Applications of strongly transitive geometric spaces to n-ary hypergroups

S. Mirvakili and B. Davvaz
Department of Mathematics, Yazd University,
Yazd, Iran
davvaz@yazd.ac.ir

Abstract

n-ary hypergroups are a generalization of Dörnte n-ary groups and a generalization of hypergroups in the sense of Marty. In this paper, we investigate some properties of n-ary hypergroups and (commutative) fundamantal relations. We determine two families $P(H)$ and $P_\sigma(H)$ of subsets of an n-ary hypergroup H such that two geometric spaces $(H, P(H))$ and $(H, P_\sigma(H))$ are strongly transitive. We prove that in every n-ary hypergroup the fundamental relation β and the commutative fundamental relation γ are strongly compatible equivalence relations.

Keywords: n-ary hyperoperation, n-ary semihypergroup, n-ary hypergroup, Strongly compatible relation, Fundamental relation, Commutative fundamental relation, Geometric spaces.

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1 Introduction

In 1934, at the 8th congress of Scandinavian Mathematicians, Marty [20] has introduced, for the first time, the notion of hypergroups, using in different context: algebraic function, rational function, non-commutative group. This moment was the first step in the history of the developments of the algebraic hyperstructure theory all over the world, especially in Europe (France, Italy, Greece, Romania), Australia, and later also in Iran, China, Japan, Korea.

This theory has been studied in the following decades and nowadays by many mathematicians, for example see [2, 5, 6, 8]. A recent book [4] contains a wealth of applications. The fundamental relation β^* was introduced on hypergroups by Koskas [17] and was studied mainly by Corsini [3], Freni [13] and Vougiouklis [23]. Also, the commutative fundamental equivalence relation γ^* was studied on hypergroups by Freni [11, 12], Davvaz and Karimian [6, 8, 15].

The notion of an n-ary group was studied by Dörnte in 1928, which is a natural generalization of the notion of a group. Since then many papers concerning various n-ary algebra have appeared in the literature, for example see [1, 16, 21, 22]. The notion of n-ary hypergroups are defined and considered by Davvaz and Vougiouklis in [7], Leoreanu-Fotea and Davvaz [19, 18] which are a generalization of hypergroups in the sense of Marty and a generalization of n-ary groups too.

Strongly compatible equivalence relations play in n-ary semihypergroup theory a role analogous to congruences in n-ary semigroup theory. Indeed, it is known (see [7]) that if ρ is a strongly compatible equivalence relation on an n-ary semihypergroup (H, f), then we can define an n-ary operation f/ρ on the quotient set H/ρ such that $(H/\rho, f/\rho)$ is an n-ary semigroup. If ρ is a relation on a set H, we denote ρ^* as the transitive closure of ρ. Davvaz and Vougiouklis [7] and Leoreanu and Davvaz [19] were introduced the relation β on an n-ary semihypergroup H such that β^* is the smallest strongly compatible equivalence relation such that the quotient $(H/\beta^*, f/\beta^*)$ is a fundamental n-ary semigroup. Mirvakili et al. [9] defined the relation γ on an n-ary semihypergroup and proved that γ^* is the smallest strongly compatible equivalence relation such that the quotient $(H/\gamma^*, f/\gamma^*)$ is a commutative fundamental n-ary semigroup. In this paper, we investigate some properties of n-ary hypergroups. Analogously to the work of Freni [12], we determine two geometric spaces $(H, P(H))$ and $(H, P_\sigma(H))$ on n-ary semihypergroups. We prove that in every n-ary hypergroup two geometric spaces $(H, P(H))$ and $(H, P_\sigma(H))$ are strongly transitive and so $\beta = \beta^*$ and $\gamma = \gamma^*$. Also, several examples are presentd. We study the fundamental relation on n-ary semihypergroups(hypergroups) derived(b-derived) from semihypergroups(hypergroups).

2 Basic definitions and results

Let H be a non-empty set and f be a mapping $f : H \times H \longrightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. Then, f is called a *binary (algebraic) hyperoperation* on H. In general, a mapping $f : H \times \ldots \times H \longrightarrow \mathcal{P}^*(H)$, where H appears n times, is called an *n-ary (algebraic) hyperoperation* and n is called the *arity* of this hyperoperation.

An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid* or an *n-ary hypersystem.* Since we identify the set $\{x\}$ with the element x, then any n-ary (binary) groupoid is an n-ary (binary) hypergroupoid.

As it is well-known we say that a binary ($n = 2$) hyperoperation f defined on H is:

- *associative*, if
$$f(f(x,y),z) = f(x, f(y,z)),$$
- *weak associative*, if
$$f(f(x,y),z) \cap f(x, f(y,z)) \neq \emptyset$$

for all $x, y, z \in H$.

A binary hypergroupoid with the (weak) associative hyperoperation is called a *hypersemigroup* (*H_v-semigroup*, respectively).

A hypergroupoid (H, f) satisfying the *reproducibility axiom:*

$$f(a, H) = f(H, a) = H \qquad \text{for all } a \in H,$$

is called a *hyperquasigroup.* A hyperquasigroup which is a semihypergroup (H_v-semigroup) is called a *hypergroup* (H_v-group).

We use the following abbreviated notation: the sequence $x_i, x_{i+1}, \cdots, x_j$ is denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. In this convention

$$f(x_1, \cdots, x_i, y_{i+1}, \cdots, y_j, z_{j+1}, \ldots, z_n)$$

is written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \ldots = y_j = y$ the last expression will be write in the form $f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^n)$.

Similarly, for non-empty subsets $A_1, \ldots, A_n$ of H, we define

$$f(A_1^n) = f(A_1, \ldots, A_n) = \bigcup \{f(x_1^n) \mid x_i \in A_i,\ i = 1, \ldots n\}.$$

An n-ary hyperoperation f is called *weakly (i,j)-associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) \cap f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \neq \emptyset,$$

and *(i,j)-associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

holds for fixed $1 \leq i < j \leq n$ and all $x_1, x_2, \ldots, x_{2n-1} \in H$. If the above condition is satisfied for all $i, j \in \{1, 2, \ldots, n\}$, then we say that f is *weakly associative* (*associative*, respectively). An n-ary hypergroupoid with the (weakly) associative operation is called an *n-ary semihypergroup*

(H_v-*semigroup*, respectively). An n-ary semihypergroup (H, f) is *weakly* i-*cancellative*, if there exist elements $a_2, \ldots, a_n \in H$ such that

$$f(a_2^i, x, a_{i+1}^n) = f(a_2^i, y, a_{i+1}^n) \quad \text{implies} \quad x = y$$

for all $x, y \in H$. If this implication is valid for all $i = 1, 2, \ldots, n$, then we say that (H, f) is *weakly cancellative* and elements $a_2, \ldots, a_n$ are called *cancellable*. An n-ary hypergroupoid in which this implication holds for all $a_2, \ldots, a_n \in H$ is called i-*cancellative*. An n-ary hypergroupoid i-cancellative for every $i = 1, 2, \ldots, n$ is called *cancellative*.

An n-ary hypergroupoid (H, f) is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $(a_1^n) \in H^n$ we have

$$f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

If $a_1^n \in H^n$ we denote $a_{\sigma(1)}^{\sigma(n)}$ as the $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. An n-ary hypergroupoid (H, f) is said weak commutative, we write COW, if for every $a_1^n \in H$

$$\bigcap_{\sigma \in S_n} f(a_{\sigma(1)}^{\sigma(n)}) \neq \emptyset.$$

We say that an n-ary hypergroupoid (H, f) has a *right (left) neutral polyad [scalar right (left) neutral polyad]* $e_2^n \in H$ if for all $x \in H$

$$x \in f(x, e_2^n) \quad (x \in f(e_2^n, x)), \ [x = f(x, e_2^n) \quad (x = f(e_2^n, x))].$$

A polyad $e_i \in H$, $1 \leq i \leq n, j \notin \{1, i, n\}$ called j-*neutral polyad (scalar* j-*neutral polyad)* if for every $x \in H$ we have

$$x \in f(e_1^{j-1}, x, e_{j+1}^n), \ (x = f(e_1^{j-1}, x, e_{j+1}^n)).$$

A right neutral polyad called a 1-neutral polyad and a left neutral polyad called an n-neutral polyad.

If e_2^n is j-neutral polyad for every $j \in \{1, 2, \ldots, n\}$ we say that e_2^n is *neutral polyad*, i.e. for every $x \in H$ we have

$$x \in f(x, e_2^n), \ x \in f(e_2, x, e_3^n), \ldots, \ x \in f(e_2^{n-1}, x, e_n), \ x \in f(e_2^n, x).$$

If e_2^n is scalar j-neutral polyad for every $j \in \{1, 2, \ldots, n\}$ we say that e_2^n is *scalar neutral polyad*, i.e., for every $x \in H$ we have

$$x = f(x, e_2^n), \ x = f(e_2, x, e_3^n), \ldots, \ x = f(e_2^{n-1}, x, e_n), \ x = f(e_2^n, x).$$

An n-ary hypergroupoid (H, f) has a *right (left) neutral element [scalar right (left) neutral element]* if there exists $e \in H$, such that for all $x \in H$

$$x \in f(x, \overset{(n-1)}{e}) \quad (x \in f(\overset{(n-1)}{e}, x)), \ [x = f(x, \overset{(n-1)}{e}) \quad (x = f(\overset{(n-1)}{e}, x))].$$

An element $e \in H$ called *j-neutral element (scalar j-neutral element)* if $x \in f(\overset{(j-1)}{e}, x, \overset{(n-j)}{e})(x = f(\overset{(j-1)}{e}, x, \overset{(n-j)}{e}))$, for every $x \in H$. If $e \in H$ there exist such that e is a i-neutral(scalar i-neutral) for every $i = 1, 2, \ldots, n$ then e is called a *neutral element(scalar neutral element)*. It is clear that polyad $e_2 = \ldots = e_n = e$, where e is (scalar)neutral element, is (scalar)neutral.

EXAMPLE 1. An n-ary hypergroupoid (H, f) with the operation $f(x_1^n) = x_1$ is a simple example of an n-ary semihypergroup in which each element is a scalar right neutral element. In this semihypergroup no left neutral elements.

EXAMPLE 2. Give an example of hypergroupoids with two or more neutral elements. Suggestion $\mathbb{Z}_{n-1}$ with an n-ary hyperoperation $f(x_1^n) = \{x_1, (x_1 + \ldots x_n)(mod n)\}$ is an example of an n-ary hypergroupoid in which each element is neutral and has only one scalar neutral element.

If for all $x_1^n \in H$ the set $f(x_1^n)$ is singleton, then f is called an n-ary operation and (H, f) is called an n-ary semigroup. If $m = k(n-1)+1$, then m-ary hyperoperation h given by

$$h(x_1^{k(n-1)+1}) = \underbrace{f(f(\cdots(f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \cdots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of m does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$ to mean $f_{(k)}$ for some $k = 1, 2, \cdots$. If $k = 0$, then $m = 1$ and we denote $f_{(0)}(z_1^m) = z_1$.

An n-ary semihypergroup (H, f) in which the equation

$$b \in f(a_1^{i-1} x_i, a_{i+1}^n) \qquad (*)$$

has the solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an n-ary hypergroup. This condition can be formulated $f(a_1^{i-1}, H, a_{i+1}^n) = H$. When (H, f) is an n-ary semigroup, (H, f) is called an n-ary hypergroup, and besides if f is an n-ary operation, then the equation $(*)$ is as follows:

$$b = f(a_1^{i-1}, x_i, a_{i+1}^n),$$

and in this case (H, f) is an n-ary group.

Let (H, f) be an n-ary hypergroup and B be a non-empty subset of H. Then, B is an n-ary subhypergroup of H if the following conditions hold:

i) B is closed under the n-ary hyperoperation f, i.e, for every $(x_1^n) \in B^n$ implies that $f(x_1^n) \subset B$.

ii) Equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has the solution $x_i \in B$ for every $i \in \{1, \ldots, n\}$ and $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, b \in B$.

Let (H, f) and (G, g) be two n-ary hypergroups. A *homomorphism* from H to G is a mapping $\phi : H \longrightarrow G$ such that $\phi(f(a_1^n)) = g(\phi(a_1), \ldots, \phi(a_n))$ holds for all $a_1, \ldots, a_n \in H$.

If ϕ is injective, then is called *embedding*. The map ϕ is an *isomorphism* if ϕ is injective and onto. We say that H is *isomorphic* to G, denote $H \cong G$, if there is an isomorphism from H to G. Let $\phi : H \longrightarrow G$ be a homomorphism, K be an n-ary subhypergroup of H and L be an n-ary subhypergroup of G. Then, $\phi(K)$ is an n-ary subhypergroup of G. If $\phi^{-1}(L)$ is non-empty, then $\phi^{-1}(L)$ is an n-ary subhypergroup of H([7]). Let $\phi : H \longrightarrow G$ be a homomorphism. Then, the kernel ϕ, written $\ker \phi$, is defined by

$$\ker \phi = \{(a, b) \in H^2 \mid \phi(a) = \phi(b)\}.$$

The next theorem proved by Ušan [22] in 1999 for n-ary groups. We prove that with weak conditions for n-ary hypergroups.

Theorem 2.1. *Let $n \geq 3$ and let (H, f) be an n-ary hypergroupoid. Then, the following statements are equivalent:*

(1) (H, f) is an n-ary hypergroup.

(2) For arbitrary $i \in \{2, \ldots, n-1\}$ the following conditions hold:

  (i) (H, f) is weak i-cancellative;

  (ii) the $(i-1, i)$-associative law hold in (H, f);

  (iii) the $(i, i+1)$-associative law hold in (H, f);

  (iv) for every $a_1^n \in H$ there is $x \in H$ such that the following equality holds

$$a_n \in f(a_1^{i-1}, x, a_i^{n-1}).$$

Proof. (1) $\Rightarrow$ (2) it is straightforward.

(2) $\Rightarrow$ (1) Let (2) holds. We prove that the following propositions hold:

(a) (H, f) is an n-ary semihypergroup.

(b) For every $a_1^n \in H$ there is $z \in H$ such that for every $j \in \{1, \ldots, n\}$, the following equality hold:

$$a_n \in f(a_1^{j-1}, z, x, a_j^{n-1}).$$

Proof of (a) :

Let $i \in \{2, \ldots, n-1\}$ and $k \in \mathbb{N}$, satisfying

$$i \leq k < n-1. \tag{1}$$

In addition, suppose that $(k,k+1)$-associative law holds in (H,f) [for $k=i$ it holds by (iii)]. We prove that (H,f) is $(k+1,k+2)$-associative. Let $b_1^{n-1}\in H$ be weak cancellable elements, by (i) and (iii), we conclude for every $a_1^{2n-1}\in H$ the following sequence of implications holds:

$$f(a_2^k, f(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}, b_i) = f(a_2^{k+1}, f(a_{k+2}^{k+n+1}), a_{k+n+2}^{2n-1}, b_i) \Rightarrow$$

$$f(b_1^{i-1}, a_1, f(a_2^k, f(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}, b_i), b_{i+1}^{n-1}) =$$
$$f(b_1^{i-1}, a_1, f(a_2^{k+1}, f(a_{k+2}^{k+n+1}), a_{k+n+2}^{2n-1}, b_i), b_{i+1}^{n-1}) \Rightarrow$$

$$f(b_1^{i-1}, f(a_1^k, f(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}), b_i^{n-1}) =$$
$$f(b_1^{i-1}, f(a_1^{k+1}, f(a_{k+2}^{k+n+1}), a_{k+n+2}^{2n-1}), b_i^{n-1}) \Rightarrow$$

$$f(a_1^k, f(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}) = f(a_1^{k+1}, f(a_{k+2}^{k+n+1}), a_{k+n+2}^{2n-1}).$$

Whence we conclude that: if the $(k,k+1)$-associative law holds in (H,f) and $k\in\mathbb{N}$ satisfies (1), then also $(k+1,k+2)$-associative law holds in (H,f).
Now, let $i\in\{2,\ldots,n-1\}$ and $j\in\mathbb{N}$, satisfying

$$2<j\le i. \qquad (2)$$

Suppose (H,f) is $(j-1,j)$-associative [for $j=i$ it holds by (ii)]. Let $b_1^{n-1}\in H$ be weak cancellable elements, for every $a_1^{2n-1}\in H$, by (i) and (ii) we have

$$f(b_{i-1}, x_1^{j-3}, f(a_{j-2}^{n+j-1}), a_{n+j}^{2n-2}) = f(b_{i-1}, a_1^{j-2}, f(a_{j-1}^{n+j-2}), a_{n+j-1}^{2n-2}) \Rightarrow$$

$$f(b_1^{i-2}, f(b_{i-1}, a_1^{j-3}, f(a_{j-2}^{n+j-1}), a_{n+j}^{2n-2}), a_{2n-1}, b_i^{n-1}) =$$
$$f(b_1^{i-2}, f(b_{i-1}, a_1^{j-2}, f(a_{j-1}^{n+j-2}), a_{n+j-1}^{2n-2}), a_{2n-1}, b_i^{n-1}) \Rightarrow$$

$$f(b_1^{i-1}, f(a_1^{j-3}, f(a_{j-2}^{n+j-1}), a_{n+j}^{2n-1}), b_i^{n-1}) =$$
$$f(b_1^{i-1}, f(a_1^{j-2}, f(a_{j-1}^{n+j-2}), a_{n+j-1}^{2n-1}), b_i^{n-1}) \Rightarrow$$

$$f(a_1^{j-3}, f(a_{j-2}^{n+j-1}), a_{n+j}^{2n-1}) = f(a_1^{j-2}, f(a_{j-1}^{n+j-2}), a_{n+j-1}^{2n-1}).$$

Therefore, $(j-2,j-1)$-associative law holds in (H,f) for every $j\in\mathbb{N}$ satisfies (2). Therefore, (H,f) is an n-ary semihypergroup.
Proof of (b):
Now, Let $i\in\{2,\ldots,n-1\}$ and for every $a_1^n\in H$ there is $x\in H$ such that the following equality holds

$$a_n\in f(a_1^{i-1}, x, a_i^{n-1}).$$

If $j < i$, then for every a_1^n and b in H there exists $y \in H$ such that

$$b \in f(a_1^{i-1}, y, f(\overset{(n-(i-j+1))}{a_1}, a_{j+1}^{i+1}), a_{i+2}^n).$$

Since (H, f) is an n-ary semihypergroup thus we obtain

$$b \in f(a_1^{j-1}, f(a_j^{i-1}, y, \overset{(n-(i-j+1))}{a_1}), a_{j+1}^n).$$

Therefore, there exists $x \in f(a_j^{i-1}, y, \overset{(n-(i-j+1))}{a_1})$ such that

$$b \in f(a_1^{j-1}, x, a_{j+1}^n).$$

If $j > i$, then in the similar way, for every a_1^n and b in H there exists $y \in H$ such that the

$$b \in f(a_1^{j-1}, y, a_{j+1}^n).$$

So, (H, f) is an n-ary hypergroup. □

Corollary 2.2. *Let $n \geq 3$ and $i \in \{2, \ldots, n-1\}$, and let (H, f) be an n-ary hypergroupoid such that $(i, i+1)$-associative law holds in (H, f). If (H, f) is i-cancellative and $i+1$-cancellative then (H, f) is an n-ary semihypergroup.*

Proof. Since (H, f) is i-cancellative and $(i, i+1)$-associative thus the proof of (a) of previous Theorem shows that for every integer $i \leq k < n-1$, $(k, k+1)$-associative law holds in (H, f). Also, since (H, f) is $i+1$-cancellative and $(i, i+1)$-associative thus by similar way (H, f) is $(j-1, j)$-associative for every integer $2 < j \leq i+1$. Therefore, (H, f) is an n-ary semihypergroup and the proof of part (b) of previous Theorem shows that (H, f) is an n-ary quasihypergroup. So (H, f) is an n-ary hypergroup. □

Corollary 2.3. *Let $n \geq 3$ and $i \in \{2, \ldots, n-1\}$, and (H, f) be a weakly cancellative (cancellative) n-ary hypergroupoid.*
If (H, f) is $(i, i+1)$-associative, then (H, f) is an n-ary semihypergroup.

Theorem 2.4. *If e_1 and e_2 are scalar neutral elements of ternary (3-ary) hypergroup (H, f), then $(\{e_1, e_2\}, f)$ is a ternary subgroup of (H, f).*

Proof. Indeed, by the assumption

$$e_j = f(e_i, e_i, e_j) = f(e_i, e_i, e_j) = f(e_i, e_i, e_j)$$

for all $i, j = 1, 2$. Moreover, if $e_i, e_j, e_k \in \{e_1, e_2\}$, then the following equations

$$e_k = f(x, e_i, e_j) = f(e_i, y, e_j) = f(e_i, e_j, z)$$

are solvable where $x = y = z = f(e_j, e_k, e_i) \in \{e_1, e_2\}$. Since $f(e_j, e_k, e_i)$ is singleton for every $i, j, k \in \{1, 2\}$, hence $(\{e_1, e_2\}, f)$ is a ternary subgroup of (H, f). □

3 Fundamental relation and commutative fundamental relation

Let ρ be an equivalence relation on the n-ary semihypergroup (H, f). We denote by $\overline{\overline{\rho}}$ the relation defined on $\mathcal{P}^*(H)$ as follows. If $A, B \in \mathcal{P}^*(H)$, then

$$A \overline{\overline{\rho}} B \Longleftrightarrow a \rho b \text{ for all } a \in A, b \in B.$$

It follows immediately that $\overline{\overline{\rho}}$ is symmetric and transitive. In general, $\overline{\overline{\rho}}$ is not reflexive. Indeed, let us take, for example, the equality relation on A, denoted here by δ_A. The relation $\overline{\overline{\delta_A}}$ is reflexive if and only if $|A| = 1$.

Definition 3.1. Let (H, f) be an n-ary semihypergroup and ρ an equivalence relation on H. Then, ρ is a *strongly compatible relation* if

$$a_i \rho b_i \quad \text{for all } 1 \leq i \leq n \text{ then, } f(a_1, \ldots, a_n) \overline{\overline{\rho}} f(b_1, \ldots, b_n).$$

Theorem 3.2. *Let (H, f) be an n-ary semihypergroup and let ρ be an equivalence relation on H. The following conditions are equivalent.*

(1) The relation ρ is strongly compatible.

(2) If $x_1^n, a, b \in H$ and $a \rho b$, then for every $i \in \{1, \ldots, n\}$ we have

$$f(x_1^{i-1}, a, x_{i+1}^n) \overline{\overline{\rho}} f(x_1^{i-1}, b, x_{i+1}^n).$$

(3) The quotient $(H/\rho, f/\rho)$ is an n-ary semigroup.

Proof. We show that (3) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2).
(1) $\Rightarrow$ (2) It is straightforward.
(2) $\Rightarrow$ (1) Let $a_i \rho b_i$ where $i = 1, \ldots, n$. By (2) we have

$$\begin{aligned} f(a_1, \ldots, a_n) \; & \overline{\overline{\rho}} \; f(a_1, \ldots, a_{n-1}, b_n) \\ & \overline{\overline{\rho}} \; f(a_1, \ldots, a_{n-2}, b_{n-1}, b_n) \\ & \vdots \\ & \overline{\overline{\rho}} \; f(a_1, b_2, \ldots, b_n) \\ & \overline{\overline{\rho}} \; f(b_1, \ldots, b_n). \end{aligned}$$

Since $\overline{\overline{\rho}}$ is transitive thus $f(a_1, \ldots, a_n) \overline{\overline{\rho}} f(b_1, \ldots, b_n)$. Therefore, ρ is strongly compatible.
(1) $\Rightarrow$ (3) Davvaz and Vougiouklis in [7] show that (1) $\Rightarrow$ (3).
(3) $\Rightarrow$ (1) Now, let $(H/\rho, f/\rho)$ be an n-ary semigroup. Suppose that $a_i \rho b_i$, where $i = 1, \ldots, n$. Since $(H/\rho, f/\rho)$ is an n-ary semigroup,

$$\begin{aligned} f/\rho(\rho(a_1), \ldots, \rho(a_n)) &= \{\rho(y) \mid y \in f(a_1, \ldots, a_n)\}, \\ f/\rho(\rho(b_1), \ldots, \rho(b_n)) &= \{\rho(z) \mid z \in f(b_1, \ldots, a_n)\}, \end{aligned}$$

are singleton. Thus, for every $y \in f(a_1, \ldots, a_n)$ and $z \in f(b_1, \ldots, b_n)$ we have $f/\rho(\rho(a_1), \ldots, \rho(a_n)) = \rho(y)$ and $f/\rho(\rho(b_1), \ldots, \rho(b_n)) = \rho(z)$. But $\rho(a_i) = \rho(b_i)$ and so we obtain $\rho(y) = \rho(z)$ for every $y \in f(a_1, \ldots, a_n)$ and $z \in f(b_1, \ldots, b_n)$. Therefore, $f(a_1, \ldots, a_n)\ \overline{\overline{\rho}}\ f(b_1, \ldots, b_n)$.

□

By Theorem 3.2, if ρ is a strongly compatible relation on an n-ary semihypergroup (H, f). Then, the quotient $(H/\rho, f/\rho)$ is an n-ary semigroup such that

$$f/\rho(\rho(a_1), \ldots, \rho(a_n)) = \rho(x) \text{ for all } x \in f(a_1, \ldots, a_n),$$

where $a_1, \ldots a_n \in H$. If (H, f) is an n-ary hypergroup and ρ is a strongly compatible relation. Then, the quotient $(H/\rho, f/\rho)$ is an n-ary group.

EXAMPLE 3. Let $H = \{a, b, c\}$ be a set with a 3-ary hyperoperation f as follows:

$f(x, x, x) = x$ for all $x \in H$,

$f(x_1, x_2, x_3) = \{a, c\}$, if exactly two elements of x_1, x_2, x_3 are equal with b,

$f(x_1, x_2, x_3) = x_i$, if exactly two elements of x_1, x_2, x_3 are equal and are not equal with x_i and b,

$f(x_1, x_2, x_3) = b$, if (x_1, x_2, x_3) is a permutation of (a, b, c).

It is easy to see that (H, f) is a 3-ary semihypergroup and

$$\rho = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$$

is a strongly compatible relation. We have $H/\rho = \{\rho(a), \rho(b)\}$ and

$f/\rho(\rho(x_1),\ \rho(x_2),\ \rho(x_3)) = \rho(x_1)$ if $\rho(x_1) = \rho(x_2) = \rho(x_3)$,

$f/\rho(\rho(x_1),\ \rho(x_2),\ \rho(x_3)) = \rho(x_i)$ if exactly two elements of $\rho(x_1)$, $\rho(x_2)$, $\rho(x_3)$ are equal and are not equal with $\rho(x_i)$.

Then, $(H/\rho, f/\rho)$ is a 3-ary semigroup.

Davvaz and Vougiouklis in [7] show that the relation β^* on an n-ary hypergroup (H, f) is the transitive closure of the relation $\beta = \bigcup\limits_{k \geq 1} \beta_k$, where β_1 is the diagonal relation and, for every integer $k > 1, \beta_k$ is the relation defined as follows:

$$x\ \beta_k\ y \quad \Leftrightarrow \quad \exists z_1^m \in H : \{x, y\} \in f_{(k)}(z_1^n), \text{ where } m = k(n-1)+1.$$

It is known that β^* is the smallest strongly compatible equivalence relation on an n-ary semihypergroup (H, f). Also, Leoreanu-Fotea and Davvaz [19] show that the relation β is transitive. Moreover, the canonical projection $\phi : H \to H/\beta^*$ is a homomorphism.

The relation β^* on an n-ary semihypergroup (hypergroup) is called *fundamental relation* and $(H/\beta^*, f/\beta^*)$ is called *fundamental n-ary semigroup (group)*. If (H, f) is an n-ary semihypergroup, Mirvakili and Davvaz [9] prove that the equivalence relation $\widehat{\gamma}$ denotes the transitive closure of the relation $\gamma = \bigcup_{k\geq 1} \gamma_k$, where γ_1 is the diagonal relation, i.e.,

$$\gamma_1 = \{(x, x) | \; x \in H\}$$

and for every integer $k > 1$, γ_k is the relation defined as follows:

$$x\gamma_k y \text{ if and only if } x \in u_{(k)} \textit{ and } y \in u^{\sigma}_{(k)}.$$

If $m = k(n-1)+1$ there exist $z_1^m \in H^m$ and there exists $\sigma \in \mathbb{S}_m$ such that $u_{(k)} = f_{(k)}(z_1^m)$ and $u^{\sigma}_{(k)} = f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$. $x\gamma_1 y$ (i.e., $x = y$) then we write $x \in u_{(0)}$ and $y \in u^{\sigma}_{(0)} = u_{(0)}$.

Let (H, f) is an n-ary semihypergroup, we define γ^* as the smallest equivalence relation such that the quotient $(H/\gamma^*, f/\gamma^*)$ is a commutative n-ary semigroup, where H/γ^* is the set of all equivalence classes. The equivalence relation γ^* is called *commutative fundamental relation* and $(H/\gamma^*, f/\gamma^*)$ is called *commutative fundamental n-ary semigroup*. If (H, f) be an n-ary semihypergroup, then the canonical projection $\varphi : H \to H/\gamma^*$ with $\varphi(x) = \gamma^*(x)$ is a homomorphism.

The relation γ (resp. γ^*) was introduced on hypergroups (2-ary hypergroups) by Freni [11, 12] and was studied mainly by Davvaz and Karimian [6, 8, 15].

Theorem 3.3. *[9] Let (H, f) be an n-ary hypergroup. Then, we have*

1) *The fundamental relation γ^* is the transitive closure of the relation γ, i.e., $(\gamma^* = \widehat{\gamma})$.*

2) *Relation γ is a strongly compatible relation on (H, f).*

Lemma 3.4. *If (H, f) is an n-ary hypergroup, then for every $k \in \mathbb{N} \cup \{0\}$ we have*

(1) $\beta_k \subset \beta_{k+1}$,

(2) $\gamma_k \subset \gamma_{k+1}$.

Proof. We prove that (2). The proof of (1) is similar. Let $x \gamma_k y$, thus $z_1^m \in H$ and $\sigma \in \mathbb{S}_m$ exist, where $m = k(n-1)+1$, such that $x \in f_{(k)}(z_1^m)$ and $y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$. Since (H, f) is an n-ary hypergroup and $z_m \in H$ thus there exist $x_1^n \in H$ such that $z_m \in f(x_1^n)$. If $\sigma(j) = m$, set $m' = (k+1)(n-1)+1$ and give elements ${z'}_1^{m'} \in H$ and a permutation $\tau \in \mathbb{S}_{m'}$ as follow:

$$\begin{cases} z'_i = z_i, & \text{if } i \in \{1, \ldots, m-1\} \\ \\ z'_{m+i-1} = x_i, & \text{if } i \in \{1, \ldots, n\} \end{cases}$$

and

$$\begin{cases} \tau(i) = \sigma(i), & \text{if } i \in \{1, \ldots, j-1\} \\ \tau(i) = \sigma(j) + i - j, & \text{if } i \in \{j, \ldots, j+n-l\} \\ \tau(i) = \sigma(i-n+1), & \text{if } i \in j+n, m'. \end{cases}$$

Therefore, $x \in f_{(k+1)}({z'}_1^{m'})$ and $y \in f_{(k+1)}({z'}_{\tau(1)}^{\tau(m')})$. Hence, $x \gamma_{k+1} y$ and so we obtain $\gamma_k \subset \gamma_{k+1}$. □

EXAMPLE 4. This example shows that Lemma 3.4 for n-ary semihypergroups is not true. Let $H = \{1, \ldots, 6\}$ and 2-ary hyperoperation $\circ$ on H defined as follows:

$\circ$	1	2	3	4	5	6
1	{1,2,3}	{1,2}	{1,3}	{1,2}	{1,3}	{1,3}
2	{1,2}	{1,2,3}	{2,3}	{2,3}	{1,2}	{1,2}
3	{1,3}	{2,3}	{1,2,3}	{1,3}	{2,3}	{2,3}
4	{1,2}	{2,3}	{1,3}	{1,2,3}	{1,2,3}	{1,2,3}
5	{1,3}	{1,2}	{2,3}	{1,2,3}	{1,2,3}	{1,2,3}
6	{1,3}	{1,2}	{2,3}	{1,2,3}	{1,2,3}	{4,5}

For every $x, y, z \in H$ we have $x \circ (y \circ z) = (x \circ y) \circ z = \{1, 2, 3\}$. Thus, $(H, \circ)$ is a 2-ary semihypergroup. It easy to see that $\gamma = \beta$ and $4 \gamma_2 5$ but $4 \not\gamma_3 5$.

Let (A, f) and (B, g) be two n-ary semihypergroups. We define $(f, g) : (A \times B)^n \longrightarrow \mathcal{P}^*(A \times B)$ by

$$(f,g)((a_1, b_1), \ldots, (a_n, b_n)) = \{(a, b) \mid a \in f(a_1, \ldots, a_n), b \in g(b_1, \ldots, b_n)\}.$$

Clearly $(A \times B, (f, g))$ is an n-ary semihypergroup and we call this n-ary semihypergroup the *direct hyperproduct* of A and B.

Theorem 3.5. *Let (A, f) and (B, g) be two n-ary hypergroups and let γ_A^*, γ_B^* and $\gamma_{A\times B}^*$ be commutative fundamental equivalence relations on A, B and $A \times B$ respectively. Then,*

$$A \times B/\gamma_{A\times B}^* \cong A/\gamma_A^* \times B/\gamma_B^*.$$

Proof. First, we define the relation $\widetilde{\gamma}$ on $A \times B$ as follows:

$$(a_1, b_1)\ \widetilde{\gamma}\ (a_2, b_2) \iff a_1\ \gamma_A^*\ a_2 \text{ and } b_1\ \gamma_B^*\ b_2.$$

$\widetilde{\gamma}$ is an equivalence relation. We define ψ on $A \times B/\widetilde{\gamma}$ as follows:

$$\psi(\widetilde{\gamma}(a_1, b_1), \ldots, \widetilde{\gamma}(a_n, b_n)) = \widetilde{\gamma}(a, b)$$

for all $a \in f(\gamma_A^*(a_1), \ldots, \gamma_A^*(a_n))$ and $b \in g(\gamma_B^*(b_1), \ldots, \gamma_B^*(b_n))$. Since f, g are associative, we see that ψ is associative and consequently $A \times B/\widetilde{\gamma}$ is a commutative n-ary semigroup. Now, let θ be an equivalence relation on $A \times B$ such that $A \times B/\theta$ is a commutative n-ary group. we prove that

$$(a_1, b_1)\ \widetilde{\gamma}\ (a_2, b_2) \implies (a_1, b_1)\ \theta\ (a_2, b_2).$$

Suppose $a_1\ \gamma_A\ a_2$ and $b_1\ \gamma_B\ b_2$, thus there exist $k, h \in \mathbb{N} \cup \{0\}$ such that $a_1\ \gamma_{Ah}\ a_2$ and $b_1\ \gamma_{Bk}\ b_2$. Set $q = \max\{k, h\}$, thus by Lemma 3.4 we obtain $a_1\ \gamma_{A_q}\ a_2$ and $b_1\ \gamma_{B_q}\ b_2$. Hence, there exist $x_1^m, y_1^m \in H$ where $m = k(n-1) + 1$, and $\sigma, \tau \in S_m$ such that

$$\begin{aligned}(a_1, b_1) \in\ & (f_{(q)}(x_1^m), g_{(q)}(y_1^m)) = (f, g)_{(q)}((x_1, y_1), \ldots, (x_m, y_m)) \\ (a_2, b_2) \in\ & (f_{(q)}(x_{\sigma(1)}^{\sigma(m)}), g_{(q)}(y_{\tau(1)}^{\tau(m)})) \\ & = (f, g)_{(q)}(x_{\sigma(1)}, y_{\tau(1)}), \ldots, (x_{\sigma(m)}, y_{\tau(m)})).\end{aligned}$$

Since θ is a compatible relation on $A \times B$ and $A \times B/\theta$ is an n-ary commutative group we have

$$\begin{aligned}\theta(a_1, b_1) &= [(f, g)/\theta]_{(q)}(\theta(x_1, y_1), \ldots, \theta(x_m, y_m)) \\ &= [(f, g)/\theta]_{(q)}(\theta(x_{\sigma(1)}, y_{\tau(1)})), \ldots, \theta((x_{\sigma(m)}, y_{\tau(m)}))) \\ &= \theta(a_2, b_2).\end{aligned}$$

This means

$$\text{if } a_1\ \gamma_A\ a_2 \text{ and } b_1\ \gamma_B\ b_2 \text{ then } (a_1, b_1)\ \theta\ (a_2, b_2). \tag{3}$$

Finally, let $a_1\ \gamma_A^*\ a_2$ and $b_1\ \gamma_B^*\ b_2$, thus there exist $u_1^l, v_1^p \in H$ such that

$$a_1 = u_1\ \gamma_A\ u_2\ \gamma_A \ldots \gamma_A\ u_l = a_2 \text{ and } b_1 = v_1\ \gamma_B\ v_2\ \gamma_B \ldots \gamma_B\ v_p = b_2.$$

Let $p > l$, for every $i \in \{1, \dots, l\}$, set $w_i = u_i$ and for every $i \in \{l+1, \dots, p\}$ set $w_i = u_l$. Then,

$$a_1 = w_1 \ \gamma_A \ w_2 \ \gamma_A \dots \gamma_A \ w_p = a_2 \text{ and } b_1 = v_1 \ \gamma_B \ v_2 \ \gamma_B \dots \gamma_B \ v_p = b_2.$$

Therefore, by (3) we have

$$(a_1, b_1) = (w_1, v_1) \ \theta \ (w_2, v_2) \ \theta \dots \theta \ (w_p, v_p) = (a_2, b_2).$$

Since θ is an equivalence relation, $(a_1, b_1) \ \theta \ (a_2, b_2)$. Hence, $(a_1, b_1) \ \widetilde{\gamma} \ (a_2, b_2)$ implies $(a_1, b_1) \ \theta \ (a_2, b_2)$. Therefore, the relation $\widetilde{\gamma}$ is the smallest equivalence relation on $A \times B$ such that $A \times B/\widetilde{\gamma}$ is an n-ary commutative semigroup, i.e., $\widetilde{\gamma} = \gamma^*_{A\times B}$. Now we consider the map $\varphi : A/\gamma^*_A \times B/\gamma^*_B \longrightarrow A \times B/\gamma^*_{A\times B}$ by

$$\varphi(\gamma^*_A(a), \gamma^*_B(b)) = \gamma^*_{A\times B}(a, b).$$

It is easy to see that φ is an isomorphism. □

By the similar way we have

Theorem 3.6. *Let (A, f) and (B, g) be two n-ary hypergroups and let β^*_A, β^*_B and $\beta^*_{A\times B}$ be fundamental equivalence relations on A, B and $A \times B$ respectively. Then,*

$$A \times B/\beta^*_{A\times B} \ \cong \ A/\beta^*_A \ \times B/\beta^*_B.$$

EXAMPLE 5. If (A, f) and (B, g) are two n-ary semihypergroups the Theorem 3.5 and 3.6 is not true. Let $H = \{a_1, \dots, a_n\}$ where $n \geq 4$. we define n-ary hyperoperation f on H as follows:

$$f(a_1, \dots, a_n) = H - \{a_1, a_2\},$$

$$f(x_1, \dots, x_n) = H - \{a_1, a_3\}, \forall (x_1, \dots, x_n) \neq (a_1, \dots, a_1),$$

(H, f) is an n-ary semihypergroup, since for every $z_1, \dots, z_n \in H$, we have $a_1 \notin f(z_1^n)$ and so

$$\begin{aligned} f(x_1^{i-1}, f(x_i^{n+i-1}), x_{i+n}^{2n-1}) &= H - \{a_1, a_3\} \\ &= f(x_1^{j-1}, f(x_j^{n+j-1}), x_{j+n}^{2n-1}), \ \forall x_1^{2n-1} \in H. \end{aligned}$$

(H, f) is not an n-ary hypergroup since for every $z_1, \dots, z_n \in H$, we have $a_1 \notin f(z_1^n)$. We obtain $\gamma^*_H = \beta^*_H$ and $\beta^*_H(a_1) = \gamma^*_H(a_1) = \{a_1\}$ and $\beta^*_H(a_2) = \gamma^*_H(a_2) = H - \{a_1\} = \gamma^*(a_j) = \beta^*(a_j)$ for every $j \geq 2$, and so $|H/\beta^*| = 2 = |H/\gamma^*|$. In the direct product $H \times H$ we have the n-ary hyperoperation:

$$(f, f)((x_1, y_1), \dots, (x_n, y_n)) = \{(a, b) \mid a \in f(x_1, \dots, x_n), b \in f(y_1, \dots, y_n)\}.$$

Therefore, we obtain

$(f,f)((a_1,a_1),\ldots,(a_1,a_1)) = (H-\{a_1,a_2\}) \times (H-\{a_1,a_2\})$,

$(f,f)((a_1,y_1),\ldots,(a_1,y_n)) = (H-\{a_1,a_2\})\times(H-\{a_1,a_3\})$, $\forall (y_1^n) \neq (\overset{(n)}{a_1})$,

$(f,f)((x_1,a_1),\ldots,(x_n,a_1)) = (H-\{a_1,a_3\})\times(H-\{a_1,a_2\})$, $\forall (x_1^n) \neq (\overset{(n)}{a_1})$,

$(f,f)((x_1,y_1),\ldots,(x_n,y_n)) = (H-\{a_1,a_3\}) \times (H-\{a_1,a_3\})$, $\forall (y_1^n),(x_1^n) \neq (\overset{(n)}{a_1})$.

Since (H,f) is an n-ary semihypergroup thus $(H\times H,(f,f))$ is an n-ary semihypergroup. It easy to see that $\beta^*_{H\times H} = \gamma^*_{H\times H}$ and

$\beta^*_{H\times H}(a_i,a_j) = \gamma^*_{H\times H}(a_i,a_j) = (H-\{a_1\}) \times (H-\{a_1\})$, if $i,j = 2,\ldots,n$.

$\beta^*_{H\times H}(a_i,a_j) = \gamma^*_{H\times H}(a_i,a_j) = \{(a_i,a_j)\}$, if $i=1$ or $j=1$.

Hence, we have $|H\times H/\beta^*_{H\times H}| = 2n = |H\times H/\gamma^*_{H\times H}|$. But $|H/\beta^*_H| = 2 = |H/\gamma^*_H|$ and so $|H/\beta^*_H \times H/\beta^*_H| = 4 = |H/\gamma^*_H \times H/\gamma^*_H|$. Since $n\geq 4$ thus $2n \neq 4$ and therefore

$$H\times H/\beta^*_{H\times H} \ncong H/\beta^*_H \times H/\beta^*_H \text{ and } H\times H/\gamma^*_{H\times H} \ncong H/\gamma^*_H \times H/\gamma^*_H.$$

Theorem 3.7. *Let (H,f) be a commutative n-ary semihypergroup. Then, $\gamma = \beta$.*

Proof. It is straightforward. □

Theorem 3.8. *Let (H,f) be a COW n-ary semihypergroup. Then, $\gamma^* = \beta^*$.*

Proof. By definition β^* is the smallest equivalence relation such that H/β^* is an n-ary semihypergroup. Since H is a COW n-ary semihypergroup, so

$$\bigcap_{\sigma\in\mathbb{S}_n} f(x_{\sigma(1)}^{\sigma(n)}) \neq \emptyset \text{ for all } x_1^n \in H.$$

Therefore, there exists $a \in \bigcap_{\sigma\in\mathbb{S}_n} f(x_{\sigma(1)}^{\sigma(n)})$ which yield that

$$\beta^*(a) = f/\beta^*(\beta^*(x_{\sigma(1)}),\ldots,\beta^*(x_{\sigma(n)})) \text{ for every } \sigma\in\mathbb{S}_n,$$

that is H/β^* is a commutative n-ary semigroup. Since $\beta\subseteq\gamma$, we get $\beta^*\subseteq\gamma^*$. Since $\beta^*\subseteq\gamma^*$ and γ^* is the smallest equivalence relation such that H/γ^* is a commutative n-ary semihypergroup, then $\gamma^*=\beta^*$. □

Theorem 3.9. *Let (H,f) be an n-ary semihypergroup, $\beta = \beta_{(H,f)}$ and $\gamma = \gamma_{(H,f)}$. If $\rho = \gamma^*_{(H/\beta^*,f/\beta^*)}$, then $(H/\beta^*)/\rho = H/\gamma^*$.*

Proof. We consider the map $\psi : H/\beta^* \longrightarrow H/\gamma^*$ by $\psi(\beta^*(x)) = \gamma^*(x)$. Obviously, ψ is well defined, since $\beta^* \subset \gamma^*$. We have

$$\begin{aligned}\psi(f/\beta^*(\beta^*(a_1),\ldots,\beta^*(a_n)) &= \psi(\beta^*(f(a_1,\ldots,a_n))) \\ &= \gamma^*(f(a_1,\ldots,a_n)) \\ &= f/\gamma^*(\gamma^*(a_1),\ldots,\gamma^*(a_n)) \\ &= f/\gamma^*(\psi(\beta^*(a_1)),\ldots,\psi(\beta^*(a_n))).\end{aligned}$$

Therefore, ψ is a homomorphism. Also, ψ is onto. Set $\theta = \ker\psi$ and consider the isomorphism $\xi : (H/\beta^*)/\theta \longrightarrow H/\gamma^*$ by $\xi(\theta(\beta^*(x)) = \gamma^*(x)$, for every $x \in H$. It is easy to see that $\theta = \ker\psi$ is a compatible relation. Since $(H/\beta^*)/\theta \cong H/\gamma^*$, so $(H/\beta^*)/\theta$ is a commutative n-ary semigroup and since $\rho = \gamma^*_{(H/\beta^*,f/\beta^*)}$ is the smallest compatible relation such that $(H/\beta^*)/\rho$ is a commutative n-ary semigroup, thus $\rho \subset \theta$. If $a\gamma^*b$. it is easy to see that $\beta^*(a)\ \gamma^*_{(H/\beta^*,f/\beta^*)}\ \beta^*(b)$, therefore

$$\begin{aligned}\theta &= \ker\psi \\ &= \{(\beta^*(a),\beta^*(b)) \in (H/\beta^*)^2 | \psi(\beta^*(a)) = \psi(\beta^*(b))\} \\ &= \{(\beta^*(a),\beta^*(b)) \in (H/\beta^*)^2 | \gamma^*(a) = \gamma^*(b)\} \\ &\subset \{(\beta^*(a),\beta^*(b)) \in (H/\beta^*)^2 | \beta^*(a)\ \gamma^*_{(H/\beta^*,f/\beta^*)}\ \beta^*(b)\} \\ &= \rho.\end{aligned}$$

Thus, $\theta = \rho$ and the proof is completed. □

4 Strongly transitive geometric space

In this Section we recall some basic definitions and propositions from [12]. A geometric space is a pair $(S,\mathcal{B})$ such that S is a non-empty set, whose elements we call *points*, and $\mathcal{B}$ is a non-empty family of subsets of S, whose elements we call *blocks*. $\mathcal{B}$ is a covering of S if for every point $y \in S$, there exists a block $B \in \mathcal{B}$ such that $y \in B$. If C is a subset of S, we say that C is a $\mathcal{B}$-part or $\mathcal{B}$-subset of S if for every $B \in \mathcal{B}$,

$$B \cap C \neq \emptyset \Rightarrow B \subset C.$$

The family $\mathcal{F}_{\mathcal{B}}(S)$ of all $\mathcal{B}$-parts of S is non-empty, since $\emptyset$ and S are elements of $\mathcal{F}_{\mathcal{B}}(S)$. Moreover, the intersection of elements of $\mathcal{F}_{\mathcal{B}}(S)$ is an element of $\mathcal{F}_{\mathcal{B}}(S)$, hence $\mathcal{F}_{\mathcal{B}}(S)$ is a closure system of S. For a subset X of S, we denote by $\Gamma(X)$, the smallest $\mathcal{B}$-part of S containing X, and it is called the *closure* of X.

The following properties are true

(1) $X \subset \Gamma(X)$.

(2) $X \subset Y \Rightarrow \Gamma(X) \subset \Gamma(Y)$.

(3) $\Gamma(\Gamma(X)) = \Gamma(X)$.

(4) $\Gamma(X) = \bigcup_{x\in X} \Gamma(x)$, where $\Gamma(x) = \Gamma(\{x\})$.

For all subsets X of S, we can associate an ascending chain of subsets $(\Gamma_n(X))_{n\in\mathbf{N}}$, called *cone* of X, defined by the following conditions:

$$\Gamma_0(X) = X;$$

and for every integer $n \geq 0$

$$\Gamma_{n+1}(X) = \Gamma_n(X) \cup \left[\bigcup\{B \in \mathcal{B} | B \cap \Gamma_n(X) \neq \emptyset\}\right].$$

Freni [12] used the notion of cone of X and obtained the closure of X, as it is shown in the next proposition.

Proposition 4.1. *Let $(S, \mathcal{B})$ be a geometric space. For every $n \in \mathbb{N}$ and for every pair (X, Y) of subsets of S we have*

(1) $X \subset Y \Rightarrow \Gamma_n(X) \subset \Gamma_n(Y)$.

(2) $\Gamma_n(X) = \bigcup_{x\in X} \Gamma_n(x)$, *where* $\Gamma_n(x) = \Gamma_n(\{x\})$.

(3) $\Gamma_n(\Gamma_m(X)) = \Gamma_{n+m}(X)$.

(4) $\Gamma(X) = \bigcup_{n\in\mathbf{N}} \Gamma_n(x)$.

(5) If the family $\mathcal{B}$ is covering of S, then

$$\Gamma_{n+1}(X) = \bigcup\{B \in \mathcal{B} | B \cap \Gamma_n(X) \neq \emptyset\}.$$

(6) If B an element of $\mathcal{B}$, we have $\Gamma(B) = \Gamma(x)$, for all $x \in B$.

(7) If $m \in \mathbb{N}$ exists such that $\Gamma_{m+1} \subset \Gamma_m(X)$, then we have $\Gamma_k(X) = \Gamma_m(X)$, for every integer $k > m$. Moreover $\Gamma(X) = \Gamma_m(X)$.

If $B_1, B_2, \ldots, B_n$ are n blocks of geometric space $(S, \mathcal{B})$ such that $B_i \cap B_{i+1} \neq \emptyset$, for any $i \in \{1, 2, \ldots, n-1\}$, then the n-tuple $(B_1, B_2, \ldots, B_n)$ is called *polygonal* of $(S, \mathcal{B})$. The concept of polygonal allows us to define on S the following relation

$x \approx y \Leftrightarrow x = y$ or a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$.

The relation $\approx$ is an equivalence relation and it is easy to see that it coincides with the transitive closure of the following relation:

$$x \sim y \Leftrightarrow x = y \text{ or there exists } B \in \mathcal{B} \text{ such that } \{x, y\} \subset B.$$

So $\approx$ is equal to $\bigcup\limits_{n \geq 1} \sim^n$, where $\sim^n = \sim \circ \sim \circ \ldots \circ \sim$ n times.

If $\mathcal{B}$ is a covering of S, the relation $\approx$ and $\sim$ is defined in the following way:

> $x \approx y \Leftrightarrow$ a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$.
>
> $x \sim y \Leftrightarrow$ there exists $B \in \mathcal{B}$ such that $\{x, y\} \subset B$.

Freni [12] proved that the $\approx$-class of x in S, that is, the equivalence class of element x modulo $\approx$, coincides with the closure $\Gamma(x)$ of x. We denote $[x]$, the $\approx$-class of x in S.

Proposition 4.2. *For every integer $n \geq 1$ and for every pair (x, y) of elements of S, we have*

(1) $y \sim^n x \Leftrightarrow y \in \Gamma_n(x)$.

(2) $[x] = \Gamma(x)$.

(3) $\sim^n$ *is transitive* $\Leftrightarrow \Gamma(x) = \Gamma_n(x)$, *for all* $x \in S$.

(4) $\sim$ *is transitive* $\Leftrightarrow \Gamma(x) = \Gamma_1(x)$, *for all* $x \in S$.

Our next proposition generalizes to arbitrary geometric spaces the content of Lemma 2.1 in [11].

Proposition 4.3. *Let $(S, \mathcal{B})$ be a geometric space. Let M be a non-empty subset of S. Then, the following conditions are equivalent:*

(1) M is a $\mathcal{B}$-part of H.

(2) If $x \in M$ and $x \sim y$, then we have $y \in M$.

(3) If $x \in M$ and $x \approx y$, then we have $y \in M$.

Proof. (1) $\Rightarrow$ (2) : Let $(x, y) \in H^2$ be a pair such that $x \in M$ and $x \sim y$, thus $x = y$ or there exists block $B \in \mathcal{B}$ such that $\{x, y\} \subset B$. If $x = y$, the proof is completed, so let $x \neq y$. Since M is $\mathcal{B}$-part and $x \in M \cap B$ so $M \cap B \neq \emptyset$, therefore $B \subset M$ and so $y \in B \subset M$.

(2) $\Rightarrow$ (3) : Let $(x, y) \in H^2$ such that $x \in M$ and $x \approx y$, thus $x = y$ or a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$. If $x = y$ proof is complete, so let $x \neq y$. Since $(B_1, B_2, \ldots, B_n)$ is a polygonal

thus there exist $w_0, \ldots, w_n \in H$ such that $w_0 = x \in B_1$ and for every $i = 1, 2, \ldots, n-1$ let $w_i \in B_i \cap B_{i+1}$ and $w_n = y \in B_n$. Thus, $x = w_0 \sim w_1 \sim \ldots \sim w_n = y$. Since $w_0 = x \in M$, applying (2) n times, we obtain $y = w_n \in M$.

(3) $\Rightarrow$ (1) : Let B be a block of geometric space $(S, \mathcal{B})$ such that $B \cap M \neq \emptyset$ and $x \in B \cap M$. For every $y \in B$ we have $\{x, y\} \subset B$ and so $x \sim y$. Thus, $x \approx y$ and $x \in M$. Finally, by (3), we obtain $y \in M$, therefore $B \subset M$ and M is a $\mathcal{B}$-part of H. □

Also, Freni [12] proved the next Theorem:

Theorem 4.4. *For every pair (A, B) of blocks of a geometric space $(S, \mathcal{B})$ and for any integer $n \in \mathbb{N}$, the following conditions are equivalent:*

(1) $A \cap B \neq \emptyset, x \in B \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subset C$.

(2) $A \cap B \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subset C$.

(3) $A \cap \Gamma(B) \neq \emptyset, x \in \Gamma(B) \Rightarrow \exists C \in \mathcal{B} : (A \cup \{x\}) \subset C$.

A geometric space $(S, \mathcal{B})$ is *strongly transitive* if the family $\mathcal{B}$ is a covering of S and moreover one of the three equivalence conditions of Theorem 4.4 is satisfied. With this definition we have (see [12]):

Theorem 4.5. *If $(S, \mathcal{B})$ is a strongly transitive geometric space, then the relation $\sim$ on S is transitive, and so $\approx = \sim$.*

REMARK 1. The converse of Theorem 4.5 is not true. Indeed, the following counterexample can also be found in [12]: If $(A^n(V), \wp)$ is a geometric space such that $A^n(V)$ is an affine space of dimension $n \geq 2$ and $\wp$ is the family of affine subspaces of dimension $n = 1$, that is the lines of $A^n(V)$, then the relation $\sim$ is transitive but the geometric space $(A^n(V), \wp)$ is not strongly transitive.

Let $(S, \mathcal{B})$ is a geometric space. For every element $x \in S$, set

$$\mathcal{B}(x) = \bigcup \{B \in \mathcal{B} | x \in B\}.$$

It is clear that $\mathcal{B}(x) = \{y \in S | x \sim y\}$. If the family $\mathcal{B}$ is a covering of S then for every $x \in S$ we have $\mathcal{B} \neq \emptyset$. From the preceding notion, it follow at once the following:

Theorem 4.6. *Let $(S, \mathcal{B})$ be a geometric space and the family $\mathcal{B}$ be a covering of S. The following conditions are equivalent:*

1) $\sim$ *is transitive i.e,* $\sim = \approx$.

2) For every $x \in S$, $[x] = \mathcal{B}(x)$ where $[x]$ is $\approx$-class of x.

3) For every $x \in S$, $\mathcal{B}(x)$ is a $\mathcal{B}$-part of a geometric space $(S, \mathcal{B})$.

Proof. (1) $\Rightarrow$ (2) : Since $\sim=\approx$ and $\mathcal{B}(x) = \{y \in S | x \sim y\}$, we have

$$[x] = \{y \in S | x \approx y\} = \{y \in S | x \sim y\} = \mathcal{B}(x)$$

(2) $\Rightarrow$ (3) : By Proposition 4.3, if M is a non-empty subset of S, then M is a $\mathcal{B}$-part of H if and only if it is union of equivalence class modulo $\approx$. Particularly, every equivalence class modulo $\approx$ is a $\mathcal{B}$-part of S.

(3) $\Rightarrow$ (1) : If $x \sim y$ and $y \sim z$, then there exist two blocks B and C of $\mathcal{B}$ such that $\{x, y\} \subset B$ and $\{y, z\} \subset C$. Since $\mathcal{B}(x)$ is a $\mathcal{B}$-part of a geometric space $(S, \mathcal{B})$, we have

$$\begin{aligned} x \in \mathcal{B}(x) \cap B \quad &\Rightarrow B \subset \mathcal{B}(x) \\ &\Rightarrow y \in \mathcal{B}(x) \cap C \\ &\Rightarrow C \subset \mathcal{B}(x) \\ &\Rightarrow z \in \mathcal{B}(x). \end{aligned}$$

But $\mathcal{B}(x) = \{y \in S | x \sim y\}$, thus $x \sim z$. Therefore, $\sim$ is transitive. □

5 Strongly transitive geometric spaces associated to n-ary hypergroups

If (H, f) is an n-ary hypergroup, we can consider the geometric space $(H, P(H))$ whose points are the elements of H and whose blocks are the n-ary hyperproducts of elements of H. Thus, if $B \in P(H)$, then there exist $k \in \mathbb{N} \cup \{0\}$ and an m-tuple $(z_1^m) \in H^m$ where $m = k(n-1)+1$, such that $B = f_{(k)}(z_1^m)$. In this section, we suppose $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.
If $n = 2$, then (H, f) is a hypergroup (or 2-ary hypergroup). Gutan (1997)[14] shows that the geometric space $(H, P(H))$ is strongly transitive, when (H, f) is a 2-ary hypergroup. In this case, the relation $\sim$ coincides with the fundamental relation β, therefore β is transitive. We notice that the transitivity of β in hypergroup has been shown by Freni in [13].

Let (H, f) be an n-ary semihypergroup and $P(H)$ be the family of H defined as follows: for every integer $k \geq 1$ and for every m-tuple $(z_1^m) \in H^m$, where $m = k(n-1)+1$, We set

(i) $B(z_1) = \{z_1\}$.

(ii) $B(z_1^m) = f_{(k)}(z_1^m)$, if $m \geq 2$.

Lemma 5.1. *Let (H, f) be an n-ary semihypergroup. Then,*

(1) For every $y_1^n \in H$ and $1 \leq l \leq n$, we have

$$f(y_1^{l-1}, B(z_1^m), y_{l+1}^n) = B(y_1^{l-1}, z_1^m, y_{l+1}^n).$$

(2) If there exist an integer $k' \in \mathbb{N}^*$, an m'-tuple $(x_1^{m'}) \in H^{m'}$ and an element $l \in \{1, 2, \ldots, m'\}$, where $m' = k'(n-1)+1$, such that $z_l \in f_{(k')}(x_1^{m'})$, then

$$B(z_1^m) \subset B(z_1^{l-1}, x_1^{m'}, z_{l+1}^n).$$

(3) Let $k \in \mathbb{N}^*$ and $(z_1^m) \in H^m$ be an m-tuple of elements of an n-ary semihypergroup (H, f), where $m = k(n-1)+1$. If an integer $k' \in \mathbb{N}^*$, an m'-tuple $(x_1^{m'}) \in H^{m'}$ and an element $l \in \{1, 2, \ldots, m'\}$ exist, where $m' = k'(n-1)+1$, such that $z_l \in B(x_1^{m'})$, then

$$B(z_1^m) \subset B(z_1^{l-1}, x_1^{m'}, z_{l+1}^n).$$

Proof. It is straightforward. □

Lemma 5.2. *Let $(k_1^n) \in \mathbb{N}^{*n}$ and for every $1 \leq i \leq n$ set $m_i = k_i(n-1)+1$. Then, for every $(m_1+m_2+\ldots+m_n)$-tuple $(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n})$ of elements of an n-ary semihypergroup (H, f), we have*

$$f(B(x_1^{m_1}), B(y_1^{m_2}), \ldots, B(z_1^{m_n})) = B(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}).$$

Proof. Since (H, f) is an n-ary semihypergroup, we have

$$\begin{aligned} & f(B(x_1^{m_1}), B(y_1^{m_2}), \ldots, B(z_1^{m_n})) \\ & = f(f_{(k_1)}(x_1^{m_1}), f_{(k_2)}(y_1^{m_2}), \ldots, f_{(k_n)}(z_1^{m_n})) \\ & = f_{(1+k_1+k_2+\ldots+k_n)}(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}) \\ & = B(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}). \end{aligned}$$

□

Lemma 5.3. *Let $(k_1^l) \in \mathbb{N}^{*l}$ and for every $1 \leq i \leq l$, where $l = q(n-1)+1$, set $m_i = k_i(n-1)+1$, then for every $(m_1+m_2+\ldots+m_l)$-tuple $(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_l})$ of elements of an n-ary semihypergroup (H, f), we have*

$$B(B(x_1^{m_1}), B(y_1^{m_2}), \ldots, B(z_1^{m_l})) = B(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_l}).$$

Proof. Since (H, f) is an n-ary semihypergroup, we have

$$\begin{aligned} & B(B(x_1^{m_1}), B(y_1^{m_2}), \ldots, B(z_1^{m_l})) \\ & = f_{(q)}(f_{(k_1)}(x_1^{m_1}), f_{(k_2)}(y_1^{m_2}), \ldots, f_{(k_n)}(z_1^{m_l})) \\ & = f_{(q+k_1+k_2+\ldots+k_l)}(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_l}) \\ & = B(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_l}). \end{aligned}$$

□

Theorem 5.4. *If (H, f) is an n-ary hypergroup, then the geometric space $(H, P(H))$ is strongly transitive.*

Proof. Let $B(z_1^m), B(x_1^{m'})$ be two blocks of $P(H)$ such that

$$B(z_1^m) \cap B(x_1^{m'}) \neq \emptyset \text{ and } x \in B(x_1^{m'}).$$

There exist $k, k' \in \mathbb{N}^*$ such that $m = k(n-1)+1$ and $m' = k'(n-1)+1$. Let $b \in B(z_1^m) \cap B(x_1^{m'})$. Since (H, f) is n-ary hypergroup thus there exist $c, y \in H$ such that

$$x \in f(b, c, \overset{(n-2)}{b}) \quad \text{and} \quad z_m \in f(\overset{(n-2)}{x}, y, x).$$

Since $x \in B(x_1^m)$, by Lemma 5.1, we have

$$\begin{aligned} x \in f(b, c, \overset{(n-2)}{b}) \;&\subset f(B(z_1^m), c, \overset{(n-2)}{b}) = f(f_{(k)}(z_1^m), c, \overset{(n-2)}{b})) \\ &\subset f_{(k+1)}(z_1^m, c, \overset{(n-2)}{b})) = B(z_1^m, c, \overset{(n-2)}{b}) \\ &\subset B(z_1^{m-1}, f(\overset{(n-2)}{x}, y, x), c, \overset{(n-2)}{b}) \\ &= B(z_1^{m-1}, \overset{(n-2)}{x}, y, x, c, \overset{(n-2)}{b}) \\ &\subset B(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b}). \end{aligned}$$

Moreover, since $b \in B(x_1^m)$, we obtain

$$\begin{aligned} B(z_1^m) \;&\subset B(z_1^{m-1}, f(\overset{(n-2)}{x}, y, x)) = B(z_1^{m-1}, \overset{(n-2)}{x}, y, x) \\ &\subset B(z_1^{m-1}, \overset{(n-2)}{x}, y, f(b, c, \overset{(n-2)}{b}) = B(z_1^{m-1}, \overset{(n-2)}{x}, y, b, c, \overset{(n-2)}{b}) \\ &\subset B(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b}). \end{aligned}$$

Therefore, $B(z_1^m) \cup \{x\} \subset B(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b})$ and the geometric space $(H, P(H))$ is strongly transitive. □

REMARK 2. If (H, f) is an n-ary hypergroup, the relation $\sim$ defined on the geometric space $(H, P(H))$ is transitive (Theorem 5.4 and 4.5) and coincides with the relation β on the n-ary hypergroup used in the paper (Davvaz an Vougiouklis [7]). Also, the relation $\approx$ defined on the geometric space $(H, P(H))$ coincides with the fundamental relation β^* on the n-ary hypergroup used in the paper Davvaz and Vougiouklis [7]. Therefore, we obtain the next result:

Theorem 5.5. *Let (H, f) be an n-ary hypergroup. Then, the relation β is an strongly compatible equivalence relation and $\beta = \beta^*$, where β^* is the fundamental relation on H.*

The following example shows that Theorems 5.4 and 5.5 are not valid for n-ary semihypergroups.

EXAMPLE 6. Let $|H| \geq 4$ and $f : H^n \longrightarrow \mathcal{P}^*(H)$ is defined as follow:

$$f(\overset{(n)}{a}) = H - \{a, b\},$$

$$f(x_1^n) = H - \{a, c\}, \forall (x_1^n) \neq (\overset{(n)}{a}),$$

where $a \neq b \neq c \neq a \in H$.

(H, f) is an n-ary semihypergroup, since for every $z_1, \ldots, z_n \in H$, we have $a \notin f(z_1^n)$ and so

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{i+n}^{2n-1}) = H - \{a, c\} = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{j+n}^{2n-1}),$$

for all $x_1^{2n-1} \in H$. Also, (H, f) is not an n-ary hypergroup, since for every $z_1, \ldots, z_n \in H$, we have $a \notin f(z_1^n)$. Let $d \in H - \{a, b, c\}$. Then, we have $b \; \beta \; d$ and $d \; \beta \; c$ so $b \; \beta^* \; c$, but not $b \; \beta \; c$.

Theorem 5.6. *Let (H, f) be an n-ary hypergroup and $\phi : H \to H/\beta^*$ be the canonical projection. Then, $\phi^{-1}(\delta)$ is a $P(H)$-part of H, where $\delta \in H/\beta^*$.*

Proof. Let $x \in \phi^{-1}(\delta)$ and $x \; \beta \; y$. Then, $\phi(x) = \delta$ and $\phi(x) = \phi(y)$. Thus, $\phi(y) = \delta$ and therefore $y \in \phi^{-1}(\delta)$. Now, Proposition 4.3 shows that $\phi^{-1}(\delta)$ is a $P(H)$-part of H. □

In [12], Freni shows that there exists a family $P_\sigma(H)$ of subsets of hypergroup H such that $\gamma = \sim$ and $\gamma^* = \approx$ on geometric space $(H, P_\sigma(H))$. Freni show that the geometric space $(H, P_\sigma(H))$ is strongly transitive, therefore γ is transitive. Also the transitivity of γ in hypergroups has been shown in [11].

Now, we see a family $P_\sigma(H)$ of subsets of an n-ary hypergroup H such that the geometric space $(H, P_\sigma(H))$ is strongly transitive. Let (H, f) be an n-ary semihypergroup and $P_\sigma(H)$ be the family of subsets of H defined as follows: for every integer $k \geq 1$ and for every m-tuple $(z_1^m) \in H^m$, where $m = k(n-1)+1$, we set:

(i) $B_\sigma(z_1) = \{z_1\}$.

(ii) $B_\sigma(z_1^m) = \bigcup \{f_{(k)}(z_{\sigma(1)}^{\sigma(m)}) | \sigma \in \mathbb{S}_m\}$, if $n \geq 2$.

Where $\mathbb{S}_m$ is the symmetric group of all permutations on set $\{1, 2, \ldots, m\}$

Lemma 5.7. *If (z_1^m) is an m-tuple of elements of an n-ary semihypergroup (H, f), where $m = k(n-1)+1$, then*

(1) For every $\delta \in \mathbb{S}_m$, we have

$$B_\sigma(z_1^m) = B_\sigma(z_{\delta(1)}^{\delta(m)}).$$

(2) For every $y_1^n \in H$ and $1 \le j \le n$, we have

$$f(y_1^{j-1}, B_\sigma(z_1^m), y_{j+1}^n) \subset B_\sigma(y_1^{j-1}, z_1^m, y_{j+1}^n)$$

(3) If there exist an integer $k' \in \mathbb{N}^*$, an m'-tuple $(x_1^{m'}) \in H^{m'}$ and an element $l \in \{1, 2, \ldots, m'\}$, where $m' = k'(n-1)+1$, such that $z_l \in f_{(k')}(x_1^{m'})$, then

$$B_\sigma(z_1^m) \subset B_\sigma(z_1^{l-1}, x_1^{m'}, z_{l+1}^n).$$

Proof. (1) For every permutation $\delta \in \mathbb{S}_m$, we have

$$\begin{aligned} x \in B_\sigma(z_{\delta(1)}^{\delta(m)}) &\Leftrightarrow \exists \tau \in \mathbb{S}_m : x \in f_{(k)}(z_{\tau(\delta(1))}^{\tau(\delta(m))}) \\ &\Leftrightarrow \exists \tau \in \mathbb{S}_m : x \in f_{(k)}(z_{\tau\circ\delta(1)}^{\tau\circ\delta(m)}) \\ &\Leftrightarrow x \in B_\sigma(z_1^m). \end{aligned}$$

(2) If $w \in f(y_1^{j-1}, B_\sigma(z_1^m), y_{j+1}^n)$, then an element $y \in B_\sigma(z_1^m)$ and a permutation $\delta \in \mathbb{S}_m$ exist such that $w \in f(y_1^{j-1}, y, y_{j+1}^n)$ and $y \in f_{(k)}(z_{\delta(1)}^{\delta(m)})$. Set:

$$\begin{cases} x_i = y_i, & \text{if } i = 1, 2, \ldots, j-1 \\ x_{j-1+i} = z_i, & \text{if } i = 1, 2, \ldots, m \\ x_{m+i} = y_{i+1}, & \text{if } i = j, j+1, \ldots, n, \end{cases}$$

and let τ be the permutation of $\mathbb{S}_{m+n-1}$ such that:

$$\begin{cases} \tau(i) = i, & \text{if } i = 1, 2, \ldots, j-1 \\ \tau(i) = \delta(i-j+1), & \text{if } i = j, j+1, \ldots, j+m-1 \\ \tau(i) = i-m+1, & \text{if } i = j+m, j+m+1, \ldots, n+m-1. \end{cases}$$

We have

$$w \in f(x_1^{j-1}, f_{(k)}(x_{\delta(j)}^{\delta(j+m-1)}), x_{j+m}^{n+m-1}) = f_{(k+1)}(x_{\tau(1)}^{\tau(m+n-1)}),$$

therefore

$$\begin{aligned} f(y_1^{j-1}, B_\sigma(z_1^m), y_{j+1}^n) &\subset f(x_1^{j-1}, B_\sigma(x_j^{j+m-1}), x_{j+m}^{m+n-1}) \\ &\subset B_\sigma(x_1^{m+n-1}) \\ &= B_\sigma(y_1^{j-1}, z_1^m, y_{j+1}^n). \end{aligned}$$

(3) If $k' = 0$, then $m' = 1$ and the proof is trivial. Thus, we suppose $k' \geq 1$ and $z_l \in f_{(k')}(x_1^{m'})$. If $w \in B_\sigma(z_1^m)$, a permutation $\delta \in \mathbb{S}_m$ exists such that $w \in f_{(k)}(z_{\delta(1)}^{\delta(m)})$. Setting $\delta(h) = l$, we have

$$\begin{aligned} w \in f_{(k)}(z_{\delta(1)}^{\delta(m)}) &= f_{(k)}(z_{\delta(1)}^{\delta(h-1)}, z_l, z_{\delta(h+1)}^{\delta(m)}) \\ &\subset f_{(k)}(z_{\delta(1)}^{\delta(h-1)}, f_{(k')}(x_1^{m'}), z_{\delta(h+1)}^{\delta(m)}) \\ &= f_{(k+k')}(z_{\delta(1)}^{\delta(h-1)}, x_1^{m'}, z_{\delta(h+1)}^{\delta(m)}) \\ &\subset B_\sigma(z_1^{l-1}, x_1^{m'}, z_{l+1}^m). \end{aligned}$$

□

Corollary 5.8. *Let $k \in \mathbb{N}^*$ and $(z_1^m) \in H^m$ be an m-tuple of elements of an n-ary semihypergroup (H, f), where $m = k(n-1)+1$. If an integer $k' \in \mathbb{N}^*$, an m'-tuple $(x_1^{m'}) \in H^{m'}$ and an element $l \in \{1, 2, \ldots, m'\}$ exist, where $m' = k'(n-1)+1$, such that $z_l \in B_\sigma(x_1^{m'})$, then*

$$B_\sigma(z_1^m) \subset B_\sigma(z_1^{l-1}, x_1^{m'}, z_{l+1}^m).$$

Proof. If $z_l \in B_\sigma(x_1^{m'})$, a permutation $\delta \in \mathbb{S}_{m'}$ exists such that $z_l \in f_{(k')}(x_{\delta(1)}^{\delta(m')})$. By Lemma 5.7, we have

$$B_\sigma(z_1^m) \subset B_\sigma(z_1^{l-1}, x_{\delta(1)}^{\delta(m')}, z_{l+1}^m) = B_\sigma(z_1^{l-1}, x_1^{m'}, z_{l+1}^m).$$

□

Corollary 5.9. *Let $(k_1^n) \in \mathbb{N}^{*n}$ and for every $1 \leq i \leq n$ set $m_i = k_i(n-1)+1$. Then, for every $(m_1+m_2+\ldots+m_n)$-tuple $(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n})$ of elements of an n-ary semihypergroup (H, f), we have*

$$f(B_\sigma(x_1^{m_1}), B_\sigma(y_1^{m_2}), \ldots, B_\sigma(z_1^{m_n})) \subset B_\sigma(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}).$$

Proof. For every $w \in f(B_\sigma(x_1^{m_1}), B_\sigma(y_1^{m_2}), \ldots, B_\sigma(z_1^{m_n}))$ there exist $y' \in B_\sigma(y_1^{m_2}), \ldots, z' \in B_\sigma(z_1^{m_n})$ such that $w \in f(B_\sigma(x_1^{m_1}), y', \ldots, z')$. Thus, by Lemma 5.7 and Corollary 5.8, we have

$$w \in f(B_\sigma(x_1^{m_1}), y', \ldots, z') \subset B_\sigma(x_1^{m_1}, y', \ldots, z') \subset B_\sigma(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}).$$

□

Corollary 5.10. *Let $k \in \mathbb{N}^*$ and $(k_1^h) \in \mathbb{N}^{*h}$, where $h = k(n-1)+1$ and for every $1 \leq i \leq h$ set $m_i = k_i(n-1)+1$. Then, for every $(m_1+m_2+\ldots+m_h)$-tuple $(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_h})$ of elements of an n-ary semihypergroup (H, f), we have*

$$B_\sigma(B_\sigma(x_1^{m_1}), B_\sigma(y_1^{m_2}), \ldots, B_\sigma(z_1^{m_n})) \subset B_\sigma(x_1^{m_1}, y_1^{m_2}, \ldots, z_1^{m_n}).$$

Proof. It is similar to the proof of Corollary 5.9. □

Theorem 5.11. *If (H, f) is an n-ary hypergroup, then the geometric space $(H, P_\sigma(H))$ is strongly transitive.*

Proof. Let $k, k' \in \mathbb{N}^*$ such that $m = k(n-1)+1$ and $m' = k'(n-1)+1$. Let $B_\sigma(z_1^m)$ and $B_\sigma(x_1^{m'})$ be two blocks of $P_\sigma(H)$ such that

$$B_\sigma(z_1^m) \cap B_\sigma(x_1^{m'}) \neq \emptyset \quad \text{and} \quad x \in B_\sigma(x_1^{m'}).$$

Thus, there exists $\delta \in \mathbb{S}_{m'}$ such that $x \in f_{(k')}(x_{\delta(1)}^{\delta(m')})$. Let $b \in B_\sigma(z_1^m) \cap B_\sigma(x_1^{m'})$, so $\tau \in \mathbb{S}_m$ exists such that $b \in f_{(k)}(z_{\tau(1)}^{\tau(m)})$. Since (H, f) is an n-ary hypergroup thus there exist $c, y \in H$ such that

$$x \in f(b, c, \overset{(n-2)}{b}) \quad \text{and} \quad z_m \in f(\overset{(n-2)}{x}, y, x).$$

Since $x \in f_{(k')}(x_{\delta(1)}^{\delta(m')})$, by part (3) of Lemma 5.7, we have

$$\begin{aligned}
x \in f(b, c, \overset{(n-2)}{b}) &\subset f(f_{(k)}(z_{\tau(1)}^{\tau(m)}), c, \overset{(n-2)}{b}) \\
&\subset B_\sigma(z_1^m, c, \overset{(n-2)}{b}) \\
&\subset B_\sigma(z_1^{m-1}, f(\overset{(n-2)}{x}, y, x), c, \overset{(n-2)}{b}) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, x, c, \overset{(n-2)}{b}) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b}).
\end{aligned}$$

Moreover, since $b \in B(x_1^m)$, we obtain

$$\begin{aligned}
B_\sigma(z_1^m) &\subset B_\sigma(z_1^{m-1}, f(\overset{(n-2)}{x}, y, x)) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, x) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, f(b, c, \overset{(n-2)}{b})) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, b, c, \overset{(n-2)}{b}) \\
&\subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b}).
\end{aligned}$$

Therefore, $B_\sigma(z_1^m) \cup \{x\} \subset B_\sigma(z_1^{m-1}, \overset{(n-2)}{x}, y, x_1^{m'}, c, \overset{(n-2)}{b})$ and the geometric space $(H, P_\sigma(H))$ is strongly transitive. □

Proposition 5.12. *Let (H, f) be an n-ary hypergroup. Then, the relation γ is a strongly compatible equivalence relation on H and so $\gamma = \gamma^*$, where γ^* is the commutative fundamental relation on H.*

Proof. Two relations $\sim$ and $\approx$ defined on the geometric space $(H, P_\sigma(H))$ are coincide with two relations γ and γ^*, i.e, $\sim = \gamma$ and $\approx = \gamma^*$. So by Theorems 4.5 and 5.11 we obtain $\gamma = \gamma^*$. □

EXAMPLE 7. Let $H = \{a_1, \ldots, a_5\}$ be a 2-ary semihypergroup with the following table

$\circ$	a_1	a_2	a_3	a_4	a_5
a_1	A	B	B	B	a_5
a_2	B	B	B	B	a_5
a_3	B	B	B	B	a_5
a_4	B	B	B	B	a_5
a_5	B	B	B	B	a_5

where $A = \{a_2, a_3\}$ and $B = \{a_3, a_4\}$. This example show that $\beta \neq \beta^* \neq \gamma^* \neq \gamma$, and Theorem 5.11 and Proposition 5.12 are not valid for n-ary semihypergroups.

EXAMPLE 8. Suppose $(H, \circ)$ is a 2-ary hypergroup with the following table:

$\circ$	a	b	c	d	e	f	g
a	$\{a,b\}$	$\{a,b\}$	c	d	e	f	g
b	$\{a,b\}$	$\{a,b\}$	c	d	e	f	g
c	c	c	$\{a,b\}$	f	g	d	e
d	d	d	g	$\{a,b\}$	f	e	c
e	e	e	f	g	$\{a,b\}$	c	d
f	f	f	e	c	e	g	$\{a,b\}$
g	g	g	d	e	c	$\{a,b\}$	f

We have $\beta^* = \beta$ and $\gamma^* = \gamma$ but $\delta \neq \beta \neq \alpha \neq \rho$, where δ is diagonal relation and $\rho = H \times H$. We have $\beta(a) = \{a, b\}$ and for every $x \in R$, $a \neq x$, $\beta(x) = \{x\}$. And $\gamma(a) = \{a, b, g, f\}$ and $\gamma(c) = \{c, d, e\}$. In fact the fundamental 2-ary group H/β^* is isomorphic to permutation group $\mathbb{S}_3$ and the commutative fundamental 2-ary group H/γ^* is isomorphic to $\mathbb{Z}_2$.

Theorem 5.13. *Let (H, f) be an n-ary hypergroup and $\varphi : H \to H/\gamma^*$ be the canonical projection. Then, $\varphi^{-1}(\delta)$ is a $P_\sigma(H)$-part of H, where $\delta \in H/\gamma^*$.*

Proof. Let $x \in \varphi^{-1}(\delta)$ and $x \ \gamma \ y$. Then, $\varphi(x) = \delta$ and $\varphi(x) = \varphi(y)$, Thus, $\varphi(y) = \delta$ and therefore $y \in \varphi^{-1}(\delta)$. Now Proposition 4.3 show that $\varphi^{-1}(\delta)$ is a $P_\sigma(H)$-part of H. □

Lemma 5.14. *Let (z_1^m) be an m-tuple of elements of an n-ary semihypergroup (H, f). If $e_1, e_2, \ldots, e_h \in H$ are neutral elements of (H, f) and $E = \{e_1, e_2, \ldots, e_h\}$, then for every $i \in \{1, \ldots, n\}$ we have*

(1) $B(z_1^m) \subset f(\overset{(i-1)}{E}, B(z_1^m), \overset{(n-i)}{E})$,

(2) $B_\sigma(z_1^m) \subset f(\overset{(i-1)}{E}, B_\sigma(z_1^m), \overset{(n-i)}{E})$.

Proof. It is straightforward. □

Lemma 5.15. *If e_2^n is a right neutral polyad of a fundamental n-ary group $(H/\beta^*, f/\beta^*)$, then there exists a right neutral polyad for the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$.*

Proof. Let $i = 2, \ldots, n$, since $e_i \in H/\beta^*$, thus there exists $z_i \in H$ such that $e_i = \beta^*(z_i)$. Set $\varepsilon_i = \gamma^*(z_i)$, we prove that $\varepsilon_2^n \in H/\gamma^*$ is a right neutral polyad of H/γ^*. Suppose $\gamma^*(z) \in H/\gamma^*$ thus we have

$$z \in \beta^*(z) = f/\beta^*(\beta^*(z), e_2^n) = f/\beta^*(\beta^*(z), \beta^*(z_2^n)) = \beta^*(f(z, z_2^n)). \quad (4)$$

But, $\beta^*(f(z, z_2^n)) \subset \gamma^*(f(z, z_2^n))$ and since $\gamma^*(f(z, z_2^n)) = f/\gamma^*(\gamma^*(z), \varepsilon_2^n)$ is singleton in the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$, by (4) we obtain

$$\gamma^*(z) = f/\gamma^*(\gamma^*(z), \varepsilon_2^n).$$

This means ε_2^n is a right neutral polyad for the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$. □

By the similar way the next Lemma is true:

Lemma 5.16. *If $j \in \{1, \ldots, n\}$ and e_2^n is a j-neutral polyad (neutral polyad) of the fundamental n-ary group $(H/\beta^*, f/\beta^*)$, then there exists a j-neutral polyad (neutral polyad) for the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$.*

Lemma 5.17. *If an n-ary semihypergroup have a neutral element e, then $\beta(e)$ and $\gamma(e)$ are neutral elements of H/β^* and H/γ^*.*

Proof. It is straightforward. □

Lemma 5.18. *Let (H, f) be an n-ary semihypergroup. If e_β is a neutral element of the fundamental n-ary group $(H/\beta^*, f/\beta^*)$, then there exists a neutral element e_γ of the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$ such that $e_\beta \subset e_\gamma$.*

Proof. Let $e_\beta \in H/\beta^*$. Thus, there exists $z \in H$ such that $\beta^*(z) = e_\beta$. Set $e_\gamma = \gamma^*(z)$, then it is easy to see that e_γ is a neutral element of the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$ and since $\beta \subset \gamma$ thus $e_\beta \subset e_\gamma$. □

Our next theorem generalizes Proposition 3.6 in [12].

Theorem 5.19. *Let (H, f) be an n-ary hypergroup and the fundamental n-ary group $(H/\beta^*, f/\beta^*)$ has a neutral element e_β. If e_γ is a neutral element of the commutative fundamental n-ary group $(H/\gamma^*, f/\gamma^*)$ in Lemma 5.18 and $z \in w_e(H) = \phi^{-1}(e_\beta)$ and $D_e(H) = \varphi^{-1}(e_\gamma)$, then*

$$D_e(H) = \bigcup\{B_\sigma \in P_\sigma(H) | z \in B_\sigma\} = \Gamma_1(z) = [z].$$

Proof. Since the geometric space $(H, P_\sigma(H))$ is strongly transitive, the relation $\sim = \gamma$ is transitive and by Proposition 4.2 we have $\Gamma_1[z] = [z] = \gamma(z)$. Since $e_\beta \in H/\beta$ so there exists $x \in H$ such that $e_\beta = \beta^*(x)$ and $e_\gamma = \gamma^*(x)$, thus if $a \in w_e(H)$ and $b \in D_e(H)$, then $a \ \beta \ x$ and $b \ \gamma \ x$. Moreover, $z \in w_e(H) = \phi^{-1}(e_\beta)$ thus

$$\begin{aligned} y \in D_e(H) = \varphi^{-1}(e_\gamma) \quad &\Leftrightarrow y \ \gamma \ z \\ &\Leftrightarrow \exists B_\sigma \in P_\sigma(H) \ : \ \{y, z\} \in B_\sigma \\ &\Leftrightarrow y \in \bigcup\{B_\sigma \in P_\sigma(H) | z \in B_\sigma\}, \end{aligned}$$

whence $D_e(H) = \bigcup\{B_\sigma \in P_\sigma(H) | z \in B_\sigma\} = \Gamma_1(z) = [z]$. □

REMARK 3. If $n = 2$, then the fundamental group H/β^* has one identity(neutral element) $1_{H/\beta^*}$ and the commutative fundamental group H/γ^* has one identity(neutral element) $1_{H/\gamma^*}$. The $w_H = \phi^{-1}(1_{H/\beta^*})$ is called *heart* of H, and $D(H) = \varphi^{-1}(1_{H/\beta^*})$ is called *derived hypergroup* of H. Freni in [11, 12] shows that for every hypergroup H,

$$D(H) = \bigcup\{B_\sigma \in P_\sigma(H) | z \in B_\sigma\}.$$

If G is a group, then

$$\begin{aligned} &G' = D(G) \\ &= \{x \in G | \exists n \in \mathbb{N}, \exists (z_1, \ldots, z_n) \in H, \exists \sigma \in \mathbb{S}_n : 1_g = \prod_{i=1}^{n} x_i, \ x = \prod_{i=1}^{n} x_{\sigma(i)}\}, \end{aligned}$$

where G' is derived subgroup of G.

Lemma 5.20. *In an n-ary semihypergroup (H, f), if $e \in H$ is a scalar right neutral and a 2-neutral element, then $e \in H$ is a neutral element.*

Proof. We know e is 1-neutral and 2-neutral element. We show that e is 3-neutral element. Indeed,

$$x \in f(e, x, \overset{(n-2)}{e}) \subset f(e, f(e, x, \overset{(n-2)}{e}), \overset{(n-2)}{e})$$
$$= f(e, e, f(x, \overset{(n-1)}{e}), \overset{(n-3)}{e}) = f(e, e, x, \overset{(n-3)}{e}).$$

Iterating this procedure we can see that e is i-neutral element for every $i = 2, \ldots, n$. So, e is a neutral element of (H, f). □

Lemma 5.21. *In an n-ary semihypergroup (H, f), if $e \in H$ is a $(n-1)$-neutral element and scalar left neutral, then it is a neutral element.*

We say that an n-ary hypergroupoid (H, f) is *b-derived* from a binary hypergroupoid $(H, \circ)$ and denote this fact by $(H, f) = der_b(H, \circ)$ if the hyperoperation f has the form

$$f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n \circ b.$$

We say that (H, f) is *derived* from $(H, \circ)$ and denote this fact by $(H, f) = der(H, \circ)$, if the hyperoperation f has the form

$$f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n.$$

It is clear that if b belongs to the center of a semihypergroup (hypergroup) $(H, \circ)$, then the n-ary hypergroupoid b-derived from $(H, \circ)$ is an n-ary semihypergroup (hypergroup). If $(H, \circ)$ is a semihypergroup (hypergroup), then the n-ary hypergroupoid derived from $(H, \circ)$ is an n-ary semihypergroup (hypergroup).

Lemma 5.22. *If (H, f) is an n-ary semihypergroup derived(b-derived), where b belongs to the center of H, from a semihypergroup $(H, \circ)$, then we have*

1) $\beta_{(H,f)} \subset \beta_{(H,\circ)}$,

2) $\gamma_{(H,f)} \subset \gamma_{(H,\circ)}$.

Proof. Let (H, f) be an n-ary semihypergroup b-derived from a hypergroup $(H, \circ)$, we prove that (2). The proof of (1) is similar. Suppose $x \ \gamma_{(H,f)} \ y$, thus there exist $k \in \mathbb{N} \cup \{0\}$, elements $z_1^m \in H$ and permutation $\sigma \in \mathbb{S}_m$, such that

$$x \in f_{(k)}(z_1^m) \quad \text{and} \quad y \in f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$$

where $m = k(n-1) + 1$. Hence, we obtain

$$x \in \ f_{(k)}(z_1^m) = z_1 \circ \ldots \circ z_n \circ b \circ z_n + 1 \circ \ldots \circ z_{2n-1} \circ b \circ \ldots \circ z_m \circ b$$
and
$$y \in \ f_{(k)}(z_{\sigma(1)}^{\sigma(m)})$$
$$= z_{\sigma(1)} \circ \ldots \circ z_{\sigma(n)} \circ b \circ z_{\sigma(n+1)} \circ \ldots \circ z_{\sigma(2n-1)} \circ b \circ \ldots \circ z_{\sigma(m)} \circ b.$$

Now, it is easy to see that $x\ \gamma_{(H,\circ)}\ y$. If (H,f) is an n-ary semihypergroup derived from a hypergroup $(H,\circ)$, the proof is similar. □

Theorem 5.23. *If (H,f) is an n-ary hypergroup derived from a hypergroup $(H,\circ)$, then we have*

1) $\beta_{(H,\circ)} = \beta_{(H,f)}$,

2) $\gamma_{(H,\circ)} = \gamma_{(H,f)}$.

Proof. Since (H,f) is an n-ary semihypergroup, then Lemma 5.22 shows that

1′) $\beta_{(H,f)} \subset \beta_{(H,\circ)}$,

2′) $\gamma_{(H,f)} \subset \gamma_{(H,\circ)}$.

Conversely, let $x\ \gamma_{(H,\circ)}\ y$ we prove that $x\ \gamma_{(H,f)}\ y$. Since $x\ \gamma_{(H,\circ)}\ y$ thus there exist $l \in \mathbb{N}$, $z_1^l \in H$ and permutation $\sigma \in \mathbb{S}_l$ such that

$$x \in \prod_{i=1}^{l} z_i \quad \text{and} \quad y \in \prod_{i=1}^{l} z_{\sigma(i)}.$$

There exists $k \in \mathbb{N} \cup \{0\}$ such that $k(n-1)+1 \leq l < (k+1)(n-1)+1$. Set $m = (k+1)(n-1)+1$, Since $(H,\circ)$ is a hypergroup thus there exist $z'_l, \ldots, z'_m \in H$ such that $z_l \in z'_l \circ \ldots \circ z'_m$. Suppose $\sigma(j) = l$ and set $z'_i = z_i$ for every $i \in \{1, \ldots, l-1\}$ and give permutation $\tau \in \mathbb{S}_m$ as follows:

$$\begin{cases} \tau(i) = \sigma(i), & \text{if } i \in \{1, \ldots, j\} \\ \tau(i) = \sigma(j) + i - j, & \text{if } i \in \{j+1, \ldots, j+m-l\} \\ \tau(i) = \sigma(i-m+l), & \text{if } i \in \{j+m-l+1, \ldots, m\}. \end{cases}$$

Therefore, we obtain

$$x \in \prod_{i=1}^{m} z'_i \quad \text{and} \quad y \in \prod_{i=1}^{m} z'_{\sigma(i)},$$

and so we have

$$x \in f_{(k+1)}(z'^{m}_{1}) \quad \text{and} \quad y \in f_{(k+1)}(z'^{\tau(m)}_{\tau(1)}).$$

Hence, $x\ \gamma_{(H,f)}\ y$. Therefore, $\gamma_{(H,\circ)} \subset \gamma_{(H,f)}$, and by (2′) we obtain $\gamma_{(H,\circ)} = \gamma_{(H,f)}$. Similarly, using (1′) we have $\beta_{(H,f)} = \beta_{(H,\circ)}$. □

REMARK 4. If $(H, \circ)$ is a semihypergroup, the preceding result is not valid in general as it is proved from the structure whose table is given below.

$\circ$	a	b	c	d
a	$\{b, c\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

If $n \geq 3$ consider an n-ary hypergroup (H, f) derived from $(H, \circ)$ with hyperoperation $f(x_1, \ldots, x_n) = \{b, d\}$ for every $x_1, \ldots, x_n \in H$. Evidently $b\ \beta_{(H,\circ)}\ c$ but not $b\ \beta_{(H,f)}\ c$. Also, $b\ \beta^*_{(H,\circ)}\ c$, but not $b\ \beta^*_{(H,f)}\ c$.

EXAMPLE 9. Let $H = \{a, b, c\}$ and $\circ$ be a hyperoperation defined on H as follows:

$\circ$	a	b	c
a	$\{b, c\}$	c	c
b	c	c	c
c	c	c	c

H is a semihypergroup, since $x \circ (y \circ z) = (x \circ y) \circ z = c$ for every $x, y, z \in H$. If $n \geq 3$, then n-ary semihypergroup (H, f) derived (b-derived) from $(H, \circ)$ is an n-ary semigroup and $f(x_1^n) = c$ for every $x_1^n \in H$. We have $\beta_{(H,\circ)} = \gamma_{(H,\circ)}$ and $b\ \gamma_{(H,\circ)}\ c$ but $b\ \not\gamma_{(H,f)}\ c$. Thus, $\gamma_{(H,f)} \neq \gamma_{(H,\circ)}$ and $\beta_{(H,f)} \neq \beta_{(H,\circ)}$.

Theorem 5.24. *Let b be an element in the center of a hypergroup $(H, \circ)$. If (H, f) is an n-ary hypergroup b-derived from the hypergroup $(H, \circ)$, then we have*

1) $\beta_{(H,\circ)} = \beta_{(H,f)}$,

2) $\gamma_{(H,\circ)} = \gamma_{(H,f)}$.

Proof. It is similar to the proof of Theorem 5.23. □

Theorem 5.25. *If $(H, \circ)$ is a semihypergroup with right(left) neutral element e, i.e, $x \in x \circ e (x \in e \circ x)$ for every $x \in H$. If (H, f) is an n-ary semihypergroup derived from $(H, \circ)$, then we have*

1) $\beta_{(H,\circ)} = \beta_{(H,f)}$,

2) $\gamma_{(H,\circ)} = \gamma_{(H,f)}$.

Proof. Let (H, f) be an n-ary semihypergroup derived from semihypergroup $(H, \circ)$ and $(H, \circ)$ has a right neutral element e. We prove (2). The proof of (1) is similar. Since (H, f) is an n-ary semihypergroup, hence

Lemma 5.22 shows that $\gamma_{(H,f)} \subset \gamma_{(H,\circ)}$. Conversely, let $x\ \gamma_{(H,\circ)}\ y$ we prove that $x\ \gamma_{(H,f)}\ y$. Since $x\ \gamma_{(H,\circ)}\ y$ thus there exist $l \in \mathbb{N}$, $z_1^l \in H$ and permutation $\sigma \in \mathbb{S}_l$ such that

$$x \in \prod_{i=1}^{l} z_i \quad \text{and} \quad y \in \prod_{i=1}^{l} z_{\sigma(i)}.$$

There exists $k \in \mathbb{N} \cup \{0\}$ such that $k(n-1)+1 \leq l < (k+1)(n-1)+1$. Set $m = (k+1)(n-1)+1$, since $(H,\circ)$ has a right neutral element e, thus

$$z_l \in z_l \circ e \subset z_l \circ e \circ e \subset \ldots \subset z_l \circ \underbrace{e \circ \ldots \circ e}_{(m-l)-times}.$$

Suppose $\sigma(j) = l$ and set z'_i and permutation $\tau \in \mathbb{S}_m$ as follow:

$$\begin{cases} z'_i = z_i, & \text{if } i \in \{1,\ldots,l\} \\ \\ z'_i = e, & \text{if } i \in \{l+1,\ldots,m\} \end{cases}$$

and

$$\begin{cases} \tau(i) = \sigma(i), & \text{if } i \in \{1,\ldots,j\} \\ \tau(i) = \sigma(j)+i-j, & \text{if } i \in \{j+1,\ldots,j+m-l\} \\ \tau(i) = \sigma(i-m+l), & \text{if } i \in \{j+m-l+1,\ldots,m\}. \end{cases}$$

Therefore, we obtain

$$x \in \prod_{i=1}^{m} z'_i \quad \text{and} \quad y \in \prod_{i=1}^{m} z'_{\sigma(i)}$$

and so we have

$$x \in f_{(k+1)}({z'}_1^m) \quad \text{and} \quad y \in f_{(k+1)}({z'}_{\tau(1)}^{\tau(m)}).$$

Hence, $x\ \gamma_{(H,f)}\ y$. Therefore, $\gamma_{(H,\circ)} = \gamma_{(H,f)}$. If (H,f) has a left neutral, then the proof is similar to the above. By the similar way, we have $\beta_{(H,f)} = \beta_{(H,\circ)}$. □

In the similar way the next Theorem is true.

Theorem 5.26. *If $(H,\circ)$ is a semihypergroup with right(left) neutral element e and b. If (H,f) is an n-ary semihypergroup b-derived from a semihypergroup $(H,\circ)$ where b is in the center of H, then we have*

1) $\beta_{(H,\circ)} = \beta_{(H,f)}$,

2) $\gamma_{(H,\circ)} = \gamma_{(H,f)}$.

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