SOME THEOREMS ON BERNOULLI AND EULER NUMBERS

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Abstract From differential operator and the generating functions of Bernoulli and Euler polynomials, we derive some new theorems on Bernoulli and Euler numbers. By using integral formulae of arithmetical properties relating to the Bernoulli and Euler polynomials, we obtain new identities on Bernoulli and Euler numbers. Finally we give some new properties on Bernoulli and Euler numbers arising from the p-adic integrals on $\mathbb{Z}_p$

1. Introduction

The Bernoulli and Euler polynomials are defined by the generating functions as follows :

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1}$$

In the special case, $x=0$, $B_n(0)=B_n$ and $E_n(0)=E_n$ are called the n-th Bernoulli and Euler numbers (see[1,2,15]).
In [4,5], Carlitz gave the integral of the product of several Bernoulli polynomials as follows:

$$\int_0^A B_{m_1}(\frac{x}{a_1})\cdots B_{m_n}(\frac{x}{a_n})dx=a_1^{1-m_1}\cdots a_n^{1-m_n}\int_0^1 B_{m_1}(x)\cdots B_{m_n}(x)dx,$$

where $a_1,a_2\cdots,a_n$ are positive integers that are relatively prime in pairs and $A=a_1a_2\cdots a_n$. For $n=2$, there is the formula

$$\int_0^1 B_p(x)B_q(x)dx=(-1)^{p+1}\frac{p!\,q!}{(p+q)!}B_{p+q},\qquad (p+q\geq 2).$$

It may be of interest in this connection to recall the formula for a product of two Bernoulli polynomials:

$$B_m(x)B_n(x)=\sum_r\{\binom{m}{2r}n+\binom{n}{2r}m\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{m!\,n!}{(m+n)!}B_{m+n},$$

where $m+n\geq 2$, (see [5]).

For $m,n,p\geq 1$, Carlitz also obtained the following equation:

$$\int_0^1 B_m(x)B_n(x)B_p(x)dx$$
$$=(-1)^{p+1}p!\sum_r\{\binom{m}{2r}n+\binom{n}{2r}m\}\frac{(m+n-2r-1)!}{(m+n+p-2r)!}B_{2r}B_{m+n+p-2r}\ (see[5]).$$

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From (1), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_l,\qquad E_n(x)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}E_l. \tag{2}$$

By (2), we get

$$B_0=1,\ B_n(1)-B_n=\delta_{1,n},\quad E_0=1,\ E_n(1)+E_n=2\delta_{0,n}, \tag{3}$$

where $\delta_{n,k}$ is kronecker symbol.

From (2), we can derive the following equations:

$$\frac{d}{dx}B_n(x)=nB_{n-1}(x),\quad \frac{d}{dx}E_n(x)=nE_{n-1}(x)\qquad (n\in\mathbb{Z}_+). \tag{4}$$

Thus, by (3) and (4), we get

$$\int_0^1 B_n(x)dx=\frac{\delta_{0,n}}{n+1},\quad \int_0^1 E_n(x)dx=-\frac{2E_{n+1}}{n+1}. \tag{5}$$

As is well known, the gamma and beta functions are defined by the following definite integrals ($\alpha>0$, $\beta>0$):

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt,\ \text{(see [8-12])}, \tag{6}$$

and

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt. \tag{7}$$

By (6) and (7), we get he following equations:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha),\quad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8}$$

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p,\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-\nu_p(p)}=\frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_1(f)=\int_{\mathbb{Z}_p}f(x)d\mu(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x),\quad \text{(see [8,10])}. \tag{9}$$

and the fermionic p-adic integral on $\mathbb{Z}_p$ is given by

$$I_{-1}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)=\lim_{n\to\infty}\sum_{x=0}^{p^n-1}f(x)(-1)^x,\quad \text{(see [9])}. \tag{10}$$

By (9) and (10), we get

$$I_1(f_1)-I_1(f)=f'(0),\quad \textit{and}\quad I_{-1}(f_1)+I_{-1}(f)=2f(0), \tag{11}$$

where $f_1(x)=f(x+1)$, (see [1-16]).

Let us take $f(y)=e^{t(x+y)}$. Then, by (11), we get

$$\int_{\mathbb{Z}_p}e^{t(x+y)}d\mu(y)=\frac{te^{xt}}{e^t-1},\quad \int_{\mathbb{Z}_p}e^{t(x+y)}d\mu_{-1}(y)=\frac{2e^{xt}}{e^t+1}. \tag{12}$$

By (1) and (12), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \qquad \int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x).$$

In this paper, we derive some new theorems on Bernoulli and Euler numbers from differential operator and the generating functions of Bernoulli and Euler polynomials. By using integral formulae of arithmetical properties relating to the Bernoulli and Euler polynomials, we obtain some interesting identities on Bernoulli and Euler numbers. Finally we give some new properties on Bernoulli and Euler numbers arising from the p-adic integrals on $\mathbb{Z}_p$.

2. Theorems on Bernoulli and Euler numbers

In this section we assume that a, b are integers. Let $F_x = F_x(t) = \frac{e^{xt}}{e^t-1}$. Then, by(1), we get

$$F_x = F_x(t) = \frac{e^{xt}}{e^t-1} = \frac{1}{t}\frac{te^{xt}}{e^t-1} = \frac{1}{t} + \sum_{l=0}^{\infty} \frac{B_{l+1}(x)}{l+1}\frac{t^l}{l!}. \tag{14}$$

Let $D = \frac{d}{dt}$. Then we easily see that

$$D(e^{at}f) = ae^{at}f + e^{at}D(f) = e^{at}(aI+D)f, \tag{15}$$

where I is the identity operator with $If = f$. Thus, by(15), we get

$$D^2(e^{at}f) = e^{at}(a^2I + 2aDI + D^2)f = e^{at}(aI+D)^2 f. \tag{16}$$

Continuing this process, we get

$$D^N(e^{at}f) = e^{at}(aI+D)^N f \quad (n \in \mathbb{N}). \tag{17}$$

From (1) and (14), we have

$$e^t F_x - F_x = \frac{e^t-1}{e^t-1}e^{xt} = e^{xt}. \tag{18}$$

Thus, by (18), we get

$$e^{(a+1)t}F_x - e^{at}F_x = e^{(x+a)t}. \tag{19}$$

From (17) and (19), we can derive the following equation:

$$e^{(a+1)t}(D+(a+1)I)^k F_x - e^{at}(D+aI)^k F_x = (x+a)^k e^{(x+a)t}. \tag{20}$$

Thus, by (20), we get

$$e^{(a+b+1)t}(D+(a+1)I)^k F_x - e^{(a+b)t}(D+aI)^k F_x = (x+a)^k e^{(x+a+b)t}. \tag{21}$$

From (17) and (21), we have

$$\begin{aligned} e^{(a+b+1)t}(D+(a+b+1)I)^m(D+(a+1)I)^k F_x - e^{(a+b)t}(D+(a+b)I)^m(D+aI)^k F_x \\ = (x+a)^k(x+a+b)^m e^{(a+b+x)t}. \end{aligned} \tag{22}$$

Dividing by $e^{(a+b)t}$ on both sides in (22), we have

$$\begin{aligned} e^t(D+(a+b+1)I)^m(D+(a+1)I)^k F_x - (D+(a+b)I)^m(D+aI)^k F_x \\ = (x+a)^k(x+a+b)^m e^{xt}. \end{aligned} \tag{23}$$

Thus, by expanding on t in (23), we have

$$\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}e^tD^{j+l}F_x$$
$$-\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}D^{j+l}F_x=(x+a)^k(x+a+b)^m e^{xt}. \tag{24}$$

Let $G[0]$(not $G(0)$)be the constant term in a Laurent series of G(t). Then, by (24), we get

$$\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}(e^tD^{j+l}F_x)[0]$$
$$-\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}(D^{j+l}F_x)[0]=(x+a)^k(x+a+b)^m. \tag{25}$$

By (14), we get

$$D^NF_x=\frac{(-1)^NN!}{t^{N+1}}+\sum_{j=N}^{\infty}\frac{B_{j+1}(x)}{j+1}\frac{t^{j-N}}{(j-N)!}=\frac{(-1)^NN!}{t^{N+1}}+\sum_{j=0}^{\infty}\frac{B_{j+N+1}(x)}{j+N+1}\frac{t^j}{j!}, \tag{26}$$

and

$$e^tD^NF_x=e^t\frac{(-1)^NN!}{t^{N+1}}+e^t\sum_{j=0}^{\infty}\frac{B_{j+N+1}(x)}{j+N+1}\frac{t^j}{j!}$$
$$=(\frac{(-1)^NN!}{t^{N+1}})(\sum_{l=0}^{\infty}\frac{t^l}{l!})+(\sum_{l=0}^{\infty}\frac{t^l}{l!})(\sum_{j=0}^{\infty}\frac{B_{j+N+1}(x)}{j+N+1}\frac{t^j}{j!}). \tag{27}$$

From (26) and (27), we note that

$$(D^NF_x)[0]=\frac{B_{N+1}(x)}{N+1},$$

and

$$(e^tD^NF_x)[0]=\frac{(-1)^NN!}{(N+1)!}+\frac{B_{N+1}(x)}{N+1}. \tag{28}$$

Therefore, by (25), (27) and (28), we obtain the following theorem.

Theorem 2.1. *For $m,k\in\mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}\frac{(-1)^{j+l}}{j+l+1}$$
$$+\sum_{j=0}^{m}\sum_{l=0}^{k}\{(a+b+1)^{m-j}(a+1)^{k-l}-(a+b)^{m-j}a^{k-l}\}\binom{m}{j}\binom{k}{l}\frac{B_{j+l+1}(x)}{j+l+1}$$
$$=(x+a)^k(x+a+b)^m.$$

.

Let us take $m = k$, $a = 0$, and $b = -2$ (or $m = k$, $a = -2$, and $b = 2$). Then, from Theorem 1, we have

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{(-1)^{j+l}}{j+l+1}+\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{B_{j+l+1}(x)}{j+l+1}$$
$$-\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{B_{j+m+1}(x)}{j+m+1}=x^m(x-2)^m. \tag{29}$$

Let us take the definite integral from 0 to 1 on both sides in (29). Then, by (5) and (29), we get

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{(-1)^{j+l}}{j+l+1}=\int_0^1 x^m(x-2)^m dx. \tag{30}$$

Let $J_m = \int_0^1 x^m(x-2)^m dx$. Then we have

$$\begin{aligned}J_m = \int_0^1 x^m(x-2)^m dx &= 2\int_0^{1/2}(2t-2)^m(2t)^m dt\\
&= (-1)^m 2^{2m}\left(2\int_0^{\frac{1}{2}} t^m(1-t)^m dt\right)\\
&= (-1)^m 2^{2m}\int_0^1 t^m(1-t)^m dt = (-1)^m 2^{2m} B(m+1,m+1)\\
&= (-1)^m 2^{2m}\frac{\Gamma(m+1)\Gamma(m+1)}{\Gamma(2m+2)}\\
&= (-1)^m 2^{2m}\frac{m!m!}{(2m+1)!}\\
&= \frac{(-1)^m 2^{2m}}{2m+1}\frac{1}{\binom{2m}{m}}.\end{aligned} \tag{31}$$

Therefore, by (30) and (31), we obtain the following theorem.

Theorem 2.2. *For $m \in \mathbb{Z}_+$ we have*

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{j}\binom{m}{l}\frac{1}{j+l+1}=\frac{(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}}.$$

Let us take $m = k$, $a = -1$, and $b = 2$ (or $m = k$, $a = 1$, and $b = -2$) in Theorem 1. Then we have

$$\sum_{j=0}^{m}2^{m-j}\binom{m}{j}\frac{(-1)^{m+j}}{m+j+1}+\sum_{j=0}^{m}2^{m-j}\binom{m}{j}\frac{B_{j+m+1}(x)}{j+m+1}$$
$$-\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{B_{j+l+1}(x)}{j+l+1}=(x^2-1)^m. \tag{32}$$

Let us take the definite integral from 0 to 1 on both sides in (32). Then, by (5) and (32), we get

$$\sum_{j=0}^{m}2^{m-j}\binom{m}{j}\frac{(-1)^{m+j}}{m+j+1}=\int_0^1 (x^2-1)^m dx. \tag{33}$$

Let $I_m = \int_0^1 (x^2-1)^m dx$. Then we see that

$$\begin{aligned} I_m &= \int_0^1 (x^2-1)^m dx \\ &= \int_0^1 (x^2-1)(x^2-1)^{m-1} dx \\ &= \int_0^1 x^2(x^2-1)^{m-1} dx - \int_0^1 (x^2-1)^{m-1} dx, \end{aligned} \tag{34}$$

and

$$\int_0^1 x^2(x^2-1)^{m-1} dx = [\frac{1}{2m}(x^2-1)^m x]_0^1 - \frac{1}{2m}\int_0^1 (x^2-1)^m dx = -\frac{I_m}{2m}. \tag{35}$$

By (34) and (35), we get

$$I_m = -\frac{1}{2m} I_m - I_{m-1}. \tag{36}$$

Thus, from (36), we have

$$I_m = \frac{-2m}{2m+1} I_{m-1} \quad (m \geq 1). \tag{37}$$

Continuing this process, we have

$$\begin{aligned} I_m &= (\frac{-2m}{2m+1})(\frac{-2(m-1)}{2m-1})\dots(-\frac{2}{3})I_0 \\ &= (-1)^m(\frac{2m}{2m+1})(\frac{2m-2}{2m-1})\dots(\frac{2}{3}) \\ &= (-1)^m \prod_{k=1}^{m}(\frac{2k}{2k+1}) = J_m. \end{aligned} \tag{38}$$

Therefore, by (33) and (38), we obtain the following theorem.

Theorem 2.3. *For $m \in \mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m} 2^{m-j}\binom{m}{j}\frac{(-1)^{m-j}}{m+j+1} = (-1)^m \prod_{l=1}^{m}(\frac{2l}{2l+1}).$$

In particular,

$$\sum_{j=0}^{m} 2^{m-j}\binom{m}{j}\frac{(-1)^{m-j}}{m+j+1} = \frac{(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}}.$$

Let us consider the generating function of Euler polynomials:

$$G_x = G_x(t) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{39}$$

From (39), we note that

$$e^{(a+1)t}G_x + e^{at}G_x = 2e^{(x+a)t}. \tag{40}$$

By (17) and (40), we get

$$e^{(a+1)t}(D+(a+1)I)^k G_x + e^{at}(D+aI)^k G_x = 2(x+a)^k e^{(x+a)t}. \tag{41}$$

From (41), we can derive the following equation:

$$\begin{aligned} &e^{(a+b+1)t}(D+(a+1)I)^k G_x + e^{(a+b)t}(D+aI)^k G_x \\ &= 2(x+a)^k e^{(x+a+b)t}. \end{aligned} \tag{42}$$

Let us take the m-th derivative D^m on both sides in (42). Then, by (17), we get

$$\begin{aligned} &e^{(a+b+1)t}(D+(a+b+1)I)^m(D+(a+1)I)^k G_x \\ &+e^{(a+b)t}(D+(a+b)I)^m(D+aI)^k G_x = 2(x+a)^k(x+a+b)^m e^{(x+a+b)t}. \end{aligned} \tag{43}$$

Dividing by $e^{(a+b)t}$ on both sides in (43), we have

$$\begin{aligned} &e^t(D+(a+b+1)I)^m(D+(a+1)I)^k G_x + (D+(a+b)I)^m(D+aI)^k G_x \\ &= 2(x+a+b)^m(x+a)^k e^{xt}. \end{aligned} \tag{44}$$

Thus, by binomial theorem, we get

$$\begin{aligned} &\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}(e^t D^{j+l}G_x) \\ &+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}(D^{j+l}G_x) = 2(x+a+b)^m(x+a)^k e^{xt}. \end{aligned} \tag{45}$$

From (45), we have

$$\begin{aligned} &\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}(e^t D^{j+l}G_x)[0] \\ &+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}(D^{j+l}G_x)[0] = 2(x+a+b)^m(x+a)^k. \end{aligned} \tag{46}$$

By (39), we get

$$(D^N G_x)[0] = E_N(x),$$

and

$$(e^t D^N G_x)[0] = E_N(x). \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 2.4. *For $m, k \in \mathbb{Z}_+$, we have*

$$\begin{aligned} &\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b+1)^{m-j}(a+1)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x) \\ &+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b)^{m-j}a^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x) = 2(x+a+b)^m(x+a)^k. \end{aligned}$$

Let $m=k$, $a=0$, and $b=-2$(or $m=k$,$a=-2$, and $b=2$) in Theorem 4. Then we have

$$\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}E_{j+l}(x)+\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}E_{j+m}(x) = 2(x-2)^m x^m. \tag{48}$$

Let us take the definite integral from 0 to 1 in (48). Then, by (5), we get

$$\begin{aligned} &-2\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1} - 2\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{E_{j+m+1}}{j+m+1} \\ &= 2\int_0^1 (x-2)^m x^m dx. \end{aligned} \tag{49}$$

From (31) and (49), we can derive the following equation:

$$
\begin{aligned}
&-2\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1}-2\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}\frac{E_{j+m+1}}{j+m+1} \\
&=\frac{2(-1)^m 2^{2m}}{(2m+1)\binom{2m}{m}}.
\end{aligned}
\tag{50}
$$

Therefore, by (50), we obtain the following theorem.

Theorem 2.5. *For $m \in \mathbb{Z}_+$, we have*

$$
\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1}+\sum_{j=0}^{m}\binom{m}{j}(-2)^{m-j}\frac{E_{j+m+1}}{j+m+1} \\
&=\frac{(-1)^{m-1}2^{2m}}{(2m+1)\binom{2m}{m}}.
\end{aligned}
$$

Let us take $m=k$, $a=1$, $b=-2$(or $m=k$, $a=-1$, and $b=2$) in Theorem 4. Then we have

$$
\sum_{l=0}^{m}2^{m-l}\binom{m}{l}E_{m+l}(x)+\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}E_{j+l}(x)=2(x^2-1)^m. \tag{51}
$$

Taking integral from 0 to 1 in (51), we get

$$
\begin{aligned}
&-2\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{E_{m+l+1}}{m+l+1}-2\sum_{j=0}^{m}\sum_{l=0}^{m}(-1)^{m-j}\binom{m}{j}\binom{m}{l}\frac{E_{j+l+1}}{j+l+1} \\
&=2\int_0^1(x^2-1)^m dx.
\end{aligned}
\tag{52}
$$

From (38) and (52), we have

$$
\begin{aligned}
&\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{E_{m+l+1}}{m+l+1}+\sum_{j=0}^{m}\sum_{l=0}^{m}\binom{m}{j}\binom{m}{l}(-1)^{m-j}\frac{E_{j+l+1}}{j+l+1} \\
&=(-1)^{m+1}\prod_{l=1}^{m}\left(\frac{2l}{2l+1}\right).
\end{aligned}
\tag{53}
$$

Therefore, by (53), we obtain the following theorem.

Theorem 2.6. *For $m \in \mathbb{Z}_+$, we have*

$$
\sum_{l=0}^{m}2^{m-l}\binom{m}{l}\frac{E_{m+l+1}}{m+l+1}+\sum_{j=0}^{m}\sum_{l=0}^{m}\binom{m}{l}\binom{m}{j}(-1)^{m-j}\frac{E_{j+l+1}}{j+l+1}=(-1)^{m+1}\prod_{l=1}^{m}\left(\frac{2l}{2l+1}\right).
$$

From (31), (38), Theorem 5 and Theorem 6, we obtain the following corollary.

Corollary 2.7. *For $m \in \mathbb{Z}_+$, we have*

$$
\sum_{l=0}^{m}\binom{m}{l}(1-(-1)^{m-l})2^{m-l}\frac{E_{l+m+1}}{l+m+1}=0.
$$

Indeed, let us take m be even, replacing m by $2m$ in Corollary 7. Then we have

$$
2\sum_{j=0}^{m}2^{2m-2j-1}\binom{2m}{2j+1}\frac{E_{2m+2j+2}}{2m+2j+2}=0. \tag{54}
$$

If we take m is odd, replacing m by $2m+1$ in Corollary 7, then we have

$$2\sum_{j=0}^{[\frac{2m+1}{2}]} 2^{2m-2j+1}\binom{2m+1}{2j}\frac{E_{2m+2j+2}}{2m+2j+2}=0. \tag{55}$$

Corollary 2.8. *For $m \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{m}\binom{m}{l}2^{m-l}\frac{E_{l+m+1}}{l+m+1}=\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}2^{m-l}\frac{E_{l+m+1}}{l+m+1},$$

where $[\cdot]$ is Gauss' symbol.

3. Note on the p-adic integral of Bernoulli and Euler polynomials

In this section, we assume that $a, b, c, d \in \mathbb{Z}$. From (10), (11), (12), (46),and (47), we can derive the following equation:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\{((a+b+1)+(x+y))^m((a+1)+(x+y))^k\}d\mu_{-1}(y)\\&+\int_{\mathbb{Z}_p}\{((a+b)+(x+y))^m(a+(x+y))^k\}d\mu_{-1}(y)=2(x+a+b)^m(x+a)^k.\end{aligned} \tag{56}$$

Thus, by (56), we get

$$\begin{aligned}&2(x+a+b)^m(x+a)^k\\&=\int_{\mathbb{Z}_p}\{((a+b-c+1)+(x+c+y))^m((a-c+1)+(x+c+y))^k\}d\mu_{-1}(y)\\&+\int_{\mathbb{Z}_p}\{((a+b-d)+(x+y+d))^m((a-d)+(x+y+d))^k\}d\mu_{-1}(y)\\&=\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-c+1)^{m-j}(a-c+1)^{k-l}\int_{\mathbb{Z}_p}(x+c+y)^{j+l}d\mu_{-1}(y)\\&+\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-d)^{m-j}(a-d)^{k-l}\int_{\mathbb{Z}_p}(x+d+y)^{j+l}d\mu_{-1}(y).\end{aligned} \tag{57}$$

By (13) and (57), we get

$$\begin{aligned}&2(x+a+b)^m(x+a)^k\\&=\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-c+1)^{m-j}(a-c+1)^{k-l}E_{j+l}(x+c)\\&+\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-d)^{m-j}(a-d)^{k-l}E_{j+l}(x+d).\end{aligned} \tag{58}$$

Therefore, by (58), we obtain the following proposition.

Proposition 3.1. *For $m, k \in \mathbb{Z}_+$, we have*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b-c+1)^{m-j}(a-c+1)^{k-l}E_{j+l}(x+c)\\&+\sum_{j=0}^{m}\sum_{l=0}^{k}(a+b-d)^{m-j}(a-d)^{k-l}\binom{m}{j}\binom{k}{l}E_{j+l}(x+d)=2(x+a+b)^m(x+a)^k.\end{aligned}$$

By (56), we also get

$$\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b+1)^{m-j}(a+1)^{k-l}\int_{\mathbb{Z}_p}(x+y)^{j+l}d\mu_{-1}(y)$$
$$+\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b)^{m-l}a^{k-l}\int_{\mathbb{Z}_p}(x+y)^{j+l}d\mu_{-1}(y)=2(x+a+b)^m(x+a)^k. \tag{59}$$

Thus, by (59), we obtain the following proposition.

Proposition 3.2. *For $m,k\in\mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b+1)^{m-j}(a+1)^{k-l}E_{j+l}(x)+\sum_{j=0}^{m}\sum_{l=0}^{k}\binom{m}{j}\binom{k}{l}(a+b)^{m-l}a^{k-l}E_{j+l}(x)$$
$$=2(x+a+b)^m(x+a)^k.$$

Let $m=k$, $a=0$, and $b=-2$(or $m=k, a=-2, b=2$) in (56). Then we have

$$\int_{\mathbb{Z}_p}((x+y)^2-1)^m d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(x+y-2)^m(x+y)^m d\mu_{-1}(y)=2(x-2)^m x^m. \tag{60}$$

From (60), we can derive the following equation:

$$2(x-2)^m x^m=\int_{\mathbb{Z}_p}((x+y)^2-1)^m d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(x+y-2)^m(x+y)^m d\mu_{-1}(y)$$
$$=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\int_{\mathbb{Z}_p}(x+y)^{2j}d\mu_{-1}(y)+\sum_{j=0}^{m}\binom{m}{j}(-2)^{m-j}\int_{\mathbb{Z}_p}(x+y)^{m+j}d\mu_{-1}(y)$$
$$=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}E_{2j}(x)+\sum_{j=0}^{m}\binom{m}{j}(-2)^{m-j}E_{m+j}(x). \tag{61}$$

Therefore, by (61), we obtain the following proposition.

Proposition 3.3. *For $m\in\mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}E_{2j}(x)+\sum_{j=0}^{m}(-2)^{m-j}\binom{m}{j}E_{m+j}(x)=2(x-2)^m x^m.$$

Let $m=k$, $a=1$, $b=-2$(or $m=k$, $a=-1$, $b=2$) in (56). Then we have

$$\int_{\mathbb{Z}_p}(x+y)^m(2+x+y)^m d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(-1+x+y)^m(1+x+y)^m d\mu_{-1}(y)=2(x^2-1)^m. \tag{62}$$

Thus, by (62), we get

$$2(x^2-1)^m=\int_{\mathbb{Z}_p}(x+y)^m(2+x+y)^m d\mu_{-1}(y)+\int_{\mathbb{Z}_p}((x+y)^2-1)^m d\mu_{-1}(y)$$
$$=\sum_{j=0}^{m}\binom{m}{j}2^{m-j}\int_{\mathbb{Z}_p}(x+y)^{m+j}d\mu_{-1}(y)+\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}\int_{\mathbb{Z}_p}(x+y)^{2j}d\mu_{-1}(y)$$
$$=\sum_{j=0}^{m}\binom{m}{j}2^{m-j}E_{m+j}(x)+\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}E_{2j}(x). \tag{63}$$

Therefore, by (63), we obtain the following proposition.

Proposition 3.4. *For $m \in \mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}\binom{m}{j}2^{m-j}E_{m+j}(x)+\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}E_{2j}(x)=2(x^2-1)^m.$$

.

By Proposition 11 and Proposition 12, we obtain the following corollary.

Corollary 3.5. *For $m \in \mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}(1-(-1)^{m-j})2^{m-j}\binom{m}{j}E_{m+j}(x)=2(x^2-1)^m-2(x-2)^m x^m.$$

Let m be even, replacing m by $2m$ in Corollary 13. Then we have

$$2\sum_{j=0}^{m}2^{2m-(2j+1)}\binom{2m}{2j+1}E_{2m+2j+1}(x)=2(x^2-1)^{2m}-2(x-2)^{2m}x^{2m}. \tag{64}$$

Therefore, by (64), we obtain the following corollary.

Corollary 3.6. *For $m \in \mathbb{Z}_+$, we have*

$$\sum_{j=0}^{m}2^{2m-2j-1}\binom{2m}{2j+1}E_{2m+2j+1}(x)=(x^2-1)^{2m}-(x-2)^{2m}x^{2m}.$$

Let m be odd, replacing m by $2m-1$ in Corollary 13. Then we have

$$2\sum_{j=0}^{[\frac{2m-1}{2}]}2^{2m-1-2j}\binom{2m-1}{2j}E_{2m-1+2j}(x)=2(x^2-1)^{2m-1}-2(x-2)^{2m-1}x^{2m-1}. \tag{65}$$

Therefore, by (65), we obtain the following corollary.

Corollary 3.7. *For $m \in \mathbb{N}$, we have*

$$\sum_{j=0}^{[\frac{2m-1}{2}]}2^{2m-1-2j}\binom{2m-1}{2j}E_{2m-1+2j}(x)=(x^2-1)^{2m-1}-(x-2)^{2m-1}x^{2m-1}.$$

Let us consider the bosonic p-adic integral on $\mathbb{Z}_p$ in Proposition 11. By (9), (12), (13) and Proposition 11, we get

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{2j}\binom{m}{j}\binom{2j}{l}(-1)^{m-j}E_{2j-l}\int_{\mathbb{Z}_p}x^l d\mu(x)\\&+\sum_{j=0}^{m}\sum_{l=0}^{m+j}(-2)^{m-j}\binom{m}{j}\binom{m+j}{l}E_{m+j-l}\int_{\mathbb{Z}_p}x^l d\mu(x)=2\sum_{l=0}^{m}\binom{m}{l}(-2)^{m-l}\int_{\mathbb{Z}_p}x^{m+l}d\mu(x).\end{aligned} \tag{66}$$

From (66) and (13), we note that

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{2j}\binom{m}{j}\binom{2j}{l}(-1)^{m-j}E_{2j-l}B_l\\&+\sum_{j=0}^{m}\sum_{l=0}^{m+j}(-2)^{m-j}\binom{m}{j}\binom{m+j}{l}E_{m+j-l}B_l=2\sum_{l=0}^{m}\binom{m}{l}(-2)^{m-l}B_{m+l}.\end{aligned} \tag{67}$$

By (9), (12), and Proposition 12, we get

$$\sum_{j=0}^{m}\sum_{l=0}^{m+j}\binom{m}{j}2^{m-j}\binom{m+j}{l}E_{m+j-l}\int_{\mathbb{Z}_p}x^l d\mu(x)$$
$$+\sum_{j=0}^{m}\sum_{l=0}^{2j}\binom{m}{j}\binom{2j}{l}(-1)^{m-j}E_{2j-l}\int_{\mathbb{Z}_p}x^l d\mu(x)=2\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}\int_{\mathbb{Z}_p}x^{2l}d\mu(x). \tag{68}$$

Thus, from (68), we have

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{m+j}\binom{m}{j}\binom{m+j}{l}2^{m-j}E_{m+j-l}B_l\\&+\sum_{j=0}^{m}\sum_{l=0}^{2j}\binom{m}{j}\binom{2j}{l}(-1)^{m-j}E_{2j-l}B_l=2\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}B_{2l}.\end{aligned} \tag{69}$$

By (67) and (69), we get

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{l=0}^{m+j}(1-(-1)^{m-j})\binom{m}{j}\binom{m+j}{l}2^{m-j}E_{m+j-l}B_l\\&=2\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}(B_{2l}-2^{m-l}B_{m+l}).\end{aligned}$$

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