Noncrossing matchings with fixed points *

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Abstract. The noncrossing matchings with each of their blocks containing a given element are introduced and studied. The enumeration of these matchings is described through a polynomial of several variables which is proved to satisfy a recursive formula. Results of the enumeration of noncrossing matchings with fixed points are connected with Catalan numbers.
Keywords: noncrossing matchings, fixed points, Catalan numbers
AMS 2000 Subject Classification: 05A15

1 Introduction

A partition $\pi = B_1/B_2/\cdots/B_m$ of a totally ordered set X is called *noncrossing partition* (n.c.p.) iff there do not exist four elements $a < b < c < d$ of X such that $a, c \in B_i$, $b, d \in B_j$ and $i \neq j$. A *(complete) matching* on a totally ordered set X ($|X| = 2n$) is a partition of X of type $(2, 2, \ldots, 2)$. A matching $\pi = B_1/B_2/\ldots/B_n$ of a totally ordered set X is called *noncrossing matching* (n.c.m.) if and only if it is a n.c.p. of X. If $|X| = 2n$, since there is an obvious order preserving bijection between X and the set $[2n] = \{1, 2, \ldots, 2n\}$, we can equivalently deal with $[2n]$ instead of X. In this case we will use the notation NCM_{2n}.

Many authors have worked on n.c.m. (see for example Lickorish [?], Simion [?] and Chen [?]).

It is well known that the number of matchings on $[2n]$ with no crossings

*The research is supported by NNSF of China (No.10771080) .
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(i. e. $|NCM_{2n}|$) is given by the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

See [?] (Exercise 6.19) for many combinatorial interpretations of Catalan numbers, where item (o) refers to noncrossing matchings.

In this paper we introduce a particular class of n.c.m. More precisely, we say that a n.c.m. $\pi \in NCM(X)$ is a *noncrossing matching with fixed points* the elements of $A \subseteq X$ ($|A| = n$), if and only if every block of π contains exactly one element of A. The set of all these n.c.m. is denoted by $NCM(X, A)$. Again if we deal with $[2n]$ instead of X we use the notation $NCM_{2n}(A)$.

A. Sapounakis and P. Tsikouras have worked in the field of noncrossing partitions with fixed points([?],[?]). Some results of this paper are related to their works, so we must thank them for their creative study.

We will see that the cardinal number $|NCM_{2n}(A)|$ is determined by the relevant positions of the elements of A. For this reason, we also consider the equivalence relation of the translation in the set of all subsets containing n elements of $[2n]$, defined as follows:
$A_1 \simeq A_2$ if and only if there exists $c \in [2n]$ with $A_1 = (A_2 + c)$ $(mod\ 2n)$.

Furthermore, for $A \subseteq [2n]$ ($|A| = n$) we define a finite sequence $\lambda_A = (x_i)$ of length n, where x_i, $i \in [n-1]$ is the number of elements of $[2n]\backslash A$ lying between the ith and the $(i+1)$st element of A and x_m is the number of elements of $[2n]$ that are either smaller or greater than every element of A.

It is easy to prove that $A_1 \simeq A_2$ if and only if each one of $\lambda_{A_1}, \lambda_{A_2}$ is a cyclic permutation of the other.

Now let $|A_1| = |A_2| = n$, $A_1 \simeq A_2$ and $c \in [2n]$; then $A_1 = (A_2 + c)$ $(mod\ 2n)$. It is obvious that the mapping $\tau : NCM_{2n}(A_1) \to NCM_{2n}(A_2)$ with $\tau(\pi) = \{(B + c)\ (mod\ 2n);\ B \in \pi\}$ is a bijection, so that we obtain the following result:

Proposition 1.1 *If $A_1, A_2 \subseteq [2n]$ ($|A_1| = |A_2| = n$) with $A_1 \simeq A_2$ then* $|NCM_{2n}(A_1)| = |NCM_{2n}(A_2)|$.

For every $n \in \mathbb{N}^*$ and for every sequence $\lambda_A = (x_i), i \in [n]$ in $\mathbb{N}$ with $\sum_{i=1}^{n} x_i = n$ there exists at least one set $A \subseteq [2n]$ ($|A| = n$) with $\lambda_A = \lambda$. Indeed, for the set $A = \{t_1, t_2, \ldots, t_n\}$ with $t_1 = 1$ and $t_{i+1} = t_i + x_i + 1$, $i \in [n-1]$ we have that $\lambda_A = \lambda$.

So, we can define a function g_n of n variables as follows:

$$g_n(x_1, x_2, \ldots, x_n) = |NCM_{2n}(A)|$$

where A is any subset containing n elements of $[2n]$, with $\lambda_A = (x_1, x_2, \ldots, x_n)$.

From the previous proposition it is clear that g_n is well defined and that $g_n(x_1, x_2, \ldots, x_n) = g_n(y_1, y_2, \ldots, y_n)$, whenever the sequence $(y_1, y_2, \ldots, y_n)$ or its reverse is a cyclic permutation of $(x_1, x_2, \ldots, x_n)$.

For the evaluation of the formula of the function g_n, which counts $|NCM_{2n}(A)|$, it is more convenient to express the problem in the following equivalent form:

Let $X = [n] \cup Y$, where $|Y| = n$ and the elements of Y are distributed in the intervals $(i, i+1)$, $i \in [n-1]$ and $(n, +\infty)$, so that $|X \cap (i, i+1)| = x_i$, $\forall i \in [n-1]$ and $|X \cap (n, +\infty)| = x_n$. We want to determine the number $g_n(x_1, x_2, \ldots, x_n) = |NCM(X, [n])|$.

In Section 2 we give a recursive formula for g_n, which is used to study the explicit formula of g_n.

In Section 3 we prove that $g_n(x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0)$ where $x_i \neq 0$ for every $i \in [k]$, can be expressed in the form $2^r C_{i_1+1} C_{i_t+1} \prod_{j=2}^{t-1} C_{i_j+2}$.

2 The recursive formula of g_n

Firstly, notice that $g_n(n, 0, \ldots, 0) = 1$. Suppose that $g_n(x_1, x_2, \ldots, x_n) = 0$, *if* $\sum_{i=1}^{n} x_i \neq n$. We give a recursive formula for g_n, $n \geq 2$.

Theorem 2.1 *For every sequence $(x_1, x_2, \ldots, x_n)$ of natural numbers, with $n \geq 2$, $x_1 \geq 1$ and $\sum_{i=1}^{n} x_i = n$ the following relation holds:*

$$g_n(x_1, x_2, \ldots, x_n) = g_{n-1}(x_2, x_3, \ldots, x_n) + g_{n-1}(x_1 + x_2 - 1, x_3, \ldots, x_n) + \sum_{t=1}^{n-2} g_t(x_2, x_3, \ldots, x_{t+1}) g_{n-t-1}(x_1 + x_{t+2} - 1, x_{t+3}, \ldots, x_n).$$

Proof. Let $r = max[(1,2) \cap X]$. We partition the set $NCM(X, [n])$ into the sets T_i, $i \in [n]$, so that each partition in T_i contains r and i in the same block. Obviously,

$$|T_1| = g_{n-1}(x_2, x_3, \ldots, x_n) \text{ and } |T_2| = g_{n-1}(x_1 + x_2 - 1, x_3, \ldots, x_n). \quad (1)$$

For $k \geq 3$, we have that

$$|T_k| = g_{k-2}(x_2, x_3, \ldots, x_{k-1}) g_{n-k+1}(x_1 + x_k - 1, x_{k+1}, \ldots, x_n). \quad (2)$$

Indeed, if $\sum_{i=2}^{k-1} x_i = k-2$, to each $\pi \in T_k$ correspond two uniquely determined matchings π_1, π_2 where π_1 is a n.c.m. of $[2,k) \cap X$ with fixed points $2, 3, \ldots, k-1$ and π_2 is a n.c.m. of $([1,r) \cup (k, +\infty)) \cap X$ with fixed points $1, k+1, \ldots, n$. Thus,

$$|T_k| = g_{k-2}(x_2, x_3, \cdots, x_{k-1}) g_{n-k+1}(x_1 + x_k - 1, x_{k+1}, \cdots, x_n).$$

If $\sum_{i=2}^{k-1} x_i \neq k-2$, then $|T_k| = 0$ and $g_{k-2}(x_2, x_3, \ldots, x_{k-1}) = 0$, so that we obtain again that

$$|T_k| = g_{k-2}(x_2, x_3, \ldots, x_{k-1}) g_{n-k+1}(x_1 + x_k - 1, x_{k+1}, \ldots, x_n).$$

From (1), (2) we obtain the required result. □

Note. Theorem 2.1 (and its proof) for g_n is closely related to Proposition 2.2 (and its proof) for f_n in [?].

Corollary 2.2 *For every $n \in \mathbb{N}^*$, we have that*

$$g_n(1, 1, \ldots, 1) = C_n.$$

Proof. Set $g_0 = 1$ and $g_n = g_n(1, 1, \ldots, 1)$. Obviously, $g_1 = 1$ and from Theorem 2.1 we obtain

$$g_{n+1} = g_n + g_n + \sum_{t=1}^{n-1} g_t g_{n-t} = \sum_{t=0}^{n} g_t g_{n-t}.$$

Hence, g_n satisfies the well-known Segner formula, i.e. $g_n = C_n$. □

Note. It is easy to show that $NCM_{2n} = NCM([2n], [n])$.

Since $|NCM([2n], [n])| = g_n(1, 1, \ldots, 1)$ and $|NCM_{2n}| = C_n$, we obtain a combinatorial proof of Corollary 2.2.

3 The expression of $g_n(x_1, x_2, \ldots, x_k, 0, \ldots, 0)$

Set $g_n^*(x_1, x_2, \ldots, x_k) = g_n(x_1, x_2, \ldots, x_k, 0, \ldots, 0)$, where $x_1 \geq 1, \forall x_i \in [k]$ and $\sum_{i=1}^{k} x_i = n$.

Lemma 3.1 *For every positive integers n, k with $k \leq n$, we have*

1) $g_n^*(x_1, x_2, \ldots, x_k) = g_{n+1-x_1}^*(1, x_2, \ldots, x_k)$.

2) $g_n^*(x_1, x_2, \ldots, x_k) = g_{n+2-x_1-x_k}^*(1, x_2, \ldots, x_{k-1}, 1)$.

Proof. If $x_1 = 1$, both results are obviously true.

1) For $x \geq 2$, it follows that for every $\pi \in NCM(X, [n])$, the number ξ for which $\{1, \xi\} \in \pi$ is the least element of $(1, 2) \cap X$ since, otherwise, $|(1, \xi) \cap [n]| < |(1, \xi) \cap Y|$, so that the elements of these two sets cannot be matched.

If we delete the pair $\{1, \xi\}$ of π, we obtain a n.c.m. $\pi' \in NCM(X \backslash \{\xi\}, [n-1])$; thus we obtain a bijection between $NCM(X, [n])$ and $NCM(X \backslash \{\xi\}, [n-1])$ which gives $g_n^*(x_1, x_2, \ldots, x_k) = g_{n-1}^*(x_1 - 1, x_2, \ldots, x_k)$.

From the above equality it follows recursively that

$$g_n^*(x_1, x_2, \ldots, x_k) = g_{n+1-x_1}^*(1, x_2, \ldots, x_k).$$

2) We apply 1) twice, using also the fact that $(1, x_2, \ldots, x_{k-1}, x_k, 0, \ldots, 0)$ is a cyclic permutation of the reverse of $(x_k, x_{k-1}, \ldots, x_2, 1, 0, \ldots, 0)$; so, we have

$$\begin{aligned} g_n^*(x_1, x_2, \ldots, x_k) &= g_{n+1-x_1}^*(1, x_2, \ldots, x_k) \\ &= g_{n+1-x_1}^*(x_k, x_{k-1}, \ldots, x_2, 1) \\ &= g_{n+2-x_1-x_k}^*(1, x_{k-1}, \ldots, x_2, 1) \\ &= g_{n+2-x_1-x_k}^*(1, x_2, \ldots, x_{k-1}, 1). \end{aligned}$$ □

Corollary 3.2 *For every $n \in \mathbb{N}^*$,*

$$g_{n+2}(2, 1, \ldots, 1, 2, 0, 0) = g_{n+1}(2, 1, \ldots, 1, 1, 0) = g_n(1, 1, \ldots, 1, 1) = C_n.$$

Lemma 3.3 *If $x_i > 0$, $\forall\, i \in [k]$, $\sum_{i=1}^{k} x_i = n$ and $\exists j \in [k]$ such that $x_j > 2$, then $g_n^*(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k)$
$= g_{n-x_j+2}^*(x_1, x_2, \ldots, x_{j-1}, 2, x_{j+1}, \ldots, x_k)$.*

Proof. In view of Lemma 3.1 we can suppose that $j \geq 2$ and $x_1 = 1$. In this case, it follows from Theorem 2.1 that for $x_j \geq 2$

$$\begin{aligned} & g_n^*(1, x_2, \ldots, x_j, \ldots, x_k) \\ &= g_n(1, x_2, \ldots, x_j, \ldots, x_n) \\ &= 2g_{n-1}(x_2, \ldots, x_j, \ldots, x_n) + \sum_{t=1}^{j-2} g_t(x_2, \ldots, x_{t+1}) g_{n-t-1}(x_{t+2}, \ldots, x_n). \quad (3) \end{aligned}$$

By the induction hypothesis, relation (3) gives

$$\begin{aligned} g_n^*(1, x_2, \ldots, x_j, \ldots, x_k) &= 2g_{n-x_j+1}(x_2, \ldots, 2, \ldots, x_{n+2-x_j}) \\ &\quad + \sum_{t=1}^{j-2} g_t(x_2, \ldots, x_{t+1}) g_{n-t+1-x_j}(x_{t+2}, \ldots, x_{n+2-x_j}) \\ &= g_{n-x_j+2}(1, x_2, \ldots, 2, \ldots, x_{n+2-x_j}) \end{aligned}$$

$$= g^*_{n-x_j+2}(1, x_2, \ldots, 2, \ldots, x_k). \qquad \square$$

Now it is enough to deal with $g^*_n(1, x_2, \ldots, x_{k-1}, 1)$, where $x_i = 1$ *or* 2, $\forall\, i \in [k-1]/\{1\}$ and $\sum_{i=1}^{k} x_i = n$.

Lemma 3.4 *If* $x_i = 1$ *or* 2, $\forall\, i \in [k]$, $\sum_{i=1}^{k} x_i = n$ *and* $\exists j \in [k-1]$ *such that* $x_j = x_{j+1} = 2$, *then*

$$g^*_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) = 2g^*_{n-2}(x_1, \ldots, x_{j+1}, \ldots, x_k).$$

Proof. If $j = 1$,

$$\begin{aligned} g^*_n(2, 2, x_3, \ldots, x_k) &= g^*_{n-1}(3, x_3, \ldots, x_k) + g^*_{n-2}(3, x_3, \ldots, x_k) \\ &= g^*_{n-2}(2, x_3, \ldots, x_k) + g^*_{n-2}(2, x_3, \ldots, x_k) \\ &= 2g^*_{n-2}(2, x_3, \ldots, x_k). \end{aligned}$$

If $j > 1$, for every $\pi \in NCM(X, [n])$, the number ξ for which $\{j, \xi\} \in \pi$ is the greater element of $(j-1, j) \cap X$ or the least element of $(j, j+1) \cap X$, we have $g_n(x_1, \ldots, 2, 2, x_{j+2}, \ldots, x_k, 0, \ldots, 0)$
$= 2g_{n-1}(x_1, \ldots, 3, x_{j+2}, \ldots, x_k, 0, \ldots, 0)$.

Then we have:

$$\begin{aligned} & g^*_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k) \\ = \;& g_n(x_1, \ldots, 2, 2, x_{j+2}, \ldots, x_k, 0, \ldots, 0) \\ = \;& 2g_{n-1}(x_1, \ldots, 3, x_{j+2}, \ldots, x_k, 0, \ldots, 0) \\ = \;& 2g_{n-2}(x_1, \ldots, 2, x_{j+2}, \ldots, x_k, 0, \ldots, 0) \\ = \;& 2g^*_{n-2}(x_1, \ldots, x_{j+1}, \ldots, x_k). \qquad \square \end{aligned}$$

Now we deal with the expression of

$$g^*_n(x_{11}, x_{12}, \ldots, x_{1i_1}, 2, x_{21}, x_{22}, \ldots, x_{2i_2}, 2, x_{31}, \ldots, 2, x_{t1}, x_{t2}, \ldots, x_{ti_t})$$

where $x_{uv} = 1$, for every $u \in [t]$, $v \in [i_u]$ and $n = \sum_{u=1}^{t} \sum_{v=1}^{i_u} x_{uv} + 2(t-1)$. We denote it by

$$g^*_n(1^{i_1}, 2, 1^{i_2}, 2, 1^{i_3}, \ldots, 2, 1^{i_t}).$$

Lemma 3.5 *For every* $n \geq 4$ *and* $i_j \geq 1$, $j \in [t]$, *we have that*

$$g^*_n(1^{i_1}, 2, 1^{i_2}, 2, 1^{i_3}, \ldots, 2, 1^{i_t}) = C_{i_1+1} C_{i_t+1} \prod_{j=2}^{t-1} C_{i_j+2}.$$

Proof. For $n = 4$, we have $i_1 = i_2 = 1$ and in this case the result holds since $g_n^*(1,2,1) = 4 = C_2C_2$.

Suppose that the result holds for every $k \geq n-1$. Then, using equation (3) we have

$$\begin{aligned}
&g_n^*(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t})\\
&= 2g_{n-1}^*(1^{i_1-1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) + \sum_{s=1}^{i_1-2} g_s^*(1^s)g_{n-1-s}^*(1^{i_1-1-s},2,\\
&\quad 1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) + g_{i_1-1}^*(1^{i_1-1})g_{n-1-i_1}^*(2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t})\\
&= 2g_{n-1}^*(1^{i_1-1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) + \sum_{s=1}^{i_1-2} g_s^*(1^s)g_{n-1-s}^*(1^{i_1-1-s},2,\\
&\quad 1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) + g_{i_1-1}^*(1^{i_1-1})g_{n-2-i_1}^*(1^{1+i_2},2,1^{i_3},\ldots,2,1^{i_t})\\
&= 2C_{i_1}C_{i_t+1}\prod_{j=2}^{t-1}C_{i_j+2} + \sum_{s=1}^{i_1-2}(C_sC_{i_1-s}C_{i_t+1}\prod_{j=2}^{t-1}C_{i_j+2})\\
&\quad +C_{i_1-1}C_{i_t+1}\prod_{j=2}^{t-1}C_{i_j+2}\\
&= (\sum_{s=0}^{i_1}C_sC_{i_1-s})C_{i_t+1}\prod_{j=2}^{t-1}C_{i_j+2}\\
&= C_{i_1+1}C_{i_t+1}\prod_{j=2}^{t-1}C_{i_j+2}.
\end{aligned}$$

□

Now we can deal with the general case $g_n(x_1,x_2,\ldots,x_k,0,\ldots,0)$, where $x_i > 0$, $\forall\, i \in [k]$ and $\sum_{i=1}^{k} x_i = n$.

Following Lemma 3.1, $g_n(x_1,x_2,\ldots,x_k,0,\ldots,0) = g_n^*(x_1,x_2,\ldots,x_k) = g_{n+2-x_1-x_k}^*(1,x_2,\ldots,x_{k-1},1)$.

We can express $g_{n+2-x_1-x_k}^*(1,x_2,\ldots,x_{k-1},1)$ as

$$g_{n+2-x_1-x_k}^*(x_{11},\ldots,x_{1i_1},y_{11},\ldots,y_{1j_1},x_{21},\ldots,y_{t-1j_{t-1}},x_{t1},\ldots,x_{ti_t})$$

where $x_{uv} = 1$, for every $u \in [t]$, $v \in [i_u]$ and $y_{pq} \geq 2$, for every $p \in [t-1]$, $q \in [j_p]$.

Because of Lemma 3.3, 3.4 we have that:

$$\begin{aligned}
&g_{n+2-x_1-x_k}^*(x_{11},\ldots,x_{1i_1},y_{11},\ldots,y_{1j_1},x_{21},\ldots,y_{t-1j_{t-1}},x_{t1},\ldots,x_{ti_t})\\
=\;& g_{n_1}^*(1^{i_1},2^{j_1},1^{i_2},2^{j_2},1^{i_3},\ldots,2^{j_{t-1}},1^{i_t})\\
=\;& 2^r g_{n_2}^*(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t})
\end{aligned}$$

where $n_1 = n+2-x_1-x_2+\sum_{p=1}^{t-1}\sum_{q=1}^{j_p}(y_{pq}-2)$, $r = \sum_{p=1}^{t-1}(j_p-1)$ and $n_2 = n_1-2r$.

We call $2^r g^*_{n_2}(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t})$ the simplest expression of

$$g_n(x_1,x_2,\ldots,x_k,0,\ldots,0).$$

For example, the simplest expression of $g_{13}(2,3,1,2,3,1,1,0,0,0,0,0,0)$ is $2g^*_8(1,2,1,2,1^2)$.

Theorem 3.6 *If the simplest expression of $g_n(x_1,x_2,\ldots,x_k,0,\ldots,0)$ is $2^r g^*_{n_2}(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t})$, then*

$$g_n(x_1,x_2,\ldots,x_k,0,\ldots,0) = 2^r C_{i_1+1}C_{i_t+1}\prod_{j=2}^{t-1} C_{i_j+2}.$$

Proof. Using Lemma 3.5, we have that $g^*_{n_2}(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) = C_{i_1+1}C_{i_t+1}\prod_{j=2}^{t-1} C_{i_j+2}$ and hence

$$\begin{aligned} g_n(x_1,x_2,\ldots,x_k,0,\ldots,0) &= 2^r g^*_{n_2}(1^{i_1},2,1^{i_2},2,1^{i_3},\ldots,2,1^{i_t}) \\ &= 2^r C_{i_1+1}C_{i_t+1}\prod_{j=2}^{t-1} C_{i_j+2}. \end{aligned}$$ □

Notice that the number r is the number of all pairs (x_i,x_{i+1}), where $x_i,x_{i+1} \geq 2$. We have completed the discussion for the expression of $g_n(x_1,x_2,\ldots,x_k,0,\ldots,0)$.

So, for example

$$\begin{aligned} g_{13}(2,3,1,2,3,1,1,0,0,0,0,0,0) &= g^*_{13}(2,3,1,2,3,1,1) \\ &= g^*_{11}(2,2,1,2,2,1,1) \\ &= g^*_{10}(1,2,1,2,2,1,1) \\ &= 2g^*_8(1,2,1,2,1^2) \\ &= 2C_2C_3C_3 \\ &= 100. \end{aligned}$$

Notice that we can easily obtain Corollary 3.2 from Theorem 3.6 too.

We give some remarks based on Theorem 3.6.

Remarks

1. For every $k, n \in \mathbb{N}^*$, if $k \leq \frac{n}{2}$, $x_i > 1\ \forall\ i \in [k]$ and $\sum_{i=1}^{k} x_i = n$ then

$$g_n(x_1, x_2, \ldots, x_k, 0, \ldots, 0) = 2^{k-1}.$$

2. If $x_i > 1,\ \forall\ i \in [k]$, $x_{k+1} = x_{k+2} = \cdots = x_{k+t} = 1$ and $\sum_{i=1}^{k+t} x_i = n$ then

$$\begin{aligned} g_n(x_1, x_2, \ldots, x_n) &= g_n(x_1, x_2, \ldots, x_k, 1, 1, \ldots, 1, 0, \ldots, 0) \\ &= 2^{k-1} \frac{1}{t+2} \binom{2t+2}{t+1}. \end{aligned}$$

3. If $x_i > 1,\ \forall\ i \in [k] \cup \{k+t+1, k+t+2, \ldots, k+t+j\}$,

$x_{k+1} = x_{k+2} = \cdots = x_{k+t} = 1$ and $\sum_{i=1}^{k+t+j} x_i = n$ then

$$\begin{aligned} & g_n(x_1, x_2, \ldots, x_n) \\ = \ & g_n(x_1, x_2, \ldots, x_k, 1, 1, \ldots, 1, x_{k+t+1}, x_{k+t+2}, \ldots, x_{k+t+j}, 0, \ldots, 0) \\ = \ & 2^{k-1} 2^{j-1} C_{t+2} \\ = \ & 2^{k+j-2} \frac{1}{t+3} \binom{2(t+2)}{t+2}. \end{aligned}$$

■

The general case for the formula of g_n where one can have zeros between nonzero variables can be dealt with, by using Theorem 2.1, but the expression of g_n may be complicated without a new method. We propose this as an open problem.

Acknowledgments

The authors thank Professor Sapounakis for his useful comments and suggestions.

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