Large sets of oriented P_3-decompositions of directed complete bipartite graphs*

Yanfang Zhang[1], Qi Wang[2], Feifei Fan[3]
1. College of Mathematics and Statistics
Hebei University of Economics and Business
Shijiazhuang 050061, P.R. China
yanfang_zh@163.com
2. Graduate School
Hebei University of Economics and Business
Shijiazhuang 050061, P.R. China
3. School of Mathematics and Physics
North China Electric Power University
Beijing 102206, P.R. China

Abstract. Let H,G be two graphs (or digraphs), where G is a subgraph of H. A G-decomposition of H, denoted by (H, G)-GD, is a partition of all the edges (or arcs) of H into subgraphs (G-blocks), each of which is isomorphic to G. A large set of (H, G)-GD, denoted by (H, G)-LGD, is a partition of all subgraphs isomorphic to G of H into (H, G)-GDs. In this paper, we obtain the existence spectrums of $(\lambda DK_{m,n}, P_3^i)$-$LGD$, where P_3^i $(i = 1, 2, 3)$ are the three types of oriented P_3.

Keywords: large set; G-decomposition; oriented path graph; directed complete bipartite graph

1 Introduction

Let $G = (V(G), E(G))$ be a graph, where each edge in $E(G)$ is denoted by an unordered pair $\{u, v\}$, $u, v \in V(G)$. The *degree* $d_G(v)$ of a vertex v

*Research supported by NSFC Grant 10901051, NSFHB Grant A2010001481, NSFHB Grant A2011207003 and the Fundamental Research Funds for the Central Universities (No.10ML37).

in G is $|\{u : \{u,v\} \in E(G)\}|$. A graph G is *r-regular* if $d_G(v) = r$ for all $v \in V(G)$; a *regular graph* is r-regular for some r. A graph G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A *spanning subgraph* of H is a subgraph G with $V(G) = V(H)$. Let $H = (V(H), A(H))$ be a digraph, where each arc in $A(H)$ is denoted by an ordered pair (u,v), $u,v \in V(H)$. A digraph G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $A(G) \subseteq A(H)$. The *indegree* $d_D^-(v)$ of a vertex v in D is $|\{x : (x,v) \in A(D)\}|$, and the *outdegree* $d_D^+(v)$ of v is $|\{y : (v,y) \in A(D)\}|$. Let G be a graph (or digraph), λ be a positive integer, we use λG to denote the multigraph obtained from G by repeating each edge (arc) λ times.

In this paper, K_n is the complete graph on n vertices, where any two distinct vertices x and y of K_n are joined by exactly one edge $\{x,y\}$, $K_{m,n}$ is the complete bipartite graph with two parts X and Y of cardinalities m and n respectively, where any vertex x in X and any vertex y in Y are joined by exactly one edge $\{x,y\}$, $C_k = (x_1, x_2, \cdots, x_k)$ is a cycle of length k, DK_n is the complete symmetric directed graph of order n, where any two distinct vertices x and y of DK_n are joined by exactly two arcs (x,y) and (y,x), $DK_{m,n}$ is the directed complete bipartite graph with two parts X and Y of cardinalities m and n respectively, where any vertex x in X and any vertex y in Y are joined by exactly two arcs (x,y) and (y,x).

Let H,G be two graphs (or digraphs), where G is a subgraph of H. A G-decomposition of H, denoted by (H,G)-*GD*, is a partition of all the edges (or arcs) of H into subgraphs (G-blocks), each of which is isomorphic to G. A large set of (H,G)-*GD*, denoted by (H,G)-*LGD*, is a partition of all subgraphs isomorphic to G of H into (H,G)-*GDs*.

For the undirected cycle C_k, if each edge is oriented, then we get the oriented C_k. There are two types of oriented C_3:

cyclic triangle $\overrightarrow{C_3}$ transitive triangle TT_3

If each edge of the path P_k is oriented, then we get the oriented P_k. There are three types of oriented P_3:

P_3^1 P_3^2 P_3^3

For the oriented pentagons and the cyclic cycle, the existence problems of their graph designs have been researched (see [1],[9],[2]). The large set (K_n, C_3)-*LGD* (that is large set of Steiner triple system $LSTS(n)$) has been completely solved (see [7],[8],[10]). For two types of oriented C_3, $\overrightarrow{C_3}$ and TT_3, the large set $(DK_n, \overrightarrow{C_3})$-*LGD* (that is large set of Mendelsohn triple system $LMTS(n)$) and the large set (DK_n, TT_3)-*LGD* (that is large set of transitive triple system $LDTS(n)$) have been completely solved(see [5] and [4]). For path P_3 and three types of oriented P_3^i (i=1,2,3), the existence spectrums of $(\lambda K_n, P_3)$-LGD and $(\lambda DK_n, P_3^i)$-LGD have been obtained in [6] and [11]. Not a long time ago, the existence problem of $(\lambda K_{m,n}, P_3)$-LGD (that is large set of P_3-decompositions of complete bipartite graph) was solved (see [12]). It is easy to know that $(\lambda DK_{m,n}, P_3^1)$-$GD$ (or $(\lambda DK_{m,n}, P_3^1)$-$LGD$) is equivalent to $(\lambda DK_{m,n}, P_3^2)$-$GD$ (or $(\lambda DK_{m,n}, P_3^2)$-$LGD$), so we only discuss the existence of $(\lambda DK_{m,n}, P_3^1)$-$LGD$ and $(\lambda DK_{m,n}, P_3^3)$-$LGD$. In this paper, we investigate the existence of $(\lambda DK_{m,n}, P_3^i)$-$LGD$ and obtain their existence spectrums, where P_3^i $(i = 1, 2, 3)$ are the three types of oriented P_3.

2 $(\lambda DK_{m,n}, P_3^1)$-$LGD$

An *r-factor* of K_v is an r-regular spanning subgraph of K_v. If all the edges of K_v can be partitioned into some r-factors, then we say that K_v has an *r-factorization*. If $k = |V(G)|$, then the cycle C_k is called a *Hamilton cycle* of the graph G. Obviously, a *Hamilton cycle* of a graph G must be a 2-factor of G.

Lemma 2.1 [3] *For any positive integer $n \geq 1$,*
(1) *there exists a 1-factorization of K_{2n} ;*
(2) *there exists a Hamilton cycle decomposition of K_{2n+1}.*

Lemma 2.2 *There exists a $(\lambda DK_{m,n}, P_3^1)$-$GD$ only if*

$$\begin{cases} \lambda \text{ odd}, & m \text{ and } n \text{ are both even;} \\ \lambda \text{ even}, & m \geq 1,\ n \geq 1 \text{ and } m+n \geq 3. \end{cases}$$

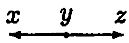
Proof. First, the digraph P_3^1 has two arcs. And, $d^-(P_3^1) = 1$, $d^+(P_3^1) = 2$, where $d^-(P_3^1)$ (or $d^+(P_3^1)$) is the greatest common divisor of all the indegrees (or outdegrees) of vertices in P_3^1. So it is easy to know that if there exists a $(\lambda DK_{m,n}, P_3^1)$-$GD$, then

$$\begin{cases} m+n \geq 3 \\ 2\lambda mn \equiv 0 \pmod 2 \\ \lambda m \equiv 0 \pmod 1 \text{ and } \lambda n \equiv 0 \pmod 1 \\ \lambda m \equiv 0 \pmod 2 \text{ and } \lambda n \equiv 0 \pmod 2 \end{cases}$$

that is

$$\begin{cases} \lambda \text{ odd}, & m \text{ and } n \text{ are both even;} \\ \lambda \text{ even}, & m \geq 1,\ n \geq 1 \text{ and } m+n \geq 3. \end{cases} \qquad \square$$

For convenience, in this section, the following P_3^1-block is denoted by $[x, y, z]_1$:



Obviously, the block $[x, y, z]_1$ contains two arcs (y, x) and (y, z).

Let $Z_m, \bar{Z}_n$ be two partite sets of $K_{m,n}$. Define two P_3^1-block families in $DK_{m,n}$ as follows:

$$\mathcal{P}(m,n) = \{[a, y, b]_1 : a \neq b \in Z_m, y \in \bar{Z}_n\}$$
$$\mathcal{Q}(m,n) = \{[c, x, d]_1 : c \neq d \in \bar{Z}_n, x \in Z_m\}$$

It is easy to see that

$$|\mathcal{P}(m,n)| = \binom{m}{2}n = \frac{mn(m-1)}{2}, \ |\mathcal{Q}(m,n)| = \binom{n}{2}m = \frac{mn(n-1)}{2}.$$

And, $|\mathcal{P}(m,n)| + |\mathcal{Q}(m,n)| = \frac{mn(m+n-2)}{2}$ is just the number of distinct P_3^1-blocks in $DK_{m,n}$. Obviously, a $(\lambda DK_{m,n}, P_3^1)$-$GD$ consists of λmn P_3^1-blocks, a $(\lambda DK_{m,n}, P_3^1)$-$LGD$ contains $\frac{m+n-2}{2\lambda}$ pairwise disjoint $(\lambda DK_{m,n}, P_3^1)$-$GD$s. Combining with Lemma 2.2, we have

Lemma 2.3 *There exists a $(\lambda DK_{m,n}, P_3^1)$-$LGD$ only if*

$$\begin{cases} 2\lambda | (m+n-2); \\ \lambda \ \textit{odd}, \ m = n \ \textit{are both even},; \\ \lambda \ \textit{even}, \ m = n \ \textit{are both even}, \ \textit{or} \ m = n \geq 3 \ \textit{are both odd}. \end{cases}$$

Proof. Firstly, a $(\lambda DK_{m,n}, P_3^1)$-$LGD$ contains $\frac{m+n-2}{2\lambda}$ pairwise disjoint $(\lambda DK_{m,n}, P_3^1)$-$GD$s, so we have $2\lambda | (m+n-2)$.

Furthermore, let $Z_m, \bar{Z}_n$ be two partite sets of $K_{m,n}$, for a fixed point x in the set Z_m, because its outdegree in each $(\lambda DK_{m,n}, P_3^1)$-$GD$ (called *small set*) is λn, a small set contains $\frac{\lambda n}{2}$ P_3^1-blocks of the type $[a, x, b]_1$, where $a \neq b$ and $a, b \in \bar{Z}_n$, the total number of the type $[a, x, b]_1$ in $DK_{m,n}$ is $\binom{n}{2}$, the number of the small set is $\frac{m+n-2}{2\lambda}$, therefore we get

$$\frac{\lambda n}{2} \times \frac{m+n-2}{2\lambda} = \binom{n}{2} \quad *$$

By $*$, we have $m = n$.

Finally, combining with the necessary conditions of a small set (i.e. Lemma 2.2), we draw the conclusion of the Lemma. □

Therefore, in order to determine the existence spectrum of $(\lambda DK_{m,n}, P_3^1)$-$LGD$, it is enough to construct $(DK_{2t,2t}, P_3^1)$-LGD and $(2DK_{2t+1,2t+1}, P_3^1)$-LGD for any positive integer t.

Lemma 2.4 *There exists a $(DK_{2t,2t}, P_3^1)$-LGD for any $t > 0$.*

Proof. By Lemma 2.1, there exist a 1-factorization $\{f_1, f_2, \cdots, f_{2t-1}\}$ of K_{2t} on Z_{2t} and a 1-factorization $\{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_{2t-1}\}$ of K_{2t} on $\bar{Z}_{2t}$. Define

$$\mathcal{A}_i = \{[a, y, b]_1 : \{a, b\} \in f_i, y \in \bar{Z}_{2t}\}, \quad i = 1, 2, \cdots, 2t-1,$$
$$\mathcal{B}_i = \{[c, x, d]_1 : \{c, d\} \in \bar{f}_i, x \in Z_{2t}\}, \quad i = 1, 2, \cdots, 2t-1.$$

It is easy to verify that each $(Z_{2t} \bigcup \bar{Z}_{2t}, \ \mathcal{A}_i \bigcup \mathcal{B}_i)$ is a $(DK_{2t,2t}, P_3^1)$-GD for $i = 1, 2, \cdots, 2t-1$.

Furthermore, the family $\{\mathcal{A}_i : i = 1, 2, \cdots, 2t-1\}$ just forms a partition of all P_3^1-blocks in $\mathcal{P}(2t, 2t)$, and the family $\{\mathcal{B}_i : i = 1, 2, \cdots, 2t-1\}$ just forms a partition of all P_3^1-blocks in $\mathcal{Q}(2t, 2t)$. Therefore, $\{\mathcal{A}_1 \bigcup \mathcal{B}_1, \mathcal{A}_2 \bigcup \mathcal{B}_2, \cdots, \mathcal{A}_{2t-1} \bigcup \mathcal{B}_{2t-1}\}$ forms a $(DK_{2t,2t}, P_3^1)$-LGD on $Z_{2t} \bigcup \bar{Z}_{2t}$. □

Example 2.5 *A* $(DK_{2,2}, P_3^1)$-$LGD = \{(Z_2 \bigcup \bar{Z}_2, \mathcal{C})\}$, *where*
$\mathcal{A}_1 = \{\ [0, \bar{0}, 1]_1, [0, \bar{1}, 1]_1\}$, $\mathcal{B}_1 = \{[\bar{0}, 0, \bar{1}]_1, \bar{0}, 1, \bar{1}]_1\}$,
$\mathcal{C} = \mathcal{A}_1 \bigcup\ \mathcal{B}_1$.

Lemma 2.6 *There exists a* $(2DK_{2t+1,2t+1}, P_3^1)$-LGD *for any* $t > 0$.

Proof. By Lemma 2.1, there exist a Hamilton cycle decomposition $\{f_1, f_2, \cdots, f_t\}$ of K_{2t+1} on Z_{2t+1} and a Hamilton cycle decomposition $\{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_t\}$ of K_{2t+1} on $\bar{Z}_{2t+1}$. Clockwise orient the edges of each Hamilton cycle so that each vertex appears once as the head of an arc and once as the tail of another arc in each Hamilton cycle. Define

$$\mathcal{A}_i = \{[a, y, b]_1 : (a, b) \in f_i, y \in \bar{Z}_{2t+1}\},\quad i = 1, 2, \cdots, t,$$
$$\mathcal{B}_i = \{[c, x, d]_1 : (c, d) \in \bar{f}_i, x \in Z_{2t+1}\},\quad i = 1, 2, \cdots, t.$$

It is easy to verify that each $(Z_{2t+1} \bigcup \bar{Z}_{2t+1},\ \mathcal{A}_i \bigcup \mathcal{B}_i)$ is a $(2DK_{2t+1,2t+1}, P_3^1)$-GD for $i = 1, 2, \cdots, t$.

Furthermore, the family $\{\mathcal{A}_i : i = 1, 2, \cdots, t\}$ just forms a partition of all P_3^1-blocks in $\mathcal{P}(2t+1, 2t+1)$, and the family $\{\mathcal{B}_i : i = 1, 2, \cdots, t\}$ just forms a partition of all P_3^1-blocks in $\mathcal{Q}(2t+1, 2t+1)$. Therefore, $\{\mathcal{A}_1 \bigcup \mathcal{B}_1, \mathcal{A}_2 \bigcup \mathcal{B}_2, \cdots, \mathcal{A}_t \bigcup \mathcal{B}_t\}$ forms a $(2DK_{2t,2t}, P_3^1)$-LGD on $Z_{2t+1} \bigcup \bar{Z}_{2t+1}$. □

Example 2.7 *A* $(2DK_{3,3}, P_3^1)$-$LGD = \{(Z_3 \bigcup \bar{Z}_3, \mathcal{C})\}$, *where*
$f_1 = (0, 1, 2)$, $\bar{f}_1 = (\bar{0}, \bar{1}, \bar{2})$,
$\mathcal{A}_1 :\ [0, \bar{0}, 1]_1,\ [1, \bar{0}, 2]_1,\ [2, \bar{0}, 0]_1,\ [0, \bar{1}, 1]_1,\ [1, \bar{1}, 2]_1,\ [2, \bar{1}, 0]_1,\ [0, \bar{2}, 1]_1,\ [1, \bar{2}, 2]_1,\ [2, \bar{2}, 0]_1,$
$\mathcal{B}_1 :\ [\bar{0}, 0, \bar{1}]_1,\ [\bar{1}, 0, \bar{2}]_1,\ [\bar{2}, 0, \bar{0}]_1,\ [\bar{0}, 1, \bar{1}]_1,\ [\bar{1}, 1, \bar{2}]_1,\ [\bar{2}, 1, \bar{0}]_1,\ [\bar{0}, 2, \bar{1}]_1,\ [\bar{1}, 2, \bar{2}]_1,\ [\bar{2}, 2, \bar{0}]_1.$
$\mathcal{C} = \mathcal{A}_1 \bigcup \mathcal{B}_1$.

Theorem 2.8 *There exists a* $(\lambda DK_{m,n}, P_3^1)$-$LGD$ *if and only if*

$$\begin{cases} 2\lambda | (m + n - 2); \\ \lambda \ \textit{odd},\ m = n \ \textit{are both even},; \\ \lambda \ \textit{even},\ m = n \ \textit{are both even, or}\ m = n \geq 3 \ \textit{are both odd}. \end{cases}$$

Proof. By Lemma 2.3, we only need to prove the sufficiency.

If $m = n$ are both even. Let $m = n = 2t$. For any $t > 0$, there exists a $(DK_{2t,2t}, P_3^1)$-$LGD = \{Z_{2t} \bigcup \bar{Z}_{2t}, C_i : 1 \leq i \leq 2t-1\}$ by Lemma 2.4. Define

$$\mathcal{D}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{C}_i,\ 0 \le k \le \tfrac{2t-1}{\lambda} - 1,$$

then $\{Z_{2t} \bigcup \bar{Z}_{2t}, \mathcal{D}_k :\ 0 \le k \le \frac{2t-1}{\lambda} - 1\}$ is a $(\lambda DK_{2t,2t}, P_3^1)$-$LGD$.

If $m = n \ge 3$ are both odd and λ even. Let $m = n = 2t + 1$. There exists a $(2DK_{2t+1,2t+1}, P_3^1)$-LGD $= \{Z_{2t+1} \bigcup \bar{Z}_{2t+1}, \mathcal{C}_i :\ 1 \le i \le t\}$ by Lemma 2.6. Define

$$\mathcal{D}_k = \bigcup_{i=k\frac{\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{C}_i,\ 0 \le k \le \tfrac{2t}{\lambda} - 1,$$

then $\{Z_{2t+1} \bigcup \bar{Z}_{2t+1}, \mathcal{D}_k :\ 0 \le k \le \frac{2t}{\lambda} - 1\}$ is a $(\lambda DK_{2t+1,2t+1}, P_3^1)$-$LGD$.
□

3 $(\lambda DK_{m,n}, P_3^3)$-$LGD$

For convenience, in this section, the following P_3^3-block is denoted by $[x, y, z]_3$:

$$x \longrightarrow y \longrightarrow z$$

Obviously, the block $[x, y, z]_3$ contains two arcs (x, y) and (y, z).

Let $Z_m, \bar{Z}_n$ be two partite sets of $K_{m,n}$. Define two P_3^3-block families in $DK_{m,n}$ as follows:

$$\mathcal{P}(m, n) = \{[a, y, b]_3 : a \ne b \in Z_m, y \in \bar{Z}_n\}$$
$$\mathcal{Q}(m, n) = \{[c, x, d]_3 : c \ne d \in \bar{Z}_n, x \in Z_m\}$$

It is easy to see that

$$|\mathcal{P}(m, n)| = n \times m(m-1) = nm(m-1),$$
$$|\mathcal{Q}(m, n)| = m \times n(n-1) = mn(n-1).$$

And, $|\mathcal{P}(m, n)| + |\mathcal{Q}(m, n)| = mn(m + n - 2)$ is just the number of distinct P_3^3-blocks in $DK_{m,n}$. Obviously, a $(\lambda DK_{m,n}, P_3^3)$-$GD$ consists of λmn P_3^3-blocks, a $(\lambda DK_{m,n}, P_3^3)$-$LGD$ contains $\frac{m+n-2}{\lambda}$ pairwise disjoint $(\lambda DK_{m,n}, P_3^3)$-$GD$s. So we have

Lemma 3.1 *There exists a $(\lambda DK_{m,n}, P_3^3)$-$LGD$ only if $\lambda | (m + n - 2)$.*

Therefore, in order to determine the existence spectrum of $(\lambda DK_{m,n}, P_3^3)$-$LGD$, it is enough to construct $(DK_{2m,2n}, P_3^3)$-LGD, $(DK_{2m,2n+1}, P_3^3)$-LGD and $(DK_{2m+1,2n+1}, P_3^3)$-LGD.

Lemma 3.2 [12] *There exist a $(K_{2m,2n}, P_3)$-LGD for any $m > 0$ and $n > 0$.*

Lemma 3.3 [12] *There exist a $(K_{2m,2n+1}, P_3)$-LGD for any $m \ge 1$ and $n \ge 0$.*

Lemma 3.4 *There exist a $(DK_{2m,2n}, P_3^3)$-LGD for any $m > 0$ and $n > 0$ and a $(DK_{2m,2n+1}, P_3^3)$-LGD for any $m \ge 1$ and $n \ge 0$.*

Proof. By Lemma 3.2, there exists a $(K_{2m,2n}, P_3)$-*LGD* for any $m > 0$ and $n > 0$, which consists of $2m + 2n - 2$ $(K_{2m,2n}, P_3)$-*GD*s. In each $(K_{2m,2n}, P_3)$-*GD*, for every P_3-block (a, b, c), we get two P_3^3-blocks $[a, b, c]_3$ and $[c, b, a]_3$ by assigning the orientation, so we can obtain a $(K_{2m,2n}, P_3^3)$-*GD*. By this means, we will obtain $2m + 2n - 2$ disjoint P_3^3-*GD*s, which form a $(DK_{2m,2n}, P_3^3)$-*LGD*.

By Lemma 3.3, there exists a $(K_{2m,2n+1}, P_3)$-*LGD* for any $m > 0$ and $n \geq 0$. With the same reason, we can obtain $(DK_{2m,2n+1}, P_3^3)$-*LGD* from $(K_{2m,2n+1}, P_3)$-*LGD*. □

Lemma 3.5 *There exists a* $(DK_{2m+1,2n+1}, P_3^3)$-*LGD for any* $m \geq 0$, $n \geq 0$ *and* $m + n > 0$.

Proof. By Lemma 2.1, there exist a Hamilton cycle decomposition $\{f_1, f_2, \cdots, f_m\}$ of K_{2m+1} on Z_{2m+1} and a Hamilton cycle decomposition $\{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_n\}$ of K_{2n+1} on $\bar{Z}_{2n+1}$. Clockwise orient the edges of each Hamilton cycle f_i, we get a directed Hamilton cycle f_i^+, Counter clockwise orient the edges of each Hamilton cycle f_i, we get a directed Hamilton cycle f_i^-. By the same means, we can obtain two directed Hamilton cycle $\bar{f}_j^+$ and $\bar{f}_j^-$ from each Hamilton cycle $\bar{f}_j$. Define

$$\mathcal{A}_i^+ = \{[a, y, b]_3 : (a, b) \in f_i^+, y \in \bar{Z}_{2n+1}\}, \quad i = 1, 2, \cdots, m,$$
$$\mathcal{A}_i^- = \{[a, y, b]_3 : (a, b) \in f_i^-, y \in \bar{Z}_{2n+1}\}, \quad i = 1, 2, \cdots, m,$$
$$\mathcal{B}_j^+ = \{[c, x, d]_3 : (c, d) \in \bar{f}_i^+, x \in Z_{2m+1}\}, \quad j = 1, 2, \cdots, n.$$
$$\mathcal{B}_j^- = \{[c, x, d]_3 : (c, d) \in \bar{f}_i^-, x \in Z_{2m+1}\}, \quad j = 1, 2, \cdots, n.$$

It is easy to verify that each of $(Z_{2m+1} \bigcup \bar{Z}_{2n+1}, \mathcal{A}_i^+)$, $(Z_{2m+1} \bigcup \bar{Z}_{2n+1}, \mathcal{A}_i^-)$, $(Z_{2m+1} \bigcup \bar{Z}_{2n+1}, \mathcal{B}_j^+)$ and $(Z_{2m+1} \bigcup \bar{Z}_{2n+1}, \mathcal{B}_j^-)$ is a $(DK_{2m+1,2n+1}, P_3^3)$-*GD* for $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$.

Furthermore, the family $\{\mathcal{A}_i^+ : i = 1, 2, \cdots, m\} \bigcup \{\mathcal{A}_i^- : i = 1, 2, \cdots, m\}$ just forms a partition of all P_3^3-blocks in $\mathcal{P}(2m + 1, 2n + 1)$, and the family $\{\mathcal{B}_j^+ : j = 1, 2, \cdots, n\} \bigcup \{\mathcal{B}_j^- : j = 1, 2, \cdots, n\}$ just forms a partition of all P_3^3-blocks in $\mathcal{Q}(2m + 1, 2n + 1)$. Therefore, $\{\mathcal{A}_1^+, \mathcal{A}_1^-, \cdots, \mathcal{A}_m^+, \mathcal{A}_m^-, \mathcal{B}_1^+, \mathcal{B}_1^-, \cdots, \mathcal{B}_n^+, \mathcal{B}_n^-\}$ forms a $(DK_{2m+1,2n+1}, P_3^3)$-*LGD* on $Z_{2m+1} \bigcup \bar{Z}_{2n+1}$. □

Theorem 3.6 *There exists a* $(\lambda DK_{m,n}, P_3^3)$-*LDGD if and only if* $\lambda | (m + n - 2)$.

Proof. By Lemma 3.1, we only need to prove the sufficiency.

By Lemma 3.4 and Lemma 3.5, there exists a $(DK_{m,n}, P_3^3)$-*LGD* $= \{Z_m \bigcup \bar{Z}_n, \mathcal{C}_i : 1 \leq i \leq m + n - 2\}$. Define

$$\mathcal{D}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{C}_i, \; 0 \leq k \leq \tfrac{m+n-2}{\lambda} - 1,$$

then $\{Z_m \bigcup \bar{Z}_n, \mathcal{D}_k : 0 \leq k \leq \frac{m+n-2}{\lambda} - 1\}$ is a $(\lambda DK_{2m+1,2n+1}, P_3^3)$-*LGD*.
□

References

[1] Brian Alspach, Katherine Heinrich and Badri N. Varma, *Decompositions of complete symmetric digraphs into the oriented pentagons*, J. Aust. Math. Soc. Ser. A 28(1979), 353-361.

[2] Brian Alspach, Heather Gavlas, Mateja Sajna and Helen Verrall, *Cycle decompositions IV: Complete directed graphs and fixed length directed cycles*, J. Combin. Theory Ser. A 103(2003), 165-208.

[3] C. J. Colbourn and J. H. Dinitz, *The CRC Handbook of Combinatorial Designs*, *CRC* Press, Inc., Boca Raton, 2006.

[4] Qingde Kang and Yanxun Chang, *A completion of the spectrum for large sets of disjoint transitive triple systems*, J. Combin. Theory Ser. A 60(1992), 287-294.

[5] Qingde Kang, Jianguo Lei and Yanxun Chang, *The spectrum for large sets of disjoint Mendelsohn triple system with any index*, J. Combin. Des. 2(1994), 351-358.

[6] Qingde Kang and Yanfang Zhang, *Large set of P_3-decompositions*, J. Combin. Des. 10(2002), 151-159.

[7] Jiaxi Lu, *On large sets of disjoint Steiner triple systems I-III*, J. Combin. Theory Ser. A 34(1983), 140-182.

[8] Jiaxi Lu, *On large sets of disjoint Steiner triple systems IV-VI*, J. Combin. Theory Ser. A 37(1984), 136-192.

[9] M. Šajna, *Cycle decompositions III: Complete graphs and fixed length cycles*, J. Combin. Des. 10(2002), 27-78.

[10] Luc Teirlinck, *A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems*, J. Combin. Theory Ser. A 57(1991), 302-305.

[11] Yanfang Zhang and Qingde Kang, *Graph designs for oriented P_3 and their large sets*, Bull. Inst. Combin. Appl. 48(2006), 82-98.

[12] Hongtao Zhao and Feifei Fan, *Large sets of P_3-decompositions of complete multipartite graphs*, Ars Combin. 104(2012), 341-352.