Some structural, metric and convex properties of the boundary of a graph*

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Abstract

Let u, v be two vertices of a connected graph G. The vertex v is said to be a *boundary vertex* of u if no neighbor of v is further away from u than v. The boundary of a graph is the set of all its boundary vertices. In this work, we present a number of properties of the boundary of a graph under different points of view: (1) a realization theorem involving different types of boundary vertex sets: extreme set, periphery, contour, and the whole boundary; (2) the contour is a monophonic set; and (3) the cardinality of the boundary is an upper bound for both the metric dimension and the determining number of a graph.

Keywords: Boundary; Contour; Extreme set; Graph convexity; Metric dimension.

1 Introduction and basic definitions

Boundary vertex sets have been studied in different contexts: *facility location*, to determine the appropriate location for certain kinds of facilities [10]; *rebuilding in graphs*, how to rebuild a graph from a certain subset of vertices by using a convex hull operator [4, 5, 7]; and *resolvability in graphs*, a

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concept introduced by Slater [25, 26] using the term *locating set* instead of resolving set. He also describes its usefulness in some practical problems. See Chartrand et al. [11] for a recent survey on this subject, where they show some applications to different topics in graph theory such as partition of the vertex set, decomposition, domination, and coloring in graphs.

The main purpose of this work is to show that the so-called, *boundary* of a graph introduced by Chartrand et al. [8] possesses a number of significant properties that make it a very interesting vertex subset to be taken into account when studying the structure of the whole graph, especially if we want to know more about it as a metric space. See the references [5, 19, 24] for the most recent works involving boundary vertices of a graph. Müller et al. [24] give the best possible lower bound, up to a constant factor, on the number of boundary vertices of a graph in terms of its minimum degree (or maximum degree). Concretely, it is an interesting question to characterize those graphs with small boundary. This problem has been considered very recently by Hasegawa and Saito [19]. Nevertheless, independently of the relevance of that problem, we need the characterization of those graphs with small number (2 or 3) of vertices in their boundaries in order to obtain a realization theorem involving different types of boundary vertex sets. This is one of the main results in this paper.

A second interesting question considered in this paper is to obtain properties of some boundary vertex sets of a graph under the point of view of two convexity criteria: the geodetic convexity and the monophonic convexity. These two criteria are close related. See the references [2, 3, 4, 5, 7, 13, 21] for recent works about the convexity of some boundary vertex sets. Actually, there are some open questions on determining whether some boundary vertex sets are geodetic or monophonic [2]. Concretely, it is still an open question to know whether the contour of a graph is a geodetic set for bipartite graphs [4]. Our second main contribution is to prove that the contour of a graph is a monophonic set, which is not so strong as its geodeticity but it directly implies some recently known result about the geodetic convexity of the contour of some graph classes [7].

The third point of view we address is to study metric properties of the boundary of a graph. It is known that determining the metric dimension of a graph is a NP-complete problem [22]. Genetic algorithms have been designed to compute the metric dimension of hypercubes and Hamming graphs up to certain number of vertices [23]. In [6] the metric dimension of Cartesian product of graphs is addressed and an exhaustive set of references on the metric dimension of graphs can be found. A simple but interesting result in this paper is that the cardinality of the boundary of a graph is an upper bound for both the metric dimension and the determining number of the graph [1, 14, 16]. Unfortunately, the boundary of a graph can be very large, but knowing its cardinality can be helpful for bounding the metric

dimension of some graphs.

All the graphs considered in this paper are finite, undirected, simple, and connected graphs $G = (V(G), E(G))$. By $d(u, v)$ we denote the distance between two vertices $u, v \in V(G)$. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [28].

Given $u, v \in V(G)$, the vertex v is said to be a *boundary vertex of* u if no neighbor of v is further away from u than v. By $\partial(u)$ we denote the set of all boundary vertices of u. A vertex v is called a *boundary vertex* of G if $v \in \partial(u)$ for some vertex $u \in V(G)$. The *boundary* of a graph was introduced by Chartrand et al. [8, 9] and originally defined to be the subgraph induced by the boundary vertices of G, not the vertices themselves. Nevertheless and since we only consider boundary vertex sets and not their induced subgraphs, we identify the *boundary* of G, $\partial(G)$, as the set of all of its boundary vertices[1]:

$$\partial(G) = \bigcup_{u \in V(G)} \partial(u) = \{v \in V(G) \mid \exists u \in V(G) \text{ such that } \forall w \in \mathrm{N}(v), d(u, w) \leq d(u, v)\}.$$

For some basic graph classes, such as complete graphs, complete bipartite graphs, cycles and hypercubes, the boundary is simply the whole vertex set. It is not difficult to see that this property holds for all vertex-transitive graphs. However, this is far from being the general rule. In fact, an interesting open question is: *to characterize the graphs in which the boundary is the whole vertex set.* As an example, notice that the boundary of a tree consists exactly of the set of its leaves. The boundary of a graph is a natural generalization of other kinds of boundary vertices which we define next.

The *eccentricity* of $u \in V(G)$, $\mathrm{ecc}(u)$, is defined as

$$\mathrm{ecc}(u) = \max\{d(u, v) \mid v \in V(G)\}.$$

Given $u, v \in V(G)$, the vertex v is called an *eccentric vertex* of u if no vertex in $V(G)$ is further away from u than v, that is, if $d(u, v) = \mathrm{ecc}(u)$. A vertex v is called an *eccentric vertex of* G if it is the eccentric vertex of some vertex $u \in V(G)$. The set of vertices of the eccentric subgraph of G or the *eccentricity* of G, $\mathrm{Ecc}(G)$, is the set of all its eccentric vertices:

$$\mathrm{Ecc}(G) = \{v \in V(G) \mid \exists u \in V(G), \mathrm{ecc}(u) = d(u, v)\}.$$

A vertex $v \in V(G)$ is called a *peripheral vertex* of G if no vertex in $V(G)$ has an eccentricity greater than $\mathrm{ecc}(v)$, i.e., if the eccentricity of v is

[1]This is usual in several papers which only deal with boundary vertex sets. Thus, the same rule applies for the set of all eccentric, peripheral, contour, and extreme vertices of a graph G to be defined next.

equal to the diameter of G. The periphery of G, $\mathrm{Per}(G)$, is the set of all its peripheral vertices:

$$\mathrm{Per}(G) = \{v \in V(G) \mid \forall u \in V(G), \mathrm{ecc}(u) \leq \mathrm{ecc}(v)\}.$$

A vertex $v \in V(G)$ is called a *contour vertex* of G if no neighbor vertex of v has an eccentricity greater than $\mathrm{ecc}(v)$ [7]. The contour of G, $\mathrm{Ct}(G)$, is the set of all its contour vertices:

$$\mathrm{Ct}(G) = \{v \in V(G) \mid \forall u \in \mathrm{N}(v), \mathrm{ecc}(u) \leq \mathrm{ecc}(v)\}.$$

Finally, a vertex $u \in V(G)$ is called *extreme vertex* of G if the subgraph induced by its neighborhood, $G[\mathrm{N}(v)]$, is a clique. The extreme set of G, $\mathrm{Ext}(G)$, is the set of all its extreme vertices:

$$\mathrm{Ext}(G) = \{v \in V(G) \mid G[\mathrm{N}(v)] \text{ is a clique}\}.$$

Figure 1 shows a graph G and its boundary vertex sets.

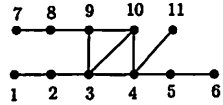

Figure 1: $\mathrm{Per}(G) = \{6,7\}$, $\mathrm{Ct}(G) = \mathrm{Ext}(G) = \{1,6,7,11\}$, $\mathrm{Ecc}(G) = \{1,6,7\}$, and $\partial(G) = \{1,6,7,10,11\}$.

Notice that every extreme vertex is a contour vertex, i.e., $\mathrm{Ext}(G) \subseteq \mathrm{Ct}(G)$. It is also clear that $\mathrm{Per}(G) \subseteq \mathrm{Ct}(G) \cap \mathrm{Ecc}(G)$ and $\mathrm{Ecc}(G) \cup \mathrm{Ct}(G) \subseteq \partial(G)$ (Figure 2).

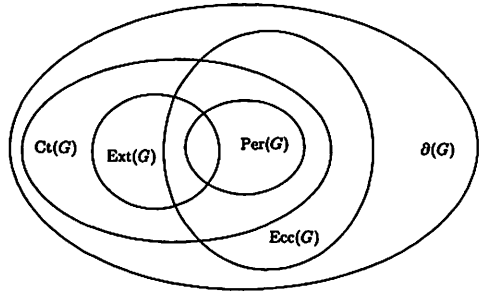

Figure 2: Boundary vertex sets.

This paper is organized as follows. In Section 2, we consider those graphs with small numbers (2 or 3) of vertices in their boundaries in order to show a realization theorem involving different sets of boundary vertices. Both geodesic and monophonic convex properties of the boundary vertex sets of a graph are considered in Section 3, in particular we show that the contour is a monophonic set. Finally, in Section 4 we focus our attention on a metric dimension property of the boundary of a graph.

2 Structural properties. A new realization theorem

In this section we show a realization theorem involving different boundary vertex sets. First we focus on their relationships and cardinalities.

Figure 3 illustrates three graphs showing that there is no general relationship between the peripheral and the extreme vertex sets. Concretely, in Figure 3a): $\mathrm{Per}(G) \cap \mathrm{Ext}(G) = \emptyset$; in Figure 3b): $\mathrm{Per}(G) \cap \mathrm{Ext}(G) \neq \emptyset$, $\mathrm{Per}(G) \not\subseteq \mathrm{Ext}(G)$ and $\mathrm{Ext}(G) \not\subseteq \mathrm{Per}(G)$; and in Figure 3c): $\mathrm{Per}(G) = \mathrm{Ext}(G)$.

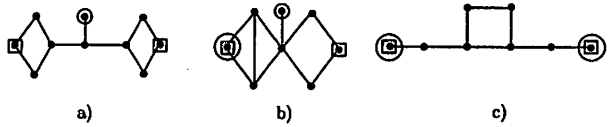

Figure 3: The vertices inside a square (circle) are peripheral (extreme) vertices.

Furthermore, Chartrand et al. [8] give a realization theorem involving the periphery, the eccentricity, and the boundary of a graph. In [5], this result was improved with the following realization theorem which also involves the contour.

Theorem 1. [5] *Let $(b, c, e, d) \in \mathbb{Z}^4$ be positive integers satisfying the following constraints:*

$$\begin{cases} 2 \leq b \leq c \leq d, \\ 2 \leq b \leq e \leq d, \\ (b, c, e, d) \neq (2, 2, 2, 3), \\ (b, c, e, d) \neq (b, c, b+1, b+1). \end{cases}$$

Then there exists a graph G such that

$$|\mathrm{Per}(G)| = b, \quad |\mathrm{Ct}(G)| = c, \quad |\mathrm{Ecc}(G)| = e, \quad \textit{and} \quad |\partial(G)| = d.$$

In this section, we will present a similar result by considering $\mathrm{Ext}(G)$ instead of $\mathrm{Ecc}(G)$. Firstly, we consider those graphs with small numbers (2 or 3) of vertices in their boundaries.

Lemma 1. [5] *If* $|\mathrm{Per}(G)| = |\mathrm{Ct}(G)| = 2$, *then* $|\partial(G)| = 2$ *or* $|\partial(G)| \geq 4$.

Hasegawa and Saito [19] characterize the graphs with boundary of order at most three. The graphs with boundary of order two is as follows. *A connected graph with exactly two boundary vertices is a path.* This characterization was also considered by Hernando et al. [20].

For the characterization of the graphs with boundary of order three, Hasegawa and Saito [19] introduce the following notation. They call a subdivision of $K_{1,3}$ a *claw-like tree*. Let H be a complete graph of order three. Take a subset X of $V(H)$, and for each $x \in X$, prepare a path P_x and join x and one of the endvertices of P_x. The resulting graph is called a *tripod*. The path P_x is called its leg. By definition, a tripod has at most three legs, and a tripod with no leg is K_3. Then, the characterization is as follows. *A connected graph G has exactly three boundary vertices if and only if G is either a claw-like tree or a tripod.*

The following corollary is an easy consequence from the characterization of the graphs with boundary of order three.

Corollary 1. *If* $|\partial(G)| = 3$, *then* $\partial(G) = \mathrm{Ct}(G) = \mathrm{Ext}(G)$.

Notice that if the boundary of the graph has few vertices, the characterization of the graphs with boundary of order two, and Corollary 1 give some restrictions on the cardinalities of both the periphery and the contour. Concretely, the restrictions are given in the following corollary.

Corollary 2. *If* $|\mathrm{Ext}(G)| = a$, $|\mathrm{Per}(G)| = b$, $|\mathrm{Ct}(G)| = c$, *and* $|\partial(G)| = d$, *then*

1. $0 \leq a \leq c \leq d$, *and* $2 \leq b \leq c \leq d$,
2. $(a, b, c, d) \neq (a, 2, 2, 3)$ *for any value of* a,
3. $(a, b, c, d) \neq (2, 2, 3, 3)$,
4. $(a, b, c, d) \neq (2, 3, 3, 3)$.

At this point, we ask whether there are more restrictions concerning the cardinalities of the sets $\mathrm{Ext}(G)$, $\mathrm{Per}(G)$, $\mathrm{Ct}(G)$, and $\partial(G)$. Next, we present a realization theorem showing the answer to be negative. First we give out a trivial lemma which proof is straightforward.

Lemma 2. *Let* $x \in V(G)$, $|V(G)| \geq 2$ *and* $\lambda \geq 1$. *Let* $\widehat{G}$ *be the graph obtained from G by replacing the vertex x by a complete graph* K_λ *and joining every vertex of* K_λ *to every neighbor of x in G. Then,*

1. *for every vertex* $y \in V(K_\lambda)$, $\mathrm{ecc}_{\widehat{G}}(y) = \mathrm{ecc}_G(x)$,

2. *for every vertex* $y \in \widehat{G} \setminus V(K_\lambda)$, $\mathrm{ecc}_{\widehat{G}}(y) = \mathrm{ecc}_G(y)$.

Theorem 2. *Let* $(a,b,c,d) \in \mathbb{Z}^4$ *be ordered 4-tuples of integers satisfying the constraints of Corollary 2. If* $a \geq 2$, *then there exists a connected graph* G *such that* $|\mathrm{Ext}(G)| = a$, $|\mathrm{Per}(G)| = b$, $|\mathrm{Ct}(G)| = c$, *and* $|\partial(G)| = d$.

Proof. Consider the list of all possible cases according to the constraints given in Corollary 2 (see Table 1). For each one of these cases in Table

Table 1: List of possible cases in the proof of Theorem 2.

(1)	$2 \leq a = b = c = d$	(2)	$2 \leq a < b = c = d$
(3)	$2 \leq a = b < c = d$	(4)	$2 \leq a = b = c < d$
(5)	$2 \leq a < b < c = d$	(6)	$2 \leq a = b < c < d$
(7)	$2 \leq a < b = c < d$	(8)	$2 \leq a < b < c < d$
(9)	$2 \leq b < a = c = d$	(10)	$2 \leq b < a < c = d$
(11)	$2 \leq b < a = c < d$	(12)	$2 \leq b < a < c < d$

1 we give a fitting graph corresponding to the smallest possible values of the parameters (a,b,c,d). Due to the constraints of these parameters, the 4-tuples $(2,3,3,3)$, $(2,2,3,3)$, and $(2,2,2,3)$ are impossible. Therefore, for each of the cases (2), (3), and (4) we give two fitting graphs. In Figures 4 and 5 we illustrate the fitting graphs of all the possible cases. Table 2 shows the corresponding vertices of the sets $\mathrm{Ext}(G)$, $\mathrm{Per}(G)$, $\mathrm{Ct}(G)$, and $\partial(G)$, and the eccentricity sequence of each fitting graph. Notice that for all the graphs in Figures 4 and 5 either $\mathrm{Ext}(G) \subseteq \mathrm{Per}(G)$, or $\mathrm{Per}(G) \subseteq \mathrm{Ext}(G)$ (see Table 2).

The proof is based on an explosion procedure which uses Lemma 2. We apply this procedure to all the cases above. In each case G will be the corresponding fitting graph in Figure 4 or 5, and $\widehat{G}$ will be the graph obtained from G by replacing some particular vertices by complete graphs according to Lemma 2, in such a way that $\widehat{G}$ satisfies the desired cardinalities for the sets $\mathrm{Ext}(\widehat{G})$, $\mathrm{Per}(\widehat{G})$, $\mathrm{Ct}(\widehat{G})$, and $\partial(\widehat{G})$. We proceed as follows.

Case (1): $\mathrm{Ext}(G) = \mathrm{Per}(G) = \mathrm{Ct}(G) = \partial(G) = \{1,3\}$. Replace vertex 3 by K_m. Then $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+1 = a$.

Case (2): $\mathrm{Ext}(G) = \{3,6\}$ and $\mathrm{Per}(G) = \mathrm{Ct}(G) = \partial(G) = \{1,3,4,6\}$. Replace vertex 6 by K_m, and replace vertex 4 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = m+1 = a$, and $|\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+2 = b$.

Case (2'): $\mathrm{Ext}(G) = \{2,3,5\}$ and $\mathrm{Per}(G) = \mathrm{Ct}(G) = \partial(G) = \{1,2,3,5\}$. Replace vertex 2 by K_m, and replace vertex 1 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = m+2 = a$, and $|\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+2 = b$.

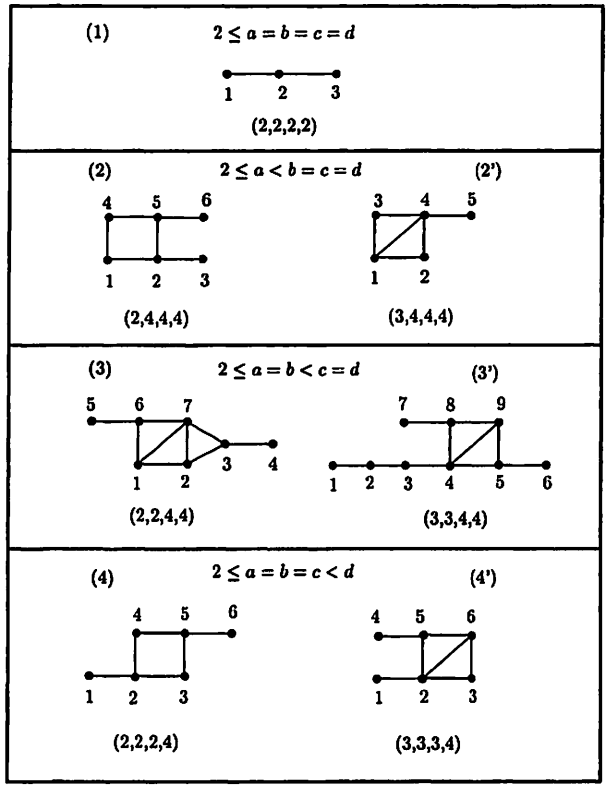

Figure 4: Fitting graphs for the cases (1) to (4).

Case (3): $\mathrm{Ext}(G) = \mathrm{Per}(G) = \{4,5\}$ and $\mathrm{Ct}(G) = \partial(G) = \{1,2,4,5\}$. Replace vertex 5 by K_m, and replace vertex 2 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Per}(\widehat{G})| = m+1 = a$, and $|\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+2 = d$.

Case (3'): $\mathrm{Ext}(G) = \mathrm{Per}(G) = \{1,6,7\}$ and $\mathrm{Ct}(G) = \partial(G) = \{1,6,7,9\}$. Replace vertex 7 by K_m, and replace vertex 9 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = m+2 = a$, and $|\partial(\widehat{G})| = m+n+2 = d$.

Case (4): $\mathrm{Ext}(G) = \mathrm{Per}(G) = \mathrm{Ct}(G) = \{1,6\}$ and $\partial(G) = \{1,3,4,6\}$. Replace vertex 6 by K_m, and replace vertex 4 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = m+1 = a$, and $|\partial(\widehat{G})| = m+n+2 = d$.

Case (4'): $\mathrm{Ext}(G) = \mathrm{Per}(G) = \mathrm{Ct}(G) = \{1,3,4\}$ and $\partial(G) = \{1,3,4,6\}$. Replace vertex 4 by K_m, and replace vertex 6 by K_n. Then $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Per}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = m+2 = a$, and $|\partial(\widehat{G})| = m+n+2 = b$.

Case (5): $\mathrm{Ext}(G) = \{1,6\}$, $\mathrm{Per}(G) = \{1,6,7\}$ and $\mathrm{Ct}(G) = \partial(G) =$

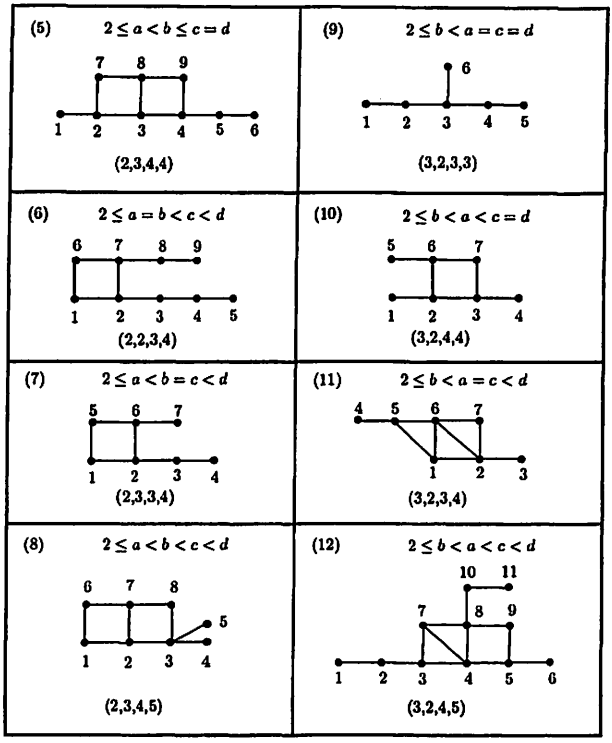

Figure 5: Fitting graphs for the cases (5) to (12).

$\{1,6,7,9\}$. Replace vertex 6 by K_m, replace vertex 7 by K_n, and replace vertex 9 by K_p. Then $|\operatorname{Ext}(\widehat{G})| = m+1 = a$, $|\operatorname{Per}(\widehat{G})| = m+n+1 = b$, and $|\operatorname{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+p+1 = d$.

Case (6): $\operatorname{Ext}(G) = \operatorname{Per}(G) = \{5,9\}$, $\operatorname{Ct}(G) = \{5,6,9\}$ and $\partial(G) = \{1,5,6,9\}$. Replace vertex 9 by K_m, replace vertex 6 by K_n, and replace vertex 1 by K_p. Then $|\operatorname{Ext}(\widehat{G})| = |\operatorname{Per}(\widehat{G})| = m+1 = a$, $|\operatorname{Ct}(\widehat{G})| = m+n+1 = c$, and $|\partial(\widehat{G})| = m+n+p+1 = d$.

Case (7): $\operatorname{Ext}(G) = \{4,7\}$, $\operatorname{Per}(G) = \operatorname{Ct}(G) = \{4,5,7\}$ and $\partial(G) = \{1,4,5,7\}$. Replace vertex 7 by K_m, replace vertex 5 by K_n, and replace vertex 1 by K_p. Then $|\operatorname{Ext}(\widehat{G})| = m+1 = a$, $|\operatorname{Per}(\widehat{G})| = |\operatorname{Ct}(\widehat{G})| = m+n+1 = b$, and $|\partial(\widehat{G})| = m+n+p+1 = d$.

Case (8): $\operatorname{Ext}(G) = \{4,5\}$, $\operatorname{Per}(G) = \{4,5,6\}$, $\operatorname{Ct}(G) = \{4,5,6,8\}$ and $\partial(G) = \{1,4,5,6,8\}$. Replace vertex 5 by K_m, replace vertex 6 by K_n, replace vertex 8 by K_p, and replace vertex 1 by K_q. Then $|\operatorname{Ext}(\widehat{G})| =$

Table 2: Boundary vertex sets of the fitting graphs.

Case	(a,b,c,d)	$\mathrm{Ext}(G)$	$\mathrm{Per}(G)$	$\mathrm{Ct}(G)$	$\partial(G)$	Eccentricity sequence
(1)	(2,2,2,2)	{1,3}	{1,3}	{1,3}	{1,3}	(2,1,2)
(2)	(2,4,4,4)	{3,6}	{1,3,4,6}	{1,3,4,6}	{1,3,4,6}	(3,2,3,3,2,3)
(2')	(3,4,4,4)	{2,3,5}	{1,2,3,5}	{1,2,3,5}	{1,2,3,5}	(2,2,2,1,2)
(3)	(2,2,4,4)	{4,5}	{4,5}	{1,2,4,5}	{1,2,4,5}	(3,3,3,4,4,3,2)
(3')	(3,3,4,4)	{1,6,7}	{1,6,7}	{1,6,7,9}	{1,6,7,9}	(5,4,3,3,4,5,5,4,4)
(4)	(2,2,2,4)	{1,6}	{1,6}	{1,6}	{1,3,4,6}	(4,3,2,2,3,4)
(4')	(3,3,3,4)	{1,3,4}	{1,3,4}	{1,3,4}	{1,3,4,6}	(3,2,3,3,2,2)
(5)	(2,3,4,4)	{1,6}	{1,6,7}	{1,6,7,9}	{1,6,7,9}	(5,4,3,3,4,5,5,4,4)
(6)	(2,2,3,4)	{5,9}	{5,9}	{5,6,9}	{1,5,6,9}	(4,3,4,5,6,5,4,5,6)
(7)	(2,3,3,4)	{4,7}	{4,5,7}	{4,5,7}	{1,4,5,7}	(3,2,3,4,4,3,4)
(8)	(2,3,4,5)	{4,5}	{4,5,6}	{4,5,6,8}	{1,4,5,6,8}	(3,2,3,4,4,4,3,3)
(9)	(3,2,3,3)	{1,5,6}	{1,5}	{1,5,6}	{1,5,6}	(4,3,2,3,4,3)
(10)	(3,2,4,4)	{1,4,5}	{4,5}	{1,4,5,7}	{1,4,5,7}	(3,2,3,4,4,3,3)
(11)	(3,2,3,4)	{3,4,7}	{3,4}	{3,4,7}	{1,3,4,7}	(2,3,4,4,3,2,3)
(12)	(3,2,4,5)	{1,6,11}	{1,11}	{1,6,9,11}	{1,6,7,9,11}	(6,5,4,3,4,5,3,4,5,5,6)

$m+1=a$, $|\mathrm{Per}(\widehat{G})| = m+n+1 = b$, $|\mathrm{Ct}(\widehat{G})| = m+n+p+1 = c$, and $|\partial(\widehat{G})| = m+n+p+q+1 = d$.

Case (9): $\mathrm{Per}(G) = \{1,5\}$ and $\mathrm{Ext}(G) = \mathrm{Ct}(G) = \partial(G) = \{1,5,6\}$. Replace vertex 5 by K_m, and replace vertex 6 by K_n. Then $|\mathrm{Per}(\widehat{G})| = m+1 = b$, $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+1 = a$.

Case (10): $\mathrm{Ext}(G) = \{1,4,5\}$, $\mathrm{Per}(G) = \{4,5\}$, $\mathrm{Ct}(G) = \partial(G) = \{1,4,5,7\}$. Replace vertex 5 by K_m, replace vertex 1 by K_n, and replace vertex 7 by K_p. Then $|\mathrm{Per}(\widehat{G})| = m+1 = b$, $|\mathrm{Ext}(\widehat{G})| = m+n+1 = a$, and $|\mathrm{Ct}(\widehat{G})| = |\partial(\widehat{G})| = m+n+p+1 = d$.

Case (11): $\mathrm{Per}(G) = \{3,4\}$, $\mathrm{Ext}(G) = \mathrm{Ct}(G) = \{3,4,7\}$ and $\partial(G) = \{1,3,4,7\}$. Replace vertex 4 by K_m, replace vertex 7 by K_n, and replace vertex 1 by K_p. Then $|\mathrm{Per}(\widehat{G})| = m+1 = b$, $|\mathrm{Ext}(\widehat{G})| = |\mathrm{Ct}(\widehat{G})| = m+n+1 = a$ and $|\partial(\widehat{G})| = m+n+p+1 = d$.

Case (12): $\mathrm{Ext}(G) = \{1,6,11\}$, $\mathrm{Per}(G) = \{1,11\}$, $\mathrm{Ct}(G) = \{1,6,9,11\}$ and $\partial(G) = \{1,6,7,9,11\}$. Replace vertex 11 by K_m, replace vertex 6 by K_n, replace vertex 9 by K_p, and replace vertex 7 by K_q. Then $|\mathrm{Ext}(\widehat{G})| = m+n+1 = a$, $|\mathrm{Per}(\widehat{G})| = m+1 = b$, $|\mathrm{Ct}(\widehat{G})| = m+n+p+1 = c$, and $|\partial(\widehat{G})| = m+n+p+q+1 = d$.

Now, given $(a,b,c,d) \in \mathbb{Z}^4$ satisfying the constraints of Corollary 2, we choose the adequate values of m, n, p and q for each of the cases and the theorem follows. □

Notice that Theorem 2 still has to be extended for $0 \leq \mathrm{Ext}(G) \leq 1$. As a first step, we conjecture that if G has no extreme vertices and $|\partial(G)| = 4$, then $\partial(G) = \mathrm{Per}(G)$.

3 Convex properties

In this section we consider both geodesic and monophonic convex properties of the boundary vertex sets of a graph. In particular we show that the contour is a monophonic set.

A *convexity* on a finite set V is a family $\mathcal{C}$ of subsets of V, to be regarded as *convex sets*, which is closed under intersection and contains both V and the empty set. The pair $(V, \mathcal{C})$ is called a *convexity space*. A finite *graph convexity space* is a pair $(G, \mathcal{C})$, formed by a finite, connected and undirected graph $G = (V(G), E(G))$; and a convexity $\mathcal{C}$ on $V(G)$ such that $(V(G), \mathcal{C})$ is a convexity space having the property that every member of $\mathcal{C}$ induces a connected subgraph of G [12, 15].

In this section, we deal with two types of graph convexities, both of them defined by a system $\mathcal{P}$ of paths in G: the *geodesic convexity* [15, 18, 21, 27] which arises when we consider shortest paths, and the *monophonic convexity* [12, 15] when we consider chordless paths. A *chord* of a path $(u_0 u_1 \cdots u_h)$ is an edge $u_i u_j$, with $j \geq i + 2$. Given $u, v \in V(G)$, a $u - v$ path ρ is called *monophonic* if it is a chordless path; and ρ is called a *geodesic* if it is a shortest $u - v$ path.

Given $u, v \in V(G)$, the *geodetic interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. Similarly, the *monophonic interval* $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $W \subseteq V(G)$, the *geodetic closure* $I[W]$ of W is defined as the union of all geodetic intervals $I[u, v]$ over all pairs $u, v \in W$. The *monophonic closure* $J[W]$ is the set formed by the union of all monophonic intervals $J[u, v]$.

A vertex set $W \subseteq V(G)$ is called *geodetically convex* (or simply g-*convex*) if $I[W] = W$, while it is said to be *geodetic* if $I[W] = V(G)$. Likewise, W is called *monophonically convex* (or simply m-*convex*) if $J[W] = W$, and is called *monophonic* if $J[W] = V$. The smallest g-convex set containing W is denoted $[W]_g$ and is called the g-*convex hull* of W. Similarly, the m-*convex hull* $[W]_m$ of W is defined as the minimum m-convex set containing W. Observe that $J[W] \subseteq [W]_m$, $I[W] \subseteq [W]_g$ and $[W]_g \subseteq [W]_m$.

The *edge-geodetic interval* $I_e[u, v]$ is the set of edges of all $u - v$ geodesics. For $W \subseteq V(G)$, the *edge-geodetic closure* $I_e[W]$ of W is the union of all edge intervals $I_e[u, v]$ over all pairs $u, v \in W$. A vertex set $W \subseteq V(G)$ is called an *edge-geodetic set* if $I_e[W] = E(G)$.

From the above definitions it is straightforward that: (1) every edge-geodetic set is geodetic set, (2) every edge-monophonic set is monophonic set, and (3) every edge-geodetic set is an edge-monophonic set. It is also easy to find examples where the converses of these statements are not true.

Next results show that all the vertices and edges of a graph lies in some shortest path between two vertices of the boundary of the graph.

Theorem 3. *For any $u \in V(G)$, $W = \{u\} \cup \partial(u)$ is a geodetic set.*

Proof. Pick any vertex $x \in V(G) \setminus W$. Any $u - x$ geodesic extends to a maximal $u - y$ geodesic. This means that $y \in \partial(u)$ and, consequently, x is in a geodesic with endpoints in W. □

As a consequence we get that $\partial(G)$ is a geodetic set (see [5]).

Theorem 4. *$\partial(G)$ is an edge-geodetic set.*

Proof. Let $e = xy \in E(G)$. Consider $P = (a \cdots xy \cdots b)$ a maximal geodesic containing e. Clearly, $b \in \partial(a)$ and $a \in \partial(b)$. Thus e is in a geodesic with endpoints in $\partial(G)$. □

Theorem 5. *Let $G = (V(G), E(G))$ be a connected bipartite graph and $u \in V(G)$. Then $W = \{u\} \cup \partial(u)$ is an edge-geodetic set.*

Proof. Let $u \in V(G)$ and $e = xy \in E(G)$. Because G is a bipartite graph, $d(x,u) \neq d(y,u)$. Suppose that $d(x,u) < d(y,u)$. Consider a $u-x$ geodesic $P = (u \cdots x)$. Then $P' = (u \cdots xy)$ is a geodesic that can be extended to a maximal $u - z$ geodesic. Then $z \in \partial(u)$ and e is in a geodesic with endpoints in W. Thus W is an edge-geodetic set. □

It is known that the contour of a graph is not a geodetic set [7, 5], i.e., not all the vertices of a graph lie necessarily in some shortest path between two vertices of the contour of the graph. In the next theorem we show that this statement is true if we consider chordless paths instead of shortest paths.

The following lemma is straightforward.

Lemma 3. $|\operatorname{ecc}(v) - \operatorname{ecc}(u)| \leq 1$, $\forall uv \in E(G)$.

Theorem 6. *$\operatorname{Ct}(G)$ is a monophonic set.*

Proof. Consider a vertex x of G. Suppose that x is not a contour vertex, i.e., x is a vertex of $V(G) \setminus \operatorname{Ct}(G)$. Since the eccentricities of two adjacent vertices differ by at most one unit, if x is not a contour vertex, then there exists a vertex $y \in V(G)$, adjacent to x, such that its eccentricity satisfies $\operatorname{ecc}(y) = \operatorname{ecc}(x)+1$. This fact implies the existence of a shortest x_0-x_r path, $\rho(x) = (x_0x_1x_2 \cdots x_r)$, such that $x = x_0$, $x_i \notin \operatorname{Ct}(G)$ for $i \in \{0, \ldots, r-1\}$, $x_r \in \operatorname{Ct}(G)$, and $\operatorname{ecc}(x_i) = \operatorname{ecc}(x_{i-1}) + 1 = l + i$ for $i \in \{1, \ldots, r\}$, where $l = \operatorname{ecc}(x)$.

Let us now consider those vertices at distance l from x. Suppose that all of them are at a distance less than $l + r$ from x_r. The vertices at a distance less than l from x are at a distance less than $l + r$ from x_r, a contradiction. Hence, the eccentricity of x_r would be less than $l + r$. This

implies the existence of a vertex z at a distance exactly l from x and $l+r$ from x_r, and x lies on a shortest path $\Psi = (z \cdots x x_1 \cdots x_r)$ between z and x_r (Figure 6).

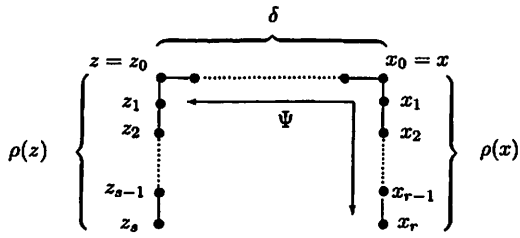

Figure 6: Ψ is a $z - x_r$ shortest path.

Suppose that z is not a contour vertex, since otherwise we are done. Let us construct a path $\rho(z) = (z_0 z_1 \cdots z_s)$ such that $z = z_0$, $z_i \notin \mathrm{Ct}(G)$ for $i \in \{0, \ldots, s-1\}$, $z_s \in \mathrm{Ct}(G)$ and $\mathrm{ecc}(z_i) = \mathrm{ecc}(z_{i-1}) + 1 = \mathrm{ecc}(z) + i$ for $i \in \{1, \ldots, s\}$ (Figure 6).

Let δ be the $z - x$ sub-path of Ψ. Notice that the vertex z satisfies $\mathrm{ecc}(z) \geq l+r$, the vertices of $V(\rho(z)) \setminus \{z\}$ have eccentricity at least $l+r+1$ and the vertices of $V(\rho(x))$ have eccentricity at most $l+r$. Therefore, the sets $V(\rho(z))$ and $V(\rho(x))$ are disjoint. Moreover, taking into account the eccentricities of all of these vertices and Lemma 3, if there is an edge joining a vertex of $V(\rho(z)) \setminus \{z\}$ with a vertex of $V(\rho(x))$, it must be $z_1 x_r$. In this case, $d(z, x_r) = 2 = l + r$, implying that $l = r = 1$ (Figure 7). Hence, the eccentricity of x is 1, the diameter of the graph is 2 and z is a contour vertex, which is a contradiction.

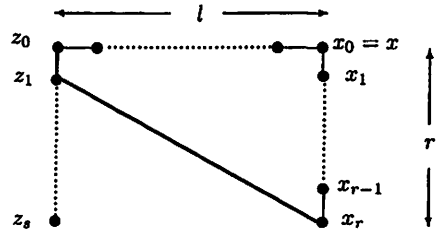

Figure 7: If $d(z_0, x_r) = 2$, then $l = 1$ and $r = 1$.

Notice that the sets of vertices $V(\rho(z)) \setminus \{z\}$ and $V(\delta)$ are not necessarily disjoint. Consider a $z_s - x$ path P contained in the walk $\rho(z) \cup \delta = (z_s \cdots z_1 z \cdots x)$. If P has a chord $e = ab$, we can replace the $a - b$ sub-path

of P with e obtaining a $z_s - x$ path P'. Since $V(P') \subsetneq V(P)$, the path P' has strictly less chords than P. We proceed in an analogous way with P', until we obtain a chordless $z_s - x$ path P^*. Recall that $\Psi = (z \cdots x \cdots x_r)$ is a shortest path, which means that there are no edges joining vertices of $V(\delta) \cup V(\rho(x))$. Therefore, $P^* \cup \rho(x)$ is a monophonic $z_s - x_r$ path through x with $z_s, x_r \in \mathrm{Ct}(G)$. □

As a consequence of Theorem 6, we obtain the following corollary, which was directly proved by Cáceres et al. [7].

Corollary 3. [7] *The contour of a distance-hereditary graph is a geodetic set.*

4 Metric dimension property

In this section we focus our attention on a metric dimension property of the boundary of a graph showing that the cardinality of the boundary is an upper bound of the metric dimension of a graph.

We say that a vertex $x \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(x,u) \neq d(x,v)$. A set of vertices $W \subseteq V(G)$ is a *resolving set* of G, if for all pair of vertices $u, v \in V(G)$ there exists a vertex in W which resolves u, v. In other words, W resolves a graph G, if every vertex of G is uniquely determined by its vector of distances to the vertices of W. The *metric dimension* of G is the minimum cardinality of a resolving set of G.

Resolving sets in general graphs were defined by Slater [25] and Harary and Melter [17], they have been widely investigated and arise in many diverse areas including network, discovery and verification, robot navigation, connected joins in graphs and strategies for the Mastermind game [6]. The next results give an upper bound of the metric dimension of a graph.

Theorem 7. *$\partial(G)$ is a resolving set of G.*

Proof. Pick any pair x, y of vertices of G. Any $x - y$ geodesic extends to a maximal $x' - y'$ geodesic. Clearly, $x' \in \partial(y') \subseteq \partial(G)$ and $d(x', x) \neq d(x', y)$. □

Corollary 4. *The metric dimension of G is at most $|\partial(G)|$.*

The metric dimension of a graph is closely related to the minimum determining set of a graph. In fact, the definition of determining set of a graph is from Harary [16] with the nomenclature of *fixing set*, see also Erwin and Harary [14], and later rediscover by Boutin [1] which use the word *determining set*, but Boutin's definition is completely equivalent to fixing set. Thus, the minimum cardinality of a fixing set is the *fixing number*, which corresponds to the Boutin's definition of determining number [1].

Definition 1. [1, 16] *A subset $W \subseteq V(G)$ in a connected graph $G = (V(G), E(G))$ is called a determining set for G if whenever $g, h \in \mathrm{Aut}(G)$ agree on the vertices of W, they agree on all the vertices of $V(G)$. That is, W is a determining set if whenever g and h are automorphisms with the property that $g(w) = h(w)$ for all $w \in W$, then $g = h$. The determining number of G is the minimum cardinality of a determining set of G.*

Intuitively, a set W is a determining set of a graph if and only if every vertex in the graph can be uniquely identified by its own graph properties and its relationship to the vertices in W. Every graph has a determining set.

Proposition 1. [1, 16] *If W is a resolving set of G, then it is a determining set of G.*

Corollary 5. *The determining number of a connected graph G is at most $|\partial(G)|$.*

Now we consider the graph G illustrated in Figure 8, where $\mathrm{Ext}(G) = \mathrm{Per}(G) = \mathrm{Ct}(G) = \mathrm{Ecc}(G) = \{1, 7\}$ and $\partial(G) = \{1, 4, 7, 8\}$. The sets $\mathrm{Ext}(G)$, $\mathrm{Per}(G)$, $\mathrm{Ct}(G)$ and $\mathrm{Ecc}(G)$ are neither resolving nor determining sets for G. On the other hand, $\partial(G)$ is both resolving and determining set for G. Nevertheless it is easy to prove that $W = \{1, 4\}$ is a resolving and a determining set of minimum cardinality for G. Thus, it is clear that $|\partial(G)|$ is not a tight upper bound for both the metric dimension and the determining number of a graph.

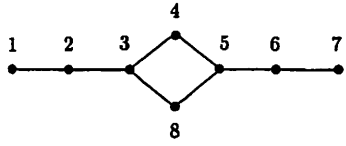

Figure 8: The graph G.

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