1, n-2\}$,

$D=\{f^*(x_1x_i)|\ i \in [4, n-2]\}= [n/2-1, n-4] \cup [n, (3n-8)/2]$.

Therefore, $f^*(E)=A \cup B \cup C \cup D =[(n-4)/2, (5n-8)/2]$. □

Theorem 2.7 *When $n \equiv 1 \pmod 4$ and $n \geq 5$, $G_1=C_n(n; 2, 3, \cdots, 2s, 2s+1)$ and $G_2= C_n(n; 2, 3, \cdots, 2s)$ for $1 \leq s \leq (n-1)/4$ are strongly $(n-1)/2$-harmonious.*

Proof The graph G_1 has $n+2s$ edges. We define f: $V \to Z_{n+2s}$ as follows: $f(x_{2i-1})=i-1$ if $i \in [1, (n+1)/2]$,

$f(x_{2i})=(n-1)/2+s+i$ if $i \in [1, (n-1)/4]$,

$f(x_{2i})=(n-1)/2+2s+i$ if $i \in [(n+3)/4, (n-1)/2]$.

Then $f(V(G_1))=[0, (n-1)/2] \cup [(n+1)/2+s, (3n-3)/4+s] \cup [(3n+1)/4+2s, (n+2s-1]$. Since $f(V(G_1)) \subseteq Z_{n+2s}$ and $|f(V(G_1))|=n$, the f is an injection from V to Z_{n+2s}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1, (n-1)/4]\}$

$=\{(n-3)/2+s+2i, (n-1)/2+s+2i|\ i \in [1, (n-1)/4]\}$

$=[(n+1)/2+s, n-1+s]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) \mid i \in [(n+3)/4, (n-1)/2]\}$

$=\{(n-3)/2+2s+2i, (n-1)/2+2s+2i \mid i \in [(n+3)/4, (n-1)/2]\}$

$=[n+2s, (3n-3)/2+2s]$,

$C=\{f^*(x_n x_1)\}=\{(n-1)/2\}$,

$D=\{f^*(x_n x_i) \mid i \in [2, 2s+1]\}=\{(n-1)/2+i, n-1+s+i \mid i \in [1, s]\}$

$= [(n+1)/2, (n-1)/2+s] \cup [n+s, n-1+2s]$.

Therefore, $f^*(E(G_1))=A \cup B \cup C \cup D = [(n-1)/2, 3(n-1)/2+2s]$.

The graph G_2 has $n+2s-1$ edges, let $f(x_{2i-1})=i-1$ if $i \in [1, (n+1)/2]$, $f(x_{2i})=(n-3)/2+s+i$ if $i \in [1, (n-1)/4]$,
$f(x_{2i})=(n-3)/2+2s+i$ if $i \in [(n+3)/4, (n-1)/2]$.

Then $f(V(G_2))=[0, (n-1)/2] \cup [(n-1)/2+s, (3n-7)/4+s] \cup [(3n-3)/4+2s, (n+2s-2]$. Since $f(V(G_2)) \subseteq Z_{n+2s-1}$ and $|f(V(G_2))|=n$, the f is an injection from V to Z_{n+2s-1}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) \mid i \in [1, (n-1)/4]\}$

$=\{(n-5)/2+s+2i, (n-3)/2+s+2i \mid i \in [1, (n-1)/4]\}$

$=[(n-1)/2+s, n-2+s]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) \mid i \in [(n+3)/4, (n-1)/2]\}$

$=\{(n-5)/2+2s+2i, (n-3)/2+2s+2i \mid i \in [(n+3)/4, (n-1)/2]\}$

$=[n+2s-1, (3n-5)/2+2s]$,

$C=\{f^*(x_n x_1)\}=\{(n-1)/2\}$,

$D=\{f^*(x_n x_i) \mid i \in [2, 2s]\}=\{(n-1)/2+i, n-2+s+j \mid i \in [1, s-1], j \in [1, s]\}$

$=[(n+1)/2, (n-3)/2+s] \cup [n+s-1, n-2+2s]$.

Therefore, $f^*(E(G_2))=A \cup B \cup C \cup D = [(n-1)/2, (3n-5)/2+2s]$. This implies that both G_1 and G_2 are strongly $(n-1)/2$-harmonious. □

Theorem 2.8 *When $n \equiv 3$ (mod 4) and $n \geq 7$, $G_1=C_n(n; 3, 4, \cdots, 2s+1, 2s+2)$ for $1 \leq s \leq (n-3)/4$ and $G_2=C_n(n; 3, 4, \cdots, 2s+1)$ for $1 \leq s \leq (n+1)/4$ are strongly $(n-1)/2$-harmonious.*

Proof The graph G_1 has $n+2s$ edges. We define f as follows:
$f(x_{2i-1})=i-1$ if $i \in [1, (n+1)/2]$, $f(x_{2i})=(n-1)/2+s+i$ if $i \in [1, (n+1)/4]$,

$f(x_{2i})=(n-1)/2+2s+i$ if $i \in [(n+5)/4,(n-1)/2]$.

Then $f(V(G_1))=[0,(n-1)/2] \cup [(n+1)/2+s,(3n-1)/4+s] \cup [(3n+3)/4+2s,n+2s-1]$. Since $f(V(G_1)) \subseteq Z_{n+2s}$ and $|f(V(G_1))|=n$, the f is an injection from V to Z_{n+2s}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i \in [1,(n+1)/4]\}$

$=\{(n-3)/2+s+2i,(n-1)/2+s+2i|\ i \in [1,(n+1)/4]\}$

$=[(n+1)/2+s,n+s]$,

$B=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i \in [(n+5)/4,(n-1)/2]\}$

$=\{(n-3)/2+2s+2i,(n-1)/2+2s+2i|\ i \in [(n+5)/4,(n-1)/2]\}$

$=[n+2s+1,(3n-3)/2+2s]$,

$C=\{f^*(x_nx_1)\}=\{(n-1)/2\}$,

$D=\{f^*(x_nx_{2i+1})|\ i \in [1,s]\} \cup \{f^*(x_nx_{2i})|\ i \in [2,s+1]\}$

$=\{(n-1)/2+i|\ i \in [1,s]\} \cup \{n-1+s+i|\ i \in [2,s+1]\}$

$=[(n+1)/2,(n-1)/2+s] \cup [n+s+1,n+2s]$.

Therefore, $f^*(E(G_1))=A \cup B \cup C \cup D=[(n-1)/2,3(n-1)/2+2s]$.

The graph G_2 has $n+2s-1$ edges, let $f(x_{2i-1})=i-1$ if $i \in [1,(n+1)/2]$,

$f(x_{2i})=(n-1)/2+s+i$ if $i \in [1,(n+1)/4]$,

$f(x_{2i})=(n-3)/2+2s+i$ if $i \in [(n+5)/4,(n-1)/2]$.

Then $f(V(G_2))=[0,(n-1)/2] \cup [(n+1)/2+s,(3n-1)/4+s] \cup [(3n-1)/4+2s,n+2s-2]$. Since $f(V(G_2)) \subseteq Z_{n+2s-1}$ and $|f(V(G_2))|=n$, the f is an injection from V to Z_{n+2s-1}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i \in [1,(n+1)/4]\}$

$=\{(n-3)/2+s+2i,(n-1)/2+s+2i|\ i \in [1,(n+1)/4]\}$

$=[(n+1)/2+s,n+s]$,

$B=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i \in [(n+5)/4,(n-1)/2]\}$

$=\{(n-5)/2+2s+2i,(n-3)/2+2s+2i|\ i \in [(n+5)/4,(n-1)/2]\}$

$=[n+2s,(3n-5)/2+2s]$,

$C=\{f^*(x_nx_1)\}=\{(n-1)/2\}$,

$D=\{f^*(x_nx_{2i+1})|\ i \in [1,s]\} \cup \{f^*(x_nx_{2i})|\ i \in [2,s]\}$

$=\{(n-1)/2+i|\ i \in [1,s]\} \cup \{n-1+s+i|\ i \in [2,s]\}$

$=[(n+1)/2, (n-1)/2+s] \cup [n+s+1, n-1+2s]$.

Therefore, $f^*(E(G_2))=A\cup B\cup C\cup D=[(n-1)/2, (3n-5)/2+2s]$. This implies that both G_1 and G_2 are strongly $(n-1)/2$-harmonious. □

3 Other C_n with some chords

Theorem 3.1 *When n is odd and $n \geq 5$, $C_n(n; 3, 5, \cdots, 2s+1)$ for $1 \leq s \leq (n-3)/2$ and $C_n((n+1)/2; (n+1)/2+2, (n+1)/2+4, \cdots, (n+1)/2+2s)$ for $1 \leq s \leq (n-1)/4$ are strongly $(n-1)/2-s$-harmonious.*

Proof The graph $C_n(n; 3, 5, \cdots, 2s+1)$ (or $C_n((n+1)/2; (n+1)/2+2, (n+1)/2+4, \cdots, (n+1)/2+2s))$ has $n+s$ edges. We define f as follows: $f(x_{2i-1})=(n+1)/2-i$ if $i \in [1, (n+1)/2]$, $f(x_{2i})=n-i$ if $i \in [1, (n-1)/2]$.

Since $f(V)=[0, n-1]$, the f is an injection from V to Z_{n+s}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i \in [1, (n-1)/2]\} \cup \{f^*(x_{2i}x_{2i+1})|\ i \in [1, (n-1)/2]\}$
$=[(n+1)/2, (3n-3)/2]$,

$B=\{f^*(x_n x_1)\}=\{(n-1)/2\}$.

For the graph $C_n(n; 3, 5, \cdots, 2s+1)$,

we have $C=\{f^*(x_n x_{2i+1})|\ i \in [1, s]\}=[(n-1)/2-s, (n-3)/2]$.

Therefore, $f^*(E)=[(n-1)/2-s, 3(n-1)/2]$.

For the graph $C_n((n+1)/2; (n+1)/2+2, (n+1)/2+4, \cdots, (n+1)/2+2s)$, we have $C=\{f^*(x_{(n+1)/2}x_{(n+1)/2+2i})|\ i \in [1, s]\}=[(n-1)/2-s, (n-3)/2]$.

Therefore, $f^*(E)=[(n-1)/2-s, 3(n-1)/2]$. □

Theorem 3.2 *When $n \equiv 1$ (mod 4) and $n \geq 5$, the graph $C_n((n+1)/2; (n+1)/2-2, (n+1)/2-4, \cdots, (n+1)/2-2s, (n+1)/2+2, (n+1)/2+4, \cdots, (n+1)/2+2t)$ for $1 \leq s \leq (n-1)/4$ and $0 \leq t \leq (n-1)/4$ is strongly $(n-1)/2-t$-harmonious.*

Proof This graph has n vertices and $n+t+s$ edges. We construct function f: $V \to Z_{n+t+s}$ as follows: $f(x_{2i-1})=(n+1)/2-i$ if $i \in [1, \frac{n+1}{2}]$, $f(x_{2i})=n+s-i$ if $i \in [1, \frac{n-1}{2}]$.

Since $|f(V)|=n$, the f is an injection from V to Z_{n+s+t}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,(n-1)/2]\}$

$=\{(3n+1)/2+s-2i, (3n-1)/2+s-2i|\ i \in [1,(n-1)/2]\}$

$=[(n+1)/2+s, (3n-3)/2+s]$,

$B=\{f^*(x_nx_1)\}=\{(n-1)/2\}$,

$C=\{f^*(x_{(n+1)/2}x_{(n+1)/2-2i})|\ i \in [1,s]\}=[(n+1)/2, (n-1)/2+s]$,

$D=\{f^*(x_{(n+1)/2}x_{(n+1)/2+2i})|\ i \in [1,t]\}=[(n-1)/2-t, (n-3)/2]$.

Therefore, $f^*(E)=A\cup B\cup C\cup D=[(n-1)/2-t, 3(n-1)/2+s]$. □

Theorem 3.3 *When $n \equiv 0 \pmod 4$ and $n \geq 8$, the graph $C_n(1;n-1,n-3,\cdots,n-2s+1)$ for $1 \leq s \leq n/2-1$ is strongly $n/2-s$-harmonious.*

Proof This graph has n vertices and $n+s$ edges. We construct function f: $V \to Z_{n+s}$ as follows: $f(x_{2i-1})=i-1$ if $i \in [1,n/2]$,

$f(x_{2i})=n/2+i$ if $i \in [1,n/4-1]$,

$f(x_{2i})=n/2+i+1$ if $i \in [n/4,n/2-1]$, $f(x_n)=n/2$.

It is not difficult to check that the f is an injection from V to the set Z_{n+s}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,n/4-1]\}$

$=\{n/2+2i-1, n/2+2i|\ i \in [1,n/4-1]\}=[(n+2)/2, n-2]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [n/4,n/2-1]\}$

$=\{n/2+2i, n/2+2i+1|\ i \in [n/4,n/2-1]\}=[n, 3n/2-1]$,

$C=\{f^*(x_nx_1), f^*(x_nx_{n-1})\}=\{n/2, n-1\}$,

$D=\{f^*(x_1x_{n+1-2i})|\ i \in [1,s]\}=[n/2-s, (n-2)/2]$.

We obtain $f^*(E)=A\cup B\cup C\cup D=[n/2-s, (3n-2)/2]$. Therefore, the f is a strongly $n/2-s$-harmonious labelling of $C_n(1;n-1,n-3,\cdots,n-2s+1)$.

□

Theorem 3.4 *When $n \equiv 3 \pmod 4$ and $n \geq 7$, the graph $C_n(n;4,6,\cdots,2+2s)$ for $1 \leq s \leq (n-3)/4$ is strongly $(n-1)/2-s$-harmonious.*

Proof This graph has n vertices and $n+s$ edges. We construct function f: $V \to Z_{n+s}$ as follows: $f(x_{2i-1})=n-i$ if $i \in [1,\frac{n+1}{4}]$,

$f(x_{2i-1})=(n+1)/2-i$ if $i\in[\frac{n+1}{4}+1,\frac{n+1}{2}]$,
$f(x_{2i})=(n+1)/2-i$ if $i\in[1,\frac{n+1}{4}]$, $f(x_{2i})=n-i$ if $i\in[\frac{n+1}{4}+1,\frac{n-1}{2}]$.

It is clear that the f is an injection from V to Z_{n+s}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[1,(n+1)/4]\}\cup\{f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-3)/4]\}$
$=[n,(3n-3)/2]$,
$B=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+5)/4,(n-1)/2]\}=[(n+1)/2,n-2]$,
$C=\{f^*(x_nx_1),f^*(x_{(n+1)/2}x_{(n+3)/2})\}=\{n-1,(n-1)/2\}$,
$D=\{f^*(x_nx_{2+2i})|\ i\in[2,s]\}=[(n-1)/2-s,(n-3)/2]$.

We obtain $f^*(E)=[(n-1)/2-s,(3n-3)/2]$. Therefore, the f is a strongly $(n-1)/2-s$-harmonious labelling of $C_n(n;4,6,\cdots,2s+2)$. □

Theorem 3.5 *When $n\equiv 1 \pmod 4$ and $n\geq 5$, the graph $C_n(1;n-1,n-2,n-4,\cdots,n-2s)$ for $1\leq s\leq(n-3)/2$ is strongly $(n-1)/2-s$-harmonious.*

Proof This graph has n vertices and $n+s+1$ edges. We construct function f: $V\to Z_{n+s+1}$ as follows: $f(x_{2i-1})=i-1$ if $i\in[1,\frac{n+1}{2}]$,
$f(x_{2i})=(n-1)/2+i$ if $i\in[1,\frac{n-1}{4}]$,
$f(x_{2i})=(n+1)/2+i$ if $i\in[\frac{n-1}{4}+1,\frac{n-1}{2}]$.

It is not difficult to check that the f is an injection from V to the set Z_{n+s+1}. Let

$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-1)/4]\}=[(n+1)/2,n-1]$,
$B=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+3)/4,(n-1)/2]\}=[n+1,(3n-1)/2]$,
$C=\{f^*(x_nx_1),f^*(x_nx_{n-1})\}=\{(n-1)/2,n\}$,
$D=\{f^*(x_1x_{n-2i})|\ i\in[1,s]\}=[(n-1)/2-s,(n-3)/2]$.

We obtain $f^*(E)=A\cup B\cup C\cup D=[(n-1)/2-s,(3n-1)/2]$. Therefore, the f is a strongly $(n-1)/2-s$-harmonious labelling of $C_n(1;n-1,n-2,n-4,\cdots,n-2s)$. □

Theorem 3.6 *When $n\equiv 2 \pmod 4$ and $n\geq 6$, the graph $C_n(3;n,n-1,n-3,\cdots,n-2s+1)$ for $2\leq s\leq n/2-2$ is strongly $(n+2)/2-s$-harmonious.*

Proof This graph has n vertices and $n+s+1$ edges. We construct function f: $V \to Z_{n+s+1}$ as follows: $f(x_{2i-1})=i-1$ if $i \in [1, n/2]$, $f(x_{2i})=n/2+i$ if $i \in [1, \frac{n+2}{4}]$, $f(x_{2i})=n/2+2+i$ if $i \in [\frac{n+2}{4}+1, \frac{n}{2}]$.

We have $f(V)=[0, n/2-1] \cup [n/2+1, (3n+2)/4] \cup [(3n+2)/4+3, n+2]$. Thus the function f is an injection from V to Z_{n+s+1}. By the definition of f, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [1, (n+2)/4]\}=[(n+2)/2, n+1]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [(n+6)/4, (n-2)/2]\}=[n+4, 3n/2]$,

$C=\{f^*(x_n x_1), f^*(x_n x_{n-1}), f^*(x_n x_3)\}=\{n+2, (3n+2)/2, n+3\}$,

$D=\{f^*(x_3 x_{n-2i+1}) | \ i \in [1, s]\}=[(n+2)/2-s, n/2]$.

We obtain $f^*(E)=A \cup B \cup C \cup D=[(n+2)/2-s, (3n+2)/2]$. Therefore, the f is a strongly $(n+2)/2-s$-harmonious labelling of $C_n(3; n, n-1, n-3, \cdots, n-2s+1)$. □

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