On the Crossing Number of the Cartesian Product of a 6-Vertex Graph with S_n *

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Abstract

Computing the crossing number of a given graph is in general an elusive problem and only the crossing numbers of few families of graphs are known. Most of them are the Cartesian product of special graphs. This paper determines the crossing number of the Cartesian product of a 6-vertex graph with star S_n.

1 Introduction

For definitions not explained here, readers are referred to [1]. Let G be a simple graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well known that the crossing number of graph is attained only in *good drawings* of the graph, which are those drawings where no edges cross itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges intersect in a common point. Let D be a good drawing of the graph G, we denote by $cr_D(G)$ the number of crossings in D. A drawing D of G is said to be *optimal* if $cr_D(G) = cr(G)$.

Let P_n and C_n be the path and cycle of length n, respectively, and the *star* S_n be the complete bipartite graph $K_{1,n}$.

Given two vertex disjoint graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ of G_1 and G_2 is defined by

$$\begin{cases} V(G_1 \times G_2) = & V(G_1) \times V(G_2) \\ E(G_1 \times G_2) = & \{(u_1, u_2)(v_1, v_2) \mid u_1 = v_1 \text{ and } u_2 v_2 \in E(G_2), \\ & \qquad \text{or } u_2 = v_2 \text{ and } u_1 v_1 \in E(G_1)\} \end{cases}$$

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Let G_1 be a graph homeomorphic to G_2, then $cr(G_1) = cr(G_2)$. And if G_1 is a subgraph of G_2, it is easy to see that $cr(G_1) \leq cr(G_2)$.

Computing the crossing number of a given graph is in general an elusive problem [2] and the crossing numbers of few families of graphs are known. Most of them are Cartesian products of special graphs, partly because of the richness of their repetitive patterns. The already known results on the crossing number of $G \times H$ fit into three categories:

(i) *G and H are two small graphs.* Harary, et al. obtained the crossing number of $C_3 \times C_3$ in 1973 [3]; Dean and Richter [4] investigated the crossing number of $C_4 \times C_4$; Klešč [5] studied the crossing number of $K_{2,3} \times C_3$. These results are usually used as the induction basis for establishing the results of type (ii):

(ii) *G is a small graph and H is a graph from some infinite family.* In [6], the crossing numbers of $G \times C_n$ for any graph G of order four except S_3 were studied by Beineke and Ringeisen, this gap was bridged by Jendrol' et al. in [7]. The crossing numbers of Cartesian products of 4-vertex graphs with P_n and S_n are determined by Klešč in [8], he also determined the crossing numbers of $G \times P_n$ for any graph G of order five [9, 10, 11]. For several special graphs of order five, the crossing numbers of their products with C_n or S_n are also known, most of which are due to Klešč [12, 13, 14, 15]. For special graphs G of order six, Peng et al. determined the crossing number of the Cartesian product of the Petersen graph $P(3,1)$ with P_n in [16], Zheng et al. gave the bound for the crossing number of $K_m \times P_n$ for $m \geq 3, n \geq 1$, and they determined the exact value for $cr(K_6 \times P_n)$, see [17], and the authors [18] established the crossing number of the Cartesian product of P_n with the complete bipartite graph $K_{2,4}$.

(iii) *Both G and H belong to some infinite family.* One very long attention-getting problem of this type is to determine the crossing number of the Cartesian product of two cycles, C_m and C_n, which was put forward by Harary et al. [3], and they conjectured that $cr(C_m \times C_n) = (m-2)n$ for $n \geq m$. In the next three decades, many authors were devoted to this problem and the conjecture has been proved true for $m = 3, 4, 5, 6, 7$, see [19, 6, 20, 21, 22]. In 2004, the problem was progressed by Glebsky and Salazar, who proved that the crossing number of $C_m \times C_n$ equals its long-conjectured value for $n \geq m(m+1)$ [23]. Besides the Cartesian product of two cycles, there are several other results. D.Bokal [24] determined the crossing number of the Cartesian product $S_m \times P_n$ for any $m \geq 3$ and $n \geq 1$ used a quite newly introduced operation: the zip product. Tang, et al. [25] and Zheng, et al. [26] independently proved that the crossing number of $K_{2,m} \times P_n$ is $2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ for arbitrary $m \geq 2$ and $n \geq 1$.

Inspired by these results, we begin to investigate the crossing number of the Cartesian product of star S_n with a 6-vertex graph G_1 shown in Figure 1, and prove that $cr(G_1 \times S_n) = Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$ for $n \geq 1$.

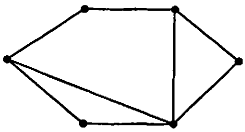

Figure 1: The graph G_1

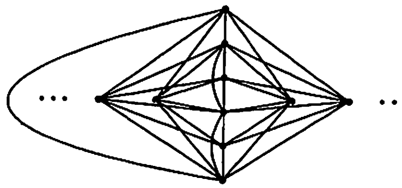

Figure 2: The graph H_n

2 Some Basic Lemmas and the Main Result

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings made by an edge in A and another edge in B is denoted by $cr_D(A, B)$. The number of crossings made by two edges in A is denoted by $cr_D(A)$, then $cr(D) = cr_D(E)$. By counting the number of crossings in D, we have Lemma 2.1.

Lemma 2.1 *Let A, B, C be mutually disjoint subsets of E. Then*

$$\begin{aligned} cr_D(A \cup B, C) &= cr_D(A, C) + cr_D(B, C); \\ cr_D(A \cup B) &= cr_D(A) + cr_D(B) + cr_D(A, B). \end{aligned} \tag{1}$$

The crossing number of the complete bipartite graph $K_{m,n}$ were determined by Kleitman [27] for the case $m \leq 6$. More precisely, he proved that

$$cr(K_{m,n}) = \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor, \quad \text{if } m \leq 6 \tag{2}$$

For convenience, $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$ in our paper. Let us denoted by H_n the graph shown in Figure 2. It is easy to verify that $H_n = G_1 \cup K_{6,n}$, where the six vertices of degree n in $K_{6,n}$ and the vertices of G_1 are the same. For $i = 1, 2, \cdots, n$, let T^i be the subgraph of $K_{6,n}$ which consists of the six edges incident with a vertex of degree six in $K_{6,n}$. Thus, we have

$$H_n = G_1 \cup K_{6,n} = G_1 \cup (\bigcup_{i=1}^{n} T^i) \tag{3}$$

Theorem 2.2 $cr(H_n) = Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$.
Proof. The good drawing in Figure 2 shows that $cr(H_n) \leq Z(6, n) + 2\lfloor \frac{n}{2} \rfloor$. Now we prove the reverse inequality by induction on n. The case $n = 1$ is trivial, and the inequality also holds when $n = 2$ since H_2 contains a subgraph isomorphic $K_{3,4}$, whose crossing number is 2. Now suppose that for $n \geq 3$,

$$cr(H_{n-2}) \geq Z(6, n-2) + 2\lfloor \tfrac{n-2}{2} \rfloor \tag{4}$$

and for a certain good drawing D of H_n, assume that

$$cr_D(H_n) < Z(6,n) + 2\lfloor \tfrac{n}{2} \rfloor \tag{5}$$

The following two cases are discussed:

Case 1. Suppose that there are at least two different subgraphs T^i and T^j that don't cross each other in D. Without loss of generality, assume $cr_D(T^{n-1}, T^n) = 0$. The induced drawing $D|_{T^{n-1} \cup T^n}$ of $T^{n-1} \cup T^n$ divides the plane into six regions that there are exactly two vertices of G_1 on the boundary of each region. Note that there is a 4-degree vertex of G_1, so one can observe that there are at least two crossings made by the edges of G_1 and the edges of $T^{n-1} \cup T^n$, that is $cr_D(G_1, T^{n-1} \cup T^n) \geq 2$. As $cr(K_{3,6}) = 6$, for all i, $i = 1, 2, \cdots, n-2$, $cr_D(T^i, T^{n-1} \cup T^n) \geq 6$. Using (1), (2), (3) and (4), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_1 \cup \bigcup_{i=1}^{n-2} T^i \cup T^{n-1} \cup T^n) \\ &= cr_D(G_1 \cup \bigcup_{i=1}^{n-2} T^i) + cr_D(T^{n-1} \cup T^n) + cr_D(G_1, T^{n-1} \cup T^n) \\ &\quad + \sum_{i=1}^{n-2} cr_D(T^i, T^{n-1} \cup T^n) \\ &\geq Z(6, n-2) + 2\lfloor \frac{n-2}{2} \rfloor + 2 + 6(n-2) \\ &= Z(6,n) + 2\lfloor \frac{n}{2} \rfloor \end{aligned}$$

This contradicts (5).

Case 2. Suppose that $cr_D(T^i, T^j) \geq 1$ for any two different subgraphs T^i and T^j, $1 \leq i \neq j \leq n$. Using (1), (2) and (3), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_1) + cr_D(\textstyle\bigcup_{i=1}^n T^i) + cr_D(G_1, \bigcup_{i=1}^n T^i) \\ &\geq cr_D(G_1) + Z(6,n) + \textstyle\sum_{i=1}^n cr_D(G_1, T^i) \end{aligned} \tag{6}$$

This, together with (5) implies that

$$cr_D(G_1) + \sum_{i=1}^{n} cr_D(G_1, T^i) < 2\lfloor \frac{n}{2} \rfloor$$

So, there is at least one subgraph T^i which doesn't cross G_1. Without loss of generality, we may assume that $cr_D(G_1, T^n) = 0$. Let us consider the 6-cycle C_6 of the graph G_1. Hence G_1 consists of C_6 and two additional edges.

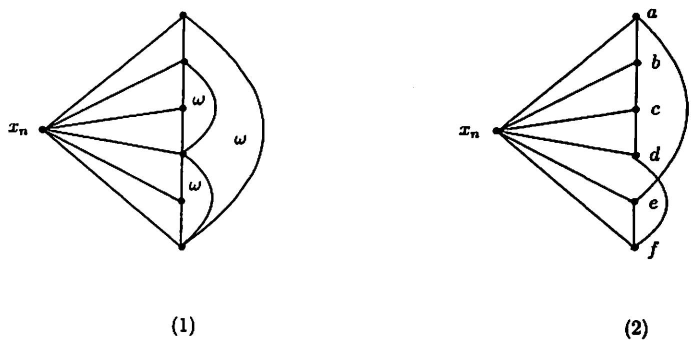

Figure 3:

Subcase 2.1. Suppose that the edges of C_6 do not cross each other in D. Since $cr_D(G_1, T^n) = 0$, then the unique possibility is $cr_D(G_1 \cup T^n) = 0$, see Figure 3(1). If, in D, x_i locates in the region labeled ω, it is easy to verify that the edges of T^i cross the edges of G_1 at least two times. Using $cr_D(T^i, T^n) \geq 1$, we have $cr_D(T^i, G_1 \cup T^n) \geq 3$. If x_i locates in the other regions, it is easy to verify that there are at least 5 crossings between the edges of T^i and the edges of $G_1 \cup T^n$, that is $cr_D(T^i, G_1 \cup T^n) \geq 5$. Let

$$M = \{T^i | x_i \text{ lies in the region labeled } \omega\}$$

Using (1), (2) and (3), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_1 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\ &= cr_D(G_1 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{T^i \in M} cr_D(G_1 \cup T^n, T^i) \\ &\quad + \sum_{T^i \notin M} cr_D(G_1 \cup T^n, T^i) \\ &\geq Z(6, n-1) + 3|M| + 5(n-1-|M|) \end{aligned}$$

Together with (5), we can get

$$2|M| \geq 5n - 5 - 2\lfloor \frac{n}{2} \rfloor - 6\lfloor \frac{n-1}{2} \rfloor \geq 2\lfloor \frac{n}{2} \rfloor \tag{7}$$

Combined with (6) and (7), we can get

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_1) + cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(G_1, \bigcup_{i=1}^{n} T^i) \\
&= cr_D(G_1) + cr_D(\bigcup_{i=1}^{n} T^i) + \sum_{T^i \in M} cr_D(G_1, T^i) + \sum_{T^i \notin M} cr_D(G_1, T^i) \\
&\geq Z(6,n) + 2|M| \\
&\geq Z(6,n) + 2\lfloor \frac{n}{2} \rfloor
\end{aligned}$$

which contradicts (5).

Subcase 2.2. Suppose that the edges of C_6 cross each other in D. We can assert that in D there must exist a subgraph T^i, $i \in \{1, 2, \cdots, n-1\}$, such that $cr_D(T^i, G_1 \cup T^n) \leq 3$. Otherwise, we have $cr_D(T^i, G_1 \cup T^n) \geq 4$ for all $i = 1, 2, \cdots, n-1$, and using (1), (2) and (3), we obtain

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_1 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\
&= cr_D(G_1 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{i=1}^{n-1} cr_D(G_1 \cup T^n, T^i) \\
&\geq 1 + Z(6, n-1) + 4(n-1) \\
&\geq Z(6,n) + 2\lfloor \frac{n}{2} \rfloor
\end{aligned}$$

a contradiction with (5).

The condition $cr_D(G_1, T^n) = 0$ implies that $cr_D(C_6, T^n) = 0$. In this case the vertex x_n of T^n lies in the region with all six vertices of C_6 on its boundary, and the condition $cr_D(T^i, G_1 \cup T^n) \leq 3$ enforces that in the subdrawing of $T^n \cup C_6$ there is a region with at least four vertices of C_6 on its boundary. In this case C_6 has only one internal crossing and the unique possibility of $T^n \cup C_6$ is as shown in Figure 3(2). In the subgraph $G_1 \cup T^n$ there are two additional edges which do not cross the edges of T^n in the considered subdrawing of $G_1 \cup T^n$. The vertices of G_1 are labeled by a, b, c, d, e, f, respectively. At least one of the vertices b and c is incident in $G_1 \cup T^n$ with an edge of G_1 not belonging to C_6. The possible edge bd separates the vertices a and c and the possible edge be crosses the edge df and separates the vertices a and c again. The possible edge ca separates the vertices b and d and the possible edge cf crosses the edge ae and separates the vertices b and d again. In all these cases $cr_D(T^i, G_1 \cup T^n) \geq 4$, which is a contradiction and completes the proof of Theorem 2.2. □

Let H be a graph isomorphic to G_1. Consider a graph G_H obtained by joining all vertices of H to six vertices of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained from G_H by contracting the edges of H.

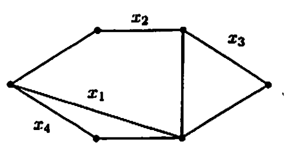

Figure 4:

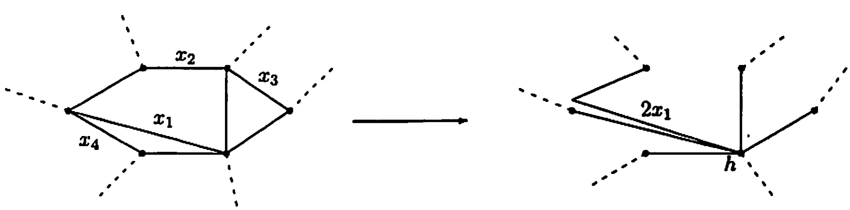

Figure 5:

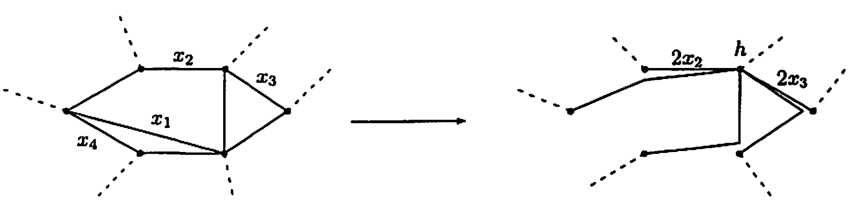

Figure 6:

Lemma 2.3 $cr(G_H^*) \leq cr(G_H)$.

Proof. Let D be an optimal drawing of G_H. The subgraph H has eight edges and let x_1, x_2, x_3, x_4 denote the numbers of crossings on the four edges of H, see Figure 4. The following two cases are discussed.

Case 1. Suppose that $x_1 \leq x_2 + x_3 + x_4$. Figure 5 shows that H can be contracted to the vertex h without increasing the number of crossings. That means $cr(G_H^*) \leq cr_D(G_H) = cr(G_H)$.

Case 2. Suppose that $x_1 > x_2+x_3+x_4$, then we have $x_1+x_4 > x_2+x_3$, Figure 6 shows that H can be contracted to the vertex h without increasing the number of crossings. This completes the proof. □

Consider now the graph $G_1 \times S_n$. For $n \geq 1$ it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies G_1^i, $i = 0, 1, \cdots, n$, and in the six stars S_n, see Figure 7.

Theorem 2.4 $cr(G_1 \times S_n) = Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$, *for* $n \geq 1$.
Proof. The drawing in Figure 7 shows that $cr(G_1 \times S_n) \leq Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$. Assume that there is an optimal drawing D of $G_1 \times S_n$ with fewer than $Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$ crossings. Contracting the edges of each G_1^i to a vertex x_i for all $i = 1, 2, \cdots, n$ in D results in a graph isomorphic to H_n, and using Lemma 2.3 repeatedly, we have $cr(H_n) \leq cr(G_1 \times S_n) = cr_D(G_1 \times S_n) < Z(6,n)+2\lfloor \frac{n}{2} \rfloor$, a contradiction with Theorem 2.2. Therefore, $cr(G_1 \times S_n) = Z(6,n) + 2\lfloor \frac{n}{2} \rfloor$. □

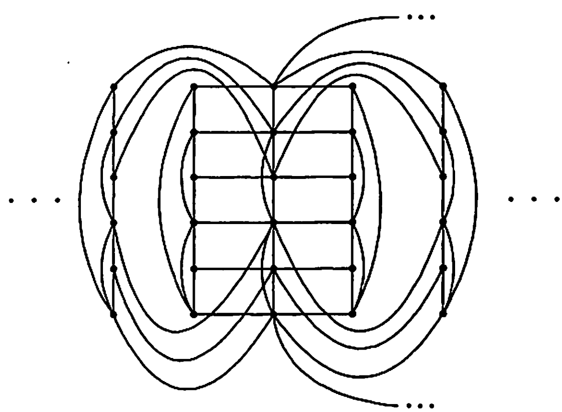

Figure 7:

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