# ON BINARY MATROIDS NOT ISOMORPHIC TO THEIR BASE MATROIDS

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**Abstract.** *Let $M = (E, \mathcal{F})$ be a matroid on a set $E$, $B$ one of its bases and $M_B$ the base matroid associated to $B$. In this paper we determine a characterization of simple binary matroids $M$ which are not isomorphic to $M_B$, for every base $B$ of $M$. We also extend to matroids some graph notions.*

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# 1 Introduction

Let $M = (E, \mathcal{I})$ be a matroid on a set $E$, having $\mathcal{I}$ as its family of independent sets. Given a set $S \subseteq E$ let us denote by $r(S)$ the rank of S, i.e. the cardinality of the largest independent set contained in $S$; moreover the closure of $S$, denoted by $cl(S)$, is the set obtained by adding to $S$ all elements $e \in E$ such that

$$r(S \cup e) = r(S).$$

A set $\theta \subseteq E$ is closed if $\theta = cl(\theta)$, i.e.

$$r(\theta \cup e) = r(\theta) + 1$$

for all $e \in E \setminus \theta$. In the following we denote by $\Xi$ the set of all closed sets of $M$.

Recall also that

$$\mathcal{I} = \{S \subseteq E : \mid S \cap \theta \mid \leq r(\theta), \forall \theta \in \Xi\}.$$

In [?] the notion of a set saturated with respect to a base has been introduced.

**Definition 1** *A set $\theta \subseteq E$ is called saturated with respect to a base $B$ of $M$, or $B$-saturated for short, if*

$$\mid \theta \cap B \mid = r(\theta).$$

*We simply denote $\theta$ as saturated when it is clear from the contest which base is involved.*

If $\theta$ belongs to $\Xi$, we have a saturated closed set. The set of all the saturated closed sets of $M$, with respect to a base $B$, is denoted by $\Xi_B$. Note that a $B$-saturated closed set $\theta$ satisfies

$$cl(\theta \cap B) = \theta;$$

in other words $\theta$ coincides with the closure of its intersection with $B$.
A circuit of $M$ is a minimal dependent set, i.e. a set $S \notin \mathcal{I}$ such that for each $i \in S$, $S \setminus i \in \mathcal{I}$. Given a base $B$ and an element $i \in E \setminus B$, the *fundamental circuit* of $i$, denoted $F(i)$, is the minimal subset of $B \cup \{i\}$ which is not in $\mathcal{I}$. A circuit is *fundamental with respect to B* (or simply *fundamental*) when it is the fundamental circuit of an element $i \in E \setminus B$.
We use the notation

$$\mathcal{I}_B = \{S \subseteq E :\mid S \cap \theta \mid \leq r(\theta), \forall \theta \in \Xi_B\}$$

and

$$M_B = (E, \mathcal{I}_B).$$

In [?] it is proved that $M_B = (E, \mathcal{I}_B)$ is a matroid , in particular a transversal matroid.
An application of these matroids, named base matroids, is in the field of inverse combinatorial optimization problems; indeed many different inverse problems have been addressed in the recent literature [?], [?].
One such problem is the inverse matroid problem: given a matroid $M = (E, \mathcal{I})$, a non-negative weighting function c on E and a *target base* B of M, find *perturbation parameters* $\delta_e$ to be added to the weight $c_e$ of each element of $E$ such that B becomes a base of maximum total weight with respect to $c'_e = c_e + \delta_e$, and $\sum_{e \in S} \mid \delta_e \mid$ is minimum. In [3] it is shown how to exploit the Linear Programming (LP) formulation of the classical *matroid optimization problem*

$$max\{\sum_{e \in S} c_e : S \in \mathcal{I}\}$$

and LP duality in order to convert the inverse matroid problem defined above into a matroid optimization problem on a suitable base-matroid.
Recall that a matroid $M$ is simple when it does not contain loops or non-trivial parallel classes. Simple binary matroids $M$ are characterized by the condition that the symmetric difference of any two different circuits is a union of disjoint circuits. Graphic and cographic matroids are examples of binary matroids. For other definitions and properties of matroids readers are referred to [?].
It is easy to see that in relation to a base $B$ of $M$, $M \simeq M_B$ if and only if every circuit of $M$ is also a circuit of $M_B$.

In [?] we proved that a matroid is isomorphic to $M_B$ for every basis $B$ if and only if $M$ is either uniform or is the direct sum of uniform matroids. In this paper we characterize simple binary matroids which are not isomorphic to $M_B$ for every base $B$. To do this we extend some graph theory notions to matroids. In particular we introduce the concepts of p-intersecting circuits and crossing chords. The main result of this paper is given is the following theorem:

**Theorem 3** A simple binary matroid $M$ is not isomorphic to $M_B$ for any base $B$ if and only if $M$ contains a pair of p-intersecting circuits or a covered circuit.

# 2 B - independent circuits

First recall the following definitions [?].

**Definition 2** *A circuit $C$ of $M$ is said independent with respect to $B$ or $B$-independent if*

$$| cl(C) \cap B | < | C | -1;$$

*$C$ is dependent with respect to $B$ or $B$-dependent if it is not independent with respect to $B$, that is*

$$| cl(C) \cap B | = | C | -1.$$

*It follows that in this case $cl(C)$ is saturated with respect to $B$.*

Notice that if a circuit $C$ of $M$ is $B$ - dependent, then it is dependent also in $M_B$; in particular it is a circuit of $M_B$.
On the contrary, if $C$ is $B$ - independent, then $C$ is independent in $M_B$ and consequently $C$ is not a circuit of $M_B$ and $M$ is not isomorphic to $M_B$.

**Definition 3** *([?])- Let $M$ be an arbitrary matroid. A circuit $C$ of $M$ has a chord $e$ if there are two circuits $C_1$ and $C_2$ such that $C_1 \cap C_2 = \{e\}$ and $C = C_1 \triangle C_2$. In this case we say that $e$ splits the circuit $C$ into the circuits $C_1$ and $C_2$.*

In other words an element $e \in E \backslash C$ is a chord of the circuit $C$ if $C \cup \{e\}$ can be decomposed into two distinct circuits $C_1$ and $C_2$, whose intersection coincides with $e$. Notice that in this case

$$| C | = | C_1 | + | C_2 | -2.$$

If $e$ belongs to $B$, it is called a *$B$-chord.* The circuit $C$ is also denoted as the *sum* of $C_1$ and $C_2$, in analogy with the similar notion for graphs [?].

**Lemma 1** *Let $B$ be a base of $M$, $C$ a $B$-dependent circuit and $e$ an edge of $cl(C)$ which does not belong to $B$. Then the fundamental circuit of $e$ with respect to $B$ is contained in $cl(C)$.*

*Proof.* As $C$ is $B$-dependent, it follows that $\mid cl(C) \cap B \mid = m - 1$, where $m = \mid C \mid$. Then $(cl(C) \cap B) \cup e$ is a dependent set of $M$. This implies the result. □

**Lemma 2** *Let $B$ be a base of a matroid $M$ and $C$ a $B$-independent circuit, sum of two circuits $C_1$ and $C_2$. Then at least one of the circuits $C_1$, $C_2$ is $B$-independent.*

*Proof.* Let $e$ be a chord of $C$ and $C_1, C_2$ the circuits such that $C_1 \cap C_2 = \{e\}$ and $C_1 \triangle C_2 = C$.
Assume that $C_1$ and $C_2$ are $B$-dependent, so that $\mid cl(C_i) \cap B \mid = \mid C_i \mid -1$, $i = 1, 2$.
Then

$$cl(C) \cap B \supseteq (cl(C_1) \cup cl(C_2)) \cap B = (cl(C_1) \cap B) \cup (cl(C_2) \cap B);$$

Thus

$$\mid cl(C) \cap B \mid \geq \mid cl(C_1) \cap B \mid + \mid cl(C_2) \cap B \mid - \mid (cl(C_1) \cap B) \cap (cl(C_2) \cap B) \mid . \quad (1)$$

If $e \in B$, then

$$\mid cl(C) \cap B \mid \geq \mid C_1 \mid -1 + \mid C_2 \mid -1 - 1 = \mid C \mid -1.$$

Because $\mid cl(C) \cap B \mid < \mid C \mid -1$, then we obtain an impossible relation.
If $e \notin B$, then we obtain that $(cl(C_1) \cap B) \cap (cl(C_2) \cap B) = \emptyset$ and then the inequality

$$\mid cl(C) \cap B \mid \geq \mid C_1 \mid -1 + \mid C_2 \mid -1 = \mid C \mid,$$

which is clearly impossible.
In any case we obtain a contradiction to the assumption; then at least one of the circuits $C_1$ and $C_2$ is $B$ - independent. □

Aa a consequence we have the following corollary.

**Corollary 1** *Let $B$ be a base of a matroid $M$ and let $M$ have a $B$-independent circuit, with a chord. Then $M$ contains at least a $B$-independent circuit without chords.*

*Proof.* Let $e$ be a chord of a $B$ - independent circuit $C$ and $C_1$, $C_2$ the circuits in which $C \cup \{e\}$ is splitted. By Lemma 2 al least one , say $C_1$, is $B$ - independent. Clearly the number of chords of $C_1$ is less than the number of $C$. By iterating, if necessary, the procedure we obtain the result. □

**Definition 4** *A circuit $C$ of $M$ is said to be* covered *if every element $e$ of $C$ belongs to a circuit $C(e)$, such that $cl(C) \cap C(e) = \{e\}$.*

**Lemma 3** *Let $C$ be a covered circuit of $M$, such that $C$ has one chord. Then $M$ contains a covered circuit $H$ such that $H$ and every circuit $H(f)$, where $f \in H$ and $H \cap H(f) = \{f\}$, are without chords.*

*Proof.* Let $f$ be a chord of the circuit $C$, and $C_1$ and $C_2$ the circuits in which $C \cup f$ is splitted. Because a chord of $C_1$ or $C_2$ is also a chord of $C$, then the number of chords of every circuit $C_i$, for $i = 1, 2$, is less than the same number of $C$. Moreover every circuit $C_i$ is covered. Indeed the elements of $C_i$, but $\{f\}$, are elements of $C$. But $f$, as element of $C_1$, belongs to $C_2$ and coincides with $cl(C_1) \cap cl(C_2)$. Thus we replace $C$ by one of these circuits, say $C_1$. Iterating the procedure the result follows. □

**Theorem 1** *Let $M$ be a matroid containing a covered circuit. Then, for every base $B$ of $M$, $M$ contains a $B$-independent circuit and $M$ is not isomorphic to $M_B$.*

*Proof.* Let $C$ be a covered circuit which we assume $B$ - dependent. Moreover let $h$ be an element of C which does not belong to B; denote $C(h) = \{h\} \cup C'$ the circuit such that $cl(C) \cap C(h) = \{h\}$. By Lemma 3 we may assume that $C$ and $C(h)$ are without chords.
If $C'$ contains another element which does not belong to $B$, then $C(h)$ is $B$ - independent because

$$| cl(C(h)) \cap B | < | C(h) | -1.$$

If $C' \subseteq B$, then there is another element of $C$, say $j$, which does not belong to $B$, because on the contrary $C \triangle C(h)$ turns out to be a circuit contained in $B$. If also the elements of $C(j) \setminus \{j\}$ belong to B, we may continue until we find an element $q$ of C which does not belong to B and such that $C(q)$ contains an element $q'$, distinct from $q$, which does not belong to B. Because $C(q)$ is without chords, it implies

$$| cl(C(q)) \cap B | < | C(q) | -1.$$

In other words $C(q)$ is independent with respect to B.

□

# 3 P-intersecting circuits

In this section we study particular conditions on the circuits of a matroid $M$ in order that, for every base $B$ of $M$, $M$ is not isomorphic to $M_B$. Recall that a *chordal path* of a cycle in a graph $G$ is a path that is edge-disjoint from the cycle and that joins two non-neighbor vertices of the cycle. Moreover two cycles $C$ and $H$ of $G$ are *intersecting in a path* when they have in common a path which connects two not consecutive vertices; this path turns out to be a chordal path of $C \triangle H$. Now we want to extend the notion of cycles intersecting in a path to matroids by introducing p-intersecting circuits.

**Definition 5** *Two circuits $C$ and $H$ of $M$ are said p-intersecting when:*

1. *$cl(C) \cap cl(H)$ is an independent set of cardinality greater than 1;*
2. *$C \triangle H$ is a circuit.*

A motivation for introducing this definition is given by the following result.

**Theorem 2** *If a matroid $M$ contains two p-intersecting circuits, then, for every base $B$ of $M$, it contains a $B$-independent circuit.*

*Proof.* Let $C$ and $H$ be two p-intersecting circuits of $M$ and $B$ a base of $M$. If at least one of them is B-independent, the result follows.
Thus assume that both are B-dependent, i.e. $| cl(C) \cap B |=| C | -1$ and $| cl(H) \cap B |=| H | -1$.
Denote $cl(C) \cap cl(H) = P$, where $| P |= t > 1$ and $P$ is independent.

1. First assume that $P \subseteq B$. This condition implies that $B$ cannot contain all the elements of $C \setminus P$ and all the elements of $H \setminus P$. In other words there is at least one element, say $e$, of $C \setminus P$ and at least one element, say $f$, of $H \setminus P$ which do not belong to B. Then the circuit $D = C \triangle H$ contains at least two elements which do not belong to $B$.
   If $D$ does not contain $B$-chords, then the following inequality holds
   $$| cl(D) \cap B |=| D \cap B |<| D | -1$$
   and D is B - independent.
   Assume that $D$ contains a $B$-chord.
   Recall that if $X$ and $Y$ are flats of $M$, then
   $$r(X)+r(Y) \geq r(X \cup Y)+r(X \cap Y) = r(cl(X \cup Y))+r(X \cap Y). \quad (2)$$

Let $X = cl(C)$ and $Y = cl(H)$. Then $r(X) = \mid C \mid -1$, $r(Y) = \mid H \mid -1$, $r(X \cap Y) = t$ and by the assumption that $P \subseteq B$:

$$r(cl(X \cup Y)) \geq \mid cl(X \cup Y) \cap B \mid \geq \mid X \cap B \mid + \mid Y \cap B \mid - \mid X \cap Y \mid +1. \quad (3)$$

Because $r(X \cup Y) = r(cl(X \cup Y))$, from (1) and (2) we obtain the impossible relation

$$\mid C \mid -1 + \mid H \mid -1 \geq \mid C \mid -1 + \mid H \mid -1 - t + t + 1.$$

2. Now suppose that at least one element of $P$, say $g$, does not belong to $B$. Let $B_C = B \cap cl(C)$ and $B_H = B \cap cl(H)$. If $g$ is added to $B_C$, we obtain a dependent set; then the fundamental circuit $F(g) \subseteq cl(C)$. In a similar way if we add $g$ to $B_H$, we obtain a dependent set and the fundamental circuit $F(g) \subseteq cl(H)$. This implies the impossible condition that $cl(C)$ and $cl(H)$ have in common a dependent set.

□

**Lemma 4** *Let $C$ and $H$ be circuits of a binary matroid $M$. If $C \triangle H$ is a set of disjoint circuits and $D$ is one of them, then $C \triangle D$ and $H \triangle D$ contain only one circuit.*

*Proof.* The circuit $D$ can be represented as $C' \cup H'$, where $C' \subset C$ and $H' \subset H$ and $C' \cap H' = \emptyset$. Thus $C \triangle D = (C \setminus C') \cup H'$ is the union of disjoint circuits because the matroid is binary; in particular it is a single circuit since otherwise the possible circuits would have been contained into $C \triangle H$ and not disjoint from $D$. □

# 4 Characterization

In this section we determine a characterization of the binary matroids $M$ not isomorphic to $M_B$ for every base $B$ in terms of the previous notions of p - intersecting circuits and covered circuits.

**Theorem 3** *A simple binary matroid $M$ is not isomorphic to $M_B$ for any base $B$ if and only if $M$ contains a pair of p-intersecting circuits or a covered circuit.*

*Proof.*

By Theorems 1 and 2 we have only to prove the necessary condition. Thus assume that $M$ is not isomorphic to $M_B$, where $B$ is a base of $M$. It follows that $M$ contains a circuit $C$ which is $B$-independent, that is it satisfies the inequality

$$| cl(C) \cap B | < | C | - 1.$$

By Corollary 1 we may assume that $C$ has no chords; this implies

$$| cl(C) \cap B | = | C \cap B | .$$

Then there are at least two elements of $C$, say $e$ and $f$, which do not belong to $B$. Let $H$ be the $B$-fundamental circuit of one of them, say $f$. Thus $H = \{f\} \cup H'$, where $H' \subseteq B$. We have to distinguish the following two cases.

1. $H' \cap C = \emptyset$. Then either $H$ is a covered cycle or there exists an element $g$ of $H'$ which does not belong to a circuit $D$, distinct from $H$ and not contained in $cl(H)$. In this case we replace $f$ by $g$ in $B$; then we obtain a new base $B'$ with respect to which $H$ remains fundamental, while the number of elements of $C$ which do not belong to the base $B'$ is less than the same number with respect to $B$. If $C$ remains $B'$-independent we may consider the same procedure in relation to another element which does not belong to $B'$; otherwise if $C$ is $B'$ - dependent by the assumption there is another independent circuit $Q$ and we may repeat the procedure in relation to $Q$. In this way we arrive to obtain a base with respect to which there are not $B$-independent circuits, a contradiction.

2. $H' \cap C \neq \emptyset$. Let $D$ be one of the circuits in which $H \triangle C$ is decomposed. Thus $D = C' \cup H^*$ where $C' \subseteq C$, $| C' | > 1$, $H^* \subseteq H'$ and $C' \cap H^* = \emptyset$. Notice that $| H^* | > 1$ because $C$ does not contain $B$ - chords. If $cl(D) \cap cl(C)$ is dependent , then there exists an element $c \in cl(C) \setminus C$, which turns out to be a chord . By the assumption this condition is impossible.
   Moreover by Lemma 4 $C \triangle D$ contains only one circuit. So $C$ and $D$ are p-intersecting.

□

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