Perfect $T(G)$-triple system for each graph G with five vertices and seven edges*

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Abstract Let G be a subgraph of K_n. The graph obtained from G by replacing each edge with a 3-cycle whose third vertex is distinct from other vertices in the configuration is called a $T(G)$-triple. An edge-disjoint decomposition of $3K_n$ into copies of $T(G)$ is called a $T(G)$-triple system of order n. If, in each copy of $T(G)$ in a $T(G)$-triple system, one edge is taken from each 3-cycle (chosen so that these edges form a copy of G) in such a way that the resulting copies of G form an edge-disjoint decomposition of K_n, then the $T(G)$-triple system is said to be perfect. The set of positive integers n for which a perfect $T(G)$-triple system exists is called its spectrum. Earlier papers by authors including Billington, Lindner, Küçükçifçi and Rosa determined the spectra for cases where G is any subgraph of K_4. Then, in our previous paper, the spectrum of perfect $T(G)$-triple systems for each graph G with five vertices and $i(\leq 6)$ edges was determined. In this paper, we will completely solve the spectrum problem 0of perfect $T(G)$-triple system for each graph G with five vertices and seven edges.

Keywords $T(G)$-triple; $T(G)$-triple system; perfect $T(G)$-triple system.

1 Introduction

Denote an edge in K_n on vertices x and y by xy or $\{x,y\}$, and denote a 3-cycle on vertices x,y,z by (x,y,z). Let G be a subgraph of K_n and $T(G)$ be-a collection of triples obtained by replacing each edge $ab \in E(G)$ with a triple (a,b,c), where $c \notin V(G)$ and c does not occur in any other triple of

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$T(G)$. The graph formed in this way, by taking a triangle or triple on each edge of G, will be called a $T(G)$-triple. In a $T(G)$-triple, the vertices and edges in G and $T(G) - G$ are called *interior* and *exterior* respectively.

A *$T(G)$-triple system* of order n is denoted by $T(G, n) = (X, \mathcal{B})$, where X is the vertex set of K_n and $\mathcal{B}$ is an edge-disjoint collection of $T(G)$-triples which partitions the edge set of $3K_n$. If the interior edges of the $T(G)$-triples, which form the copies of G, partition the edge set of K_n on X, then $(X, \mathcal{B})$ is said to be a *perfect* $T(G)$-triple system. The *spectrum* for perfect $T(G)$-triple systems is the set of all positive integers n for which there exists a perfect $T(G)$-triple system of order n. The concepts of $T(G)$-triple, $T(G)$-triple system and perfect $T(G)$-triple system were first introduced by S. Küçükçifçi and C. C. Lindner in [4].

A *holey* $T(G)$-triple system with m h-holes, denoted by $T(G, h^m)$ briefly, is a pair $(\{S_1, \cdots, S_m\}, \mathcal{A})$, where each S_i is a h-set (or hole), these S_i are pairwise disjoint, and $\mathcal{A}$ is an edge-disjoint collection of $T(G)$-triples which partitions the edges joining the vertices in distinct holes. An *incomplete* $T(H)$-triple system on the set $X - Y$, denoted by $T(H, v : h)$, is a triple (X, Y, C), where $Y \subset X$, $|X| = v$, $|Y| = h$ and C is an edge-disjoint collection of $T(H)$-triples that has at least one end in X/Y.

To date, the spectrum for perfect $T(G)$-triple system has been determined for any subgraph G of K_4, see [1,2,4,5]. Then, for any prime power q, the spectrum problem for perfect $T(K_{1,q})$-triple systems and perfect $T(K_{1,2q})$-triple systems have been completely solved, see [6,7]. Recently, the spectrum of perfect $T(G)$-triple systems for each graph G with five vertices and $i(\leq 6)$ edges was determined in [8]. In this paper, we will completely solve the spectrum problem of perfect $T(G)$-triple system for each graph G with five vertices and seven edges.

Lemma 1.1 [6] *Let G be a simple graph with e edges. There exists a $T(G, v)$ only if v is odd and $2e|v(v-1)$. Specifically, the orders $v \equiv 1 \mod 2e$ and the orders $v \equiv e \mod 2e$ (for odd e) satisfy the necessary conditions.*

Lemma 1.2 [1] *Let G be any subgraph of K_4, with e edges. There exists a $T(G, v)$ if and only if $2e|v(v-1)$ and v is odd.*

For integers $a, b > 0$ and $r, s \geq 0$, a *group-divisible design K-$GDD(a^r b^s)$* is a trio $(X, \mathcal{G}, \mathcal{B})$, where X is a $(ar + bs)$-set, $\mathcal{G}$ is a partition of X into r a-sets and s b-sets, called *groups*, and $\mathcal{B}$ is a family of some subsets from X, called *blocks*, such that $|B| \in K$, $|B \cap G| \leq 1$ for any $B \in \mathcal{B}, G \in \mathcal{G}$, and such that any 2-subset T from X with $|T \cap G| \leq 1$ for any $G \in \mathcal{G}$, is contained in exactly one block in $\mathcal{B}$. If the block set $\mathcal{B}$ is a union of some disjoint $\mathcal{B}_i$, and each $\mathcal{B}_i$ forms a partition of X, then the GDD is named *resolvable* and denoted by K-$RGDD(a^r b^s)$.

Let K be a set of positive integers, and r be a positive integer. A PBD (pairwise balanced design) $B[K\bigcup\{r^*\},1;v]$ is a pair $(V,\mathcal{A})$, where V is a v-set, $\mathcal{A}$ is a collection of some subsets from V, called *blocks*, such that any 2-subset of V is contained in exactly one block, and the size of each block belongs to $K\bigcup\{r\}$, where if $r\notin K$ then there is exact one block with size r, if $r\in K$ then there is at least one block with size r.

Lemma 1.3 [3]

(1) *There exist* $\{4,5\}$*-*$GDD(t^4u^1)$ *for* $t\geq 4,\ t\neq 6,10$ *and* $0\leq u\leq t$.

(2) *A* 4-$RGDD(3^u)$ *exists if and only if* $4|u, u\geq 8$, *with the possible exceptions* $u\in\{28,44,88,152,184,220,284,288\}$.

2 Recursive methods

Theorem 2.1 [6] *Let G be a simple graph with e edges. If there exist* $T(G,2e+1)$, $T(G,e^3)$ *and* $T(G,4e+1)$, *then a* $T(G,2me+1)$ *exists for any integer* $m>0$.

Theorem 2.2 [6] *Let G be a simple graph with odd e edges. If there exist* $T(G,3e)$, $T(G,5e)$, $T(G,e^3)$ *and* $T(G,3e:e)$, *then a* $T(G,2me+e)$ *exists for any integer* $m>0$.

Theorem 2.3 *Let G be a simple graph with e edges. If there exist* $B[K,1;m]$, $T(G,2e+1)$ *and* $T(G,(2e)^k)$ $\forall\ k\in K$, *then there exists a* $T(G,2me+1)$.

Construction. Let $(Z_m,\mathcal{B})$ be a $B[K,1;m]$. By the given systems, there exist

$$T(G,2e+1)=((\{x\}\times Z_{2e})\cup\{\infty\},\mathcal{A}_x) \text{ for each } x\in Z_m;$$
$$T(G,(2e)^{|B|})=(B\times Z_{2e},\mathcal{C}_B) \text{ for each } B\in\mathcal{B}, |B|\in K.$$

Then,

$$(\bigcup_{B\in\mathcal{B}}\mathcal{C}_B)\cup(\bigcup_{x\in Z_m}\mathcal{A}_x)$$

forms a $T(G,2me+1)$ on the set $(Z_m\times Z_{2e})\bigcup\{\infty\}$.

Proof. $\forall\ x\in Z_m, i\in Z_{2e}$, $\{(x,i),\infty\}$ appears in three blocks of $\mathcal{A}_x$, where exactly one is an interior edge. And, $\forall\ (x,i)\neq(x',i')\in Z_m\times Z_{2e}$,

if $x\neq x'$, then $\exists\ B\in\mathcal{B}$ such that $x,x'\in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{C}_B$, where exactly one is an interior edge;

if $x=x'$, then $i\neq i'$, and $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{A}_x$, where exactly one is an interior edge. ■

Theorem 2.4 *Let G be a simple graph with e edges. Suppose that there exist* $B[K\bigcup\{r^*\},1;m]$, $T(G,2e+1)$, $T(G,2re+1)$ *and* $T(G,(2e)^k)$ $\forall\ k\in K$, *then there exists a* $T(G,2me+1)$.

Construction. Let $(Z_m, \mathcal{B})$ be a $B[K \bigcup \{r^*\}, 1; m]$. By the given systems, there exist

$T(G, 2re+1) = ((B_0 \times Z_{2e}) \cup \{\infty\}, \mathcal{D})$ for the r-block $B_0 \in \mathcal{B}$;
$T(G, 2e+1) = ((\{x\} \times Z_{2e}) \cup \{\infty\}, \mathcal{A}_x)$ for each $x \in Z_m$ and $x \notin B_0$;
$T(G, (2e)^{|B|}) = (B \times Z_{2e}, \mathcal{C}_B)$ for each $B \in \mathcal{B} \backslash \{B_0\}$.

Then,

$$(\bigcup_{B \in \mathcal{B} \backslash \{B_0\}} \mathcal{C}_B) \cup (\bigcup_{x \in Z_m \backslash B_0} \mathcal{A}_x) \cup \mathcal{D}$$

forms a $T(G, 2me+1)$ on the set $(Z_m \times Z_{2e}) \bigcup \{\infty\}$.

Proof.

$\forall\, x \in Z_m, i \in Z_{2e}$, $\{(x,i), \infty\}$ appears in three blocks of $\mathcal{D}$ (if $x \in B_0$) or $\mathcal{A}_x$ (if $x \notin B_0$), where exactly one is an interior edge.

$\forall\, (x,i) \neq (x',i') \in Z_m \times Z_{2e}$,

if $x \neq x'$, then $\exists\, B \in \mathcal{B}$ such that $x, x' \in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{D}$ (if $B = B_0$) or $\mathcal{C}_B$ (if $B \neq B_0$), where exactly one is an interior edge;

if $x = x'$, then $i \neq i'$, and $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{D}$ (if $x \in B_0$) or $\mathcal{A}_x$ (if $x \notin B_0$), where exactly one is an interior edge. ■

Theorem 2.5 *Let G be a simple graph with odd e edges. If there exist $B[K,1;m]$, $T(G,3e)$, $T(G,3e:e)$ and $T(G,(2e)^k)$ $\forall\, k \in K$, then there exist $T(G,(2e)^m)$ and $T(G,2me+e)$.*

Construction. Let $(Z_m, \mathcal{B})$ be a $B[K,1;m]$. By the given systems, there exist

$T(G,3e) = ((\{0\} \times Z_{2e}) \cup (\{\infty\} \times Z_e), \mathcal{A}_0)$ for $0 \in Z_m$;
$T(G,3e:e) = ((\{x\} \times Z_{2e}) \cup (\{\infty\} \times Z_e), \{\infty\} \times Z_e, \mathcal{A}_x)$ for each $x \in Z_m^*$;
$T(G,(2e)^{|B|}) = (B \times Z_{2e}, \mathcal{C}_B)$ for each $B \in \mathcal{B}$.

Then, $\bigcup_{B \in \mathcal{B}} \mathcal{C}_B$ forms a $T(G,(2e)^m)$ on the set $Z_m \times Z_{2e}$, and $(\bigcup_{B \in \mathcal{B}} \mathcal{C}_B) \cup (\bigcup_{x \in Z_m} \mathcal{A}_x)$ forms a $T(G, 2me+e)$ on the set $X = (Z_m \times Z_{2e}) \bigcup (\{\infty\} \times Z_e)$.

Proof.

$\forall\, i \neq i' \in Z_e$, $\{(\infty,i),(\infty,i')\}$ appears in three blocks of $\mathcal{A}_0$, where exactly one is an interior edge.

$\forall\, x \in Z_m, i \in Z_{2e}, i' \in Z_e$, $\{(x,i),(\infty,i')\}$ appears in three blocks of $\mathcal{A}_x$, where exactly one is an interior edge.

$\forall\, (x,i) \neq (x',i') \in Z_m \times Z_{2e}$,

if $x \neq x'$, then $\exists\, B \in \mathcal{B}$ such that $x, x' \in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{C}_B$, where exactly one is an interior edge;

if $x = x'$, then $i \neq i'$, and $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{A}_x$, where exactly one is an interior edge. ■

Theorem 2.6 *Let G be a simple graph with odd e edges. If there exist $B[K\bigcup\{r^*\},1;m]$, $T(G,2re+e)$, $T(G,3e:e)$ and $T(G,(2e)^k)$ $\forall\ k\in K$, then there exists a $T(G,2me+e)$.*

Construction. Let $(Z_m,\mathcal{B})$ be a $B[K\bigcup\{r^*\},1;m]$. By the given systems, there exist

$T(G,2re+e)=((B_0\times Z_{2e})\cup(\{\infty\}\times Z_e),\mathcal{D})$ for the r-block $B_0\in\mathcal{B}$;

$T(G,3e:e)=((\{x\}\times Z_{2e})\cup(\{\infty\}\times Z_e),\{\infty\}\times Z_e,\mathcal{A}_x)$ for each $x\in Z_m$ and $x\notin B_0$;

$T(G,(2e)^{|B|})=(B\times Z_{2e},\mathcal{C}_B)$ for each $B\in\mathcal{B}\backslash\{B_0\}$.

Then,

$$(\bigcup_{B\in\mathcal{B}\backslash\{B_0\}}\mathcal{C}_B)\cup(\bigcup_{x\in Z_m\backslash B_0}\mathcal{A}_x)\cup\mathcal{D}$$

forms a $T(G,2me+e)$ on the set $(Z_m\times Z_{2e})\bigcup(\{\infty\}\times Z_e)$.

Proof.

$\forall\ i\neq i'\in Z_e$, $\{(\infty,i),(\infty,i')\}$ appears in three blocks of $\mathcal{D}$, where exactly one is an interior edge.

$\forall\ x\in Z_m, i\in Z_{2e}, i'\in Z_e$, $\{(x,i),(\infty,i')\}$ appears in three blocks of $\mathcal{D}$ (if $x\in B_0$) or $\mathcal{A}_x$ (if $x\notin B_0$), where exactly one is an interior edge.

$\forall\ (x,i)\neq(x',i')\in Z_m\times Z_{2e}$,

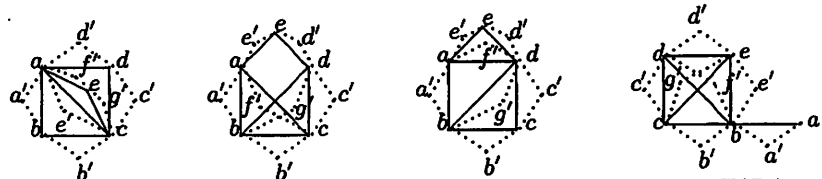
if $x\neq x'$, then $\exists\ B\in\mathcal{B}$ such that $x,x'\in B$, so $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{D}$ (if $B=B_0$) or $\mathcal{C}_B$ (if $B\neq B_0$), where exactly one is an interior edge;

if $x=x'$, then $i\neq i'$, and $\{(x,i),(x',i')\}$ appears in three blocks of $\mathcal{D}$ (if $x\in B_0$) or A_x (if $x\notin B_0$), where exactly one is an interior edge. ■

In each section, the element (x,a) in $Z_n\times Z_t$ can be denoted by x_a. Generally, the base block $B=(a,b,\cdots,c)$ in automorphism group Z_n will produce a family of blocks $B+x=(a+x,b+x,\cdots,c+x)$, $x\in Z_n$. In the following base blocks, the notation $B=(a,b(x),c,d,e;a',b',c',d'(y),e',f',g')$ means that the blocks $B+i$, $i\in Z_n$ are taken as

$$\begin{cases}(a,b,c,d,e;a',b',c',d',e',f',g')+i & \text{for } 0\leq i\leq\lfloor\frac{n}{2}\rfloor-1\\(a,x,c,d,e;a',b',c',y,e',f',g')+i & \text{for } \lfloor\frac{n}{2}\rfloor\leq i\leq n-1\end{cases}.$$

In this paper, all graphs with five vertices (no acnode) and seven edges will be discussed. These graphs are listed as follows. In order to express the blocks in each $T(R_i,v)$ briefly, use the uniform labeled form $(a,b,c,d,e;a',b',c',d',e',f',g')$ as follows. And by Lemma 1.1, there exists a $T(R_i,v)$ only if $v\equiv 1,7$ mod 14 and $v\geq 15$ for $1\leq i\leq 4$.



$T(R_1)$ $T(R_2)$ $T(R_3)$ $T(R_4)$

3 Construction for graphs R_1, R_2 and R_3

By Theorems 2.1 and 2.2, for $1 \le i \le 3$, we only need to construct $T(R_i, 7^3)$ and

$$T(R_i, 15), T(R_i, 29) \qquad \text{for } v \equiv 1 \bmod 14;$$
$$T(R_i, 21), T(R_i, 35), T(R_i, 21:7) \quad \text{for } v \equiv 7 \bmod 14.$$

Theorem 3.1 *There exists a $T(R_i, 7^3)$ for $1 \le i \le 3$.*

Proof. On the set $Z_7 \times Z_3$, total 21 blocks can be generated by one base block module (7,3).

$R_1 : (0_0, 1_2, 0_1, 3_2, 2_2; 2_1, 1_0, 4_0, 1_1, 4_2, 4_1, 2_0);$
$R_2 : (1_1, 1_0, 3_2, 4_1, 0_0; 4_2, 5_1, 2_0, 6_2, 1_2, 3_0, 0_2);$
$R_3 : (1_1, 1_0, 0_1, 3_2, 0_0; 4_2, 1_2, 2_0, 2_1, 0_2, 5_0, 6_1).$ ■

Theorem 3.2 *There exist $T(R_i, v)$ for $v \in \{15, 21, 29, 35\}$ and $1 \le i \le 3$.*

Proof.
$\underline{T(R_i, 15)}$ On the set Z_{15}, one base block module 15.

$R_1 : (0, 3, 1, 9, 5; 4, 12, 11, 10, 7, 2, 13);$
$R_2 : (3, 5, 10, 4, 0; 13, 8, 1, 6, 14, 11, 9);$
$R_3 : (3, 9, 2, 4, 0; 12, 7, 1, 8, 13, 10, 5).$

$\underline{T(R_i, 21)}$ On the set $(Z_{10} \times Z_2) \bigcup \{\infty\}$, three base blocks module $(10, -)$:

$R_1 : (0_1, \infty, 0_0, 8_0, 1_0; 3_0, 4_1, 6_1, 7_0, 9_1, 4_0, 6_0),$
$(3_1, 6_0, 0_0, 5_0(5_1), 7_0; 0_1, 7_1, \infty(2_0), 3_0, 1_1, 2_1, 1_0),$
$(3_1, 4_1, 0_1, 5_1(5_0), 9_0; 7_1, 6_1, \infty(7_0), 3_0, 8_0, 8_1, 2_0);$
$R_2 : (0_0, 3_0, 7_1, 9_1, \infty; 0_1, 4_0, 5_1, 8_0, 6_1, 6_0, 1_0),$
$(3_0, 3_1, 4_0, 0_0, 5_0(5_1); 2_1, 7_0, 9_1, \infty(2_0), 7_1, 9_0, 1_0),$
$(3_0, 1_1, 4_1, 0_1, 5_1(5_0); 8_1, 4_0, 3_1, \infty(7_0), 7_1, 6_0, 6_1);$
$R_3 : (0_0, 6_1, 5_1, 8_1, \infty; 8_0, 9_0, 3_1, 3_0, 0_1, 1_1, 2_1),$
$(3_0, 4_0, 3_1, 0_0, 5_0(5_1); 9_0, 7_0, 1_0, \infty(2_0), 7_1, 0_1, 9_1),$
$(3_0, 4_1, 0_0, 0_1, 5_1(5_0); 6_0, 1_0, 9_1, \infty(7_0), 7_1, 9_0, 3_1).$

$\underline{T(R_i, 29)}$ On the set Z_{29}, two base blocks module 29.

$R_1 : (0, 8, 1, 12, 10; 20, 18, 4, 11, 3, 24, 23), (0, 5, 2, 15, 6; 1, 7, 22, 21, 25, 16, 17);$
$R_2 : (2, 20, 12, 7, 0; 4, 8, 17, 21, 3, 23, 14), (1, 7, 16, 4, 0; 16, 11, 5, 17, 3, 10, 14);$
$R_3 : (2, 6, 21, 13, 0; 8, 9, 1, 14, 7, 10, 17), (3, 13, 7, 12, 0; 27, 26, 20, 2, 4, 21, 6).$

$\underline{T(R_i, 35)}$ On the set $\{Z_{17} \times Z_2\} \bigcup \{\infty\}$, five base blocks module $(17, -)$:

$R_1: (0_1, 9_1, 4_1, 9_0, 6_0; 11_0, 8_0, 10_1, 1_0, 5_0, 0_0, 14_1),$
$(0_0, 6_0, 1_0, 7_1, 8_0; 7_0, 14_0, 5_1, \infty, 4_1, 14_1, 13_1),$
$(0_1, 3_1, 1_1, 8_0, 7_1; 13_1, 4_1, 11_0, 7_0, 4_0, 2_1, 15_1),$
$(0_0, 4_1, 2_0, 5_1, 1_1; 6_0, 4_0, 9_1, 12_0, 11_1, 2_1, 15_0),$
$(0_0, \infty, 0_1, 4_0, 3_0; 5_1, 6_0, 6_1, 7_0, 8_1, 5_0, 12_1);$
$R_2: (0_1, 11_0, 4_1, 6_1, 14_0; 3_0, 1_0, 9_1, 13_0, 10_0, 2_0, 10_1),$
$(0_0, 1_0, 15_1, 3_1, \infty; 9_1, 8_0, 3_0, 10_0, 6_1, 14_1, 9_0),$
$(0_0, 7_1, 8_0, 3_0, 8_1; 4_1, 13_1, 4_0, 16_1, 6_1, 2_0, 6_0),$
$(0_1, 3_1, 9_1, 2_1, 6_0; 7_1, 1_1, 8_1, 1_0, \infty, 0_0, 8_0),$
$(0_0, 3_0, 7_0, 1_0, 1_1; 2_1, 9_0, 5_1, 6_1, 8_1, 3_1, 6_0);$
$R_3: (6_0, 10_0, 2_1, 0_0, 10_1; 12_0, 2_0, 8_0, 7_0, 9_0, 4_1, 13_1),$
$(0_0, 2_0, 14_1, 3_0, 8_1; 5_0, 1_1, 8_0, 2_1, 6_1, 13_1, 11_1),$
$(0_1, 1_0, 10_0, 7_1, 1_1; 6_1, 16_0, 0_0, 9_1, 6_0, 8_1, \infty),$
$(0_1, 0_0, 7_1, 3_1, 8_1; 5_1, 4_1, 5_0, 8_0, 7_0, 13_1, 13_0),$
$(0_0, 5_0, 3_1, 1_1, \infty; 1_0, 2_1, 6_1, 16_0, 9_1, 8_1, 4_0).$ ∎

Theorem 3.3 *There exists a $T(R_i, 21:7)$ for $1 \le i \le 3$.*

Proof. Total 27 blocks on the vertex set $Z_7 \times Z_3$ with a hole $Z_7 \times \{3\}$.
R_1:
$(0_0, 1_2, 0_1, 3_2, 2_2; 2_0, 2_1, 6_1, 6_0, 0_2, 4_0, 4_1), (0_1, 1_0, 0_2, 3_0, 2_0; 6_2, 6_0, 4_0, 4_2, 0_0, 5_2, 5_0),$
$(0_2, 1_1, 0_0, 3_1, 2_1; 6_1, 2_2, 6_2, 4_1, 0_1, 5_1, 4_2)$, mod $(7, -)$,
$(0_0, 1_0, 3_0, 6_0, 4_0; 3_1, 0_1, 4_1, 2_1, 1_1, 5_1, 6_1), (1_0, 2_0, 4_0, 6_0, 5_0; 4_1, 1_1, 3_1, 5_1, 2_1, 6_1, 0_1),$
$(2_0, 3_0, 5_0, 6_0, 0_0; 5_1, 2_1, 1_1, 0_1, 3_1, 6_1, 4_1), (0_1, 1_1, 3_1, 6_1, 4_1; 3_0, 0_0, 4_0, 2_0, 1_0, 5_0, 6_0),$
$(1_1, 2_1, 4_1, 6_1, 5_1; 4_0, 1_0, 3_0, 5_0, 2_0, 6_0, 0_0), (2_1, 3_1, 5_1, 6_1, 0_1; 5_0, 2_0, 1_0, 0_0, 3_0, 6_0, 4_0);$
R_2:
$(1_1, 1_0, 3_2, 4_1, 0_0; 4_2, 2_0, 2_1, 6_0, 0_2, 5_1, 3_1), (1_2, 1_1, 3_0, 4_2, 0_1; 6_1, 2_2, 6_0, 2_0, 4_0, 5_0, 5_1),$
$(1_0, 1_2, 3_1, 4_0, 0_2; 1_1, 4_1, 5_2, 0_0, 3_0, 3_2, 5_1)$, mod $(7, -)$,
$(1_0, 3_0, 4_0, 2_0, 0_0; 5_1, 2_2, 6_1, 4_1, 6_2, 0_1, 1_2), (3_0, 5_0, 6_0, 4_0, 0_0; 0_1, 4_2, 1_1, 3_1, 6_1, 2_1, 3_2),$
$(5_0, 1_0, 2_0, 6_0, 0_0; 4_1, 0_2, 5_1, 5_2, 2_1, 1_1, 3_1), (1_1, 3_1, 4_1, 2_1, 0_1; 2_0, 1_2, 3_0, 1_0, 5_2, 4_0, 0_2),$
$(3_1, 5_1, 6_1, 4_1, 0_1; 4_0, 3_2, 5_0, 0_0, 3_0, 6_0, 2_2), (5_1, 1_1, 2_1, 6_1, 0_1; 1_0, 6_2, 2_0, 4_2, 6_0, 5_0, 0_0);$
R_3:
$(1_1, 1_0, 0_1, 3_2, 0_0; 2_2, 3_0, 3_1, 2_0, 0_2, 5_0, 6_1), (1_2, 1_1, 0_2, 3_0, 0_1; 2_1, 3_1, 0_0, 5_2, 5_0, 2_0, 6_1),$
$(1_0, 1_2, 0_0, 3_1, 0_2; 4_0, 1_1, 6_2, 2_1, 2_0, 5_2, 0_1)$, mod $(7, -)$,
$(4_0, 6_0, 3_0, 1_0, 0_0; 5_1, 3_1, 2_1, 3_2, 4_1, 1_1, 0_1), (6_0, 5_0, 1_0, 2_0, 0_0; 1_2, 5_1, 4_2, 1_1, 2_2, 6_1, 2_1),$
$(5_0, 4_0, 2_0, 3_0, 0_0; 0_2, 3_1, 5_2, 0_1, 6_1, 4_1, 6_2), (4_1, 6_1, 3_1, 1_1, 0_1; 6_0, 4_0, 3_0, 3_2, 5_0, 2_0, 1_0),$
$(6_1, 5_1, 1_1, 2_1, 0_1; 1_2, 6_0, 4_2, 2_0, 2_2, 0_0, 3_0), (5_1, 4_1, 2_1, 3_1, 0_1; 0_2, 4_0, 5_2, 1_0, 0_0, 5_0, 6_2).$
∎

4 Construction for graph R_4

Theorem 4.1 *There exist $T(R_4, 14^t)$ for $t = 4, 5, 8, 9$ and 12.*

Proof. Take the vertex set Z_{14t} with holes $\{i + jt : 0 \le j \le 13\}$, $0 \le i \le t-1$.

$\underline{T(R_4, 14^4)}$ Three base blocks module 56 :
(13, 0, 1, 3, 10; 6, 11, 26, 24, 7, 18, 27),(29, 0, 5, 19, 30; 51, 27, 20, 28, 13, 21, 16),
(22, 0, 6, 21, 39; 27, 25, 12, 20, 26, 18, 29).

$\underline{T(R_4, 14^5)}$ Four base blocks module 70 :
(28, 0, 1, 19, 33; 31, 34, 47, 46, 29, 32, 27), (2, 0, 3, 27, 39; 11, 19, 34, 15, 6, 38, 46),
(44, 0, 3, 33, 38; 5, 20, 39, 15, 13, 12, 72), (26, 0, 4, 13, 21; 18, 23, 25, 35, 22, 29, 38).

$\underline{T(R_4, 14^8)}$ Seven base blocks module 112 :
(23, 0, 2, 17, 45; 49, 53, 48, 67, 52, 47, 46),(106, 0, 1, 42, 52; 110, 10, 4, 7, 11, 14, 71),
(36, 0, 9, 21, 55; 57, 54, 60, 83, 53, 52, 58),(14, 0, 7, 25, 54; 47, 37, 50, 71, 92, 44, 49),
(19, 0, 4, 31, 53; 34, 27, 40, 60, 35, 57, 26),(20, 0, 11, 37, 50; 14, 12, 22, 91, 13, 4, 21),
(11, 0, 6, 29, 22; 2, 4, 3, 1, 8, 47, 5).

$\underline{T(R_4, 14^9)}$ Eight base blocks module 126 :
(25, 0, 1, 7, 58; 19, 56, 3, 64, 61, 107, 65), (29, 0, 13, 37, 59; 53, 32, 36, 51, 28, 16, 34),
(47, 0, 12, 32, 60; 35, 41, 45, 70, 40, 39, 97), (40, 0, 4, 19, 53; 83, 60, 33, 2, 52, 58, 30),
(14, 0, 5, 35, 43; 16, 15, 55, 11, 13, 77, 83), (55, 0, 3, 26, 42; 22, 28, 70, 73, 80, 37, 77),
(2, 0, 10, 41, 62; 66, 61, 67, 96, 3, 58, 54), (56, 0, 11, 44, 61; 22, 53, 59, 91, 50, 49, 57).

$\underline{T(R_4, 14^{12})}$ Eleven base blocks module 168 :
(28, 0, 7, 15, 56; 51, 50, 40, 83, 103, 52, 109), (6, 0, 3, 14, 71; 11, 21, 34, 7, 13, 22, 32),
(40, 0, 22, 67, 76; 51, 57, 66, 105, 49, 52, 56), (52, 0, 18, 37, 80; 109, 44, 2, 7, 10, 71, 17),
(66, 0, 20, 47, 81; 21, 25, 101, 5, 26, 6, 100), (38, 0, 23, 53, 82; 47, 56, 98, 96, 50, 90, 69),
(50, 0, 16, 33, 79; 19, 56, 31, 30, 17, 61, 12), (13, 0, 1, 65, 70; 68, 77, 62, 123, 66, 63, 60),
(77, 0, 25, 51, 83; 70, 79, 90, 115, 14, 69, 92), (21, 0, 4, 39, 94; 83, 79, 45, 16, 17, 85, 20),
(10, 0, 2, 44, 75; 38, 82, 84, 117, 97, 74, 83). ■

Theorem 4.2 *There exist* $T(R_4, 14^t)$ *for* $t \equiv 0, 1 \bmod 4$ *and* $t \ge 4$.

Proof. From [3], there is a $B[\{4, 5, 8, 9, 12\}, 1; t]$ for any $t \equiv 0, 1 \bmod 4$, $t \ge 4$. By Theorem 4.1, there exist $T(R_4, 14^r), r \in \{4, 5, 8, 9, 12\}$. Thus, by Theorem 2.5, a $T(R_4, 14^t)$ exists for any $t \equiv 0, 1 \bmod 4$, $t \ge 4$. ■

Theorem 4.3 *There exist* $T(R_4, 15)$, $T(R_4, 21)$ *and* $T(R_4, 21 : 7)$.

Proof.
$\underline{T(R_4, 15)}$ On the set Z_{15}, one base block module 15 :
(6, 4, 0, 3, 10; 13, 7, 14, 1, 9, 8, 12).

$\underline{T(R_4, 21)}$ On the set $(Z_5 \times Z_4) \bigcup \{\infty\}$, six base blocks module $(5, -)$:
$(0_1, 3_1, 0_2, 0_0, 1_0; 2_1, 4_0, \infty, 3_0, 3_2, 2_3, 4_1)$, $(4_0, 0_1, 0_0, 1_2, 3_2; 4_2, 0_3, 1_3, 2_2, 3_3, 3_0, 4_3)$,
$(0_1, 2_3, 2_0, 0_0, 3_3; \infty, 4_3, 1_0, 1_1, 4_2, 0_2, 3_1)$, $(0_1, 1_3, 3_3, 0_2, 1_2; 3_0, 2_2, 2_0, 3_2, 2_1, 4_1, 4_0)$,
$(3_0, 0_2, 1_1, 0_1, 4_3; 2_1, 1_0, 4_1, 2_2, 3_1, 3_3, 2_3)$, $(0_2, \infty, 0_0, 4_1, 4_3; 2_1, 1_2, 4_2, 4_0, 0_3, 1_0, 3_3)$.

$\underline{T(R_4, 21 : 7)}$ Total 27 blocks on the set $Z_7 \times Z_3$ with a hole $Z_7 \times \{3\}$:

$(2_1,1_0,0_2,5_1,0_0;4_2,1_1,4_1,1_2,3_1,2_2,5_0)$, $(5_0,0_2,4_1,4_0,2_0;1_0,6_1,4_2,1_2,5_1,3_0,0_1)$, $(6_1,0_2,2_1,6_0,3_0;3_1,0_1,1_0,5_2,4_0,2_0,3_2)$, mod $(7,-)$;
$(6_1,0_1,0_2,3_1,1_1;5_0,2_0,3_0,4_0,6_0,1_2,2_1)$, $(6_2,2_1,1_2,1_1,4_1;2_0,3_1,3_0,2_2,5_0,0_0,4_0)$, $(0_1,2_1,5_1,3_1,2_2;3_0,3_2,6_0,4_1,4_0,1_0,5_0)$, $(6_2,6_1,3_2,3_1,4_1;1_0,6_0,5_0,2_0,0_0,4_2,5_1)$, $(6_2,0_1,4_2,4_1,5_1;1_1,0_0,6_0,3_0,1_0,5_2,6_1)$, $(2_1,6_1,5_2,5_1,1_1;0_2,0_1,0_0,6_2,2_0,4_0,1_0)$.
■

Theorem 4.4 *There exist* $T(R_4, 14m+s)$ *for* $s = 1, 7$, $m \equiv 0, 1 \bmod 4$ *and* $m \geq 1$.

Proof. When $m = 1$, a $T(R_4, 15)$ and a $T(R_4, 21)$ exist by Theorem 4.3. When $m > 1$ and $m \equiv 0, 1 \bmod 4$, there exist $T(R_4, 14^m)$ by Theorem 4.2. Thus, a $T(R_4, 14m+1)$ exists by Theorem 2.3. Furthermore, there exists a $T(R_4, 21:7)$ by Theorem 4.3, so a $T(R_4, 14m+7)$ exists by Theorem 2.5.
■

Theorem 4.5 *There exist* $T(R_4, 14m+1)$ *for* $m = 2, 3, 6, 7, 10, 11, 14, 15$.

Proof.
$T(R_4, 29)$ On the set Z_{29}, two base blocks module 29 :
(6, 0, 1, 3, 11; 12, 14, 24, 21, 9, 8, 10),(14, 0, 4, 9, 16; 10, 1, 2, 23, 5, 12, 20).

$T(R_4, 43)$ On the set Z_{43}, three base blocks module 43 :
(19, 0, 2, 17, 20; 21, 18, 7, 32, 40, 36, 37),(12, 0, 1, 5, 11; 2, 4, 6, 17, 7, 13, 10), (8, 0, 7, 16, 29; 17, 22, 36, 5, 11, 30, 13).

$T(R_4, 85)$ On the set Z_{85}, six base blocks module 85 :
(26, 0, 42, 19, 1; 41, 2, 6, 38, 43, 39, 4), (13, 0, 31, 9, 2; 34, 49, 55, 32, 14, 37, 15), (17, 0, 39, 27, 11; 26, 12, 1, 3, 25, 4, 21), (36, 0, 40, 25, 5; 1, 7, 8, 33, 34, 31, 10), (10, 0, 37, 33, 3; 7, 2, 4, 8, 9, 11, 18), (21, 0, 38, 14, 6; 11, 17, 18, 46, 22, 41, 7).

$T(R_4, 99)$ On the set Z_{99}, seven base blocks module 99 :
(27, 0, 49, 11, 4; 2, 16, 21, 18, 17, 32, 75), (34, 0, 39, 19, 6; 15, 69, 78, 60, 23, 55, 58), (16, 0, 48, 23, 1; 47, 45, 15, 44, 49, 46, 43), (18, 0, 46, 17, 2; 40, 42, 80, 37, 6, 39, 83), (31, 0, 36, 26, 12; 20, 41, 35, 60, 50, 2, 82), (28, 0, 43, 3, 8; 55, 14, 70, 15, 26, 16, 19), (57, 0, 67, 30, 9; 1, 3, 5, 18, 15, 38, 77).

$T(R_4, 141)$ On the set Z_{141}, ten base blocks module 141 :
(19, 0, 44, 9, 4; 80, 22, 97, 77, 2, 75, 24), (54, 0, 66, 36, 15; 27, 33, 51, 96, 78, 18, 111), (38, 0, 64, 16, 2; 19, 32, 40, 9, 1, 8, 33), (27, 0, 65, 37, 20; 84, 103, 51, 99, 10, 89, 113), (57, 0, 68, 25, 7; 99, 34, 117, 16, 74, 83, 101), (8, 0, 99, 49, 10; 4, 120, 74, 100, 5, 95, 125), (58, 0, 70, 24, 1; 29, 35, 47, 83, 71, 12, 106), (26, 0, 63, 32, 3; 13, 102, 118, 88, 72, 16, 33), (13, 0, 59, 47, 6; 77, 100, 53, 97, 3, 94, 103), (55, 0, 67, 33, 11; 98, 104, 50, 22, 76, 87, 39).

$T(R_4, 155)$ On the set Z_{155}, eleven base blocks module 155 :
(72, 0, 28, 43, 61; 76, 77, 100, 116, 9, 75, 102), (55, 0, 26, 76, 77; 74, 78, 3, 1, 5, 70, 6), (12, 0, 91, 49, 3; 22, 28, 35, 15, 50, 16, 32), (16, 0, 34, 56, 73; 25, 8, 3, 12, 109, 91, 13),

(2, 0, 14, 58, 66; 44, 48, 51, 23, 39, 17, 50), (27, 0, 30, 41, 62; 28, 43, 26, 80, 22, 33, 60), (17, 0, 23, 47, 60; 20, 42, 10, 2, 6, 24, 30), (68, 0, 19, 48, 57; 69, 70, 86, 119, 144, 66, 84), (36, 0, 45, 65, 70; 57, 59, 103, 9, 15, 60, 99), (74, 0, 10, 63, 69; 49, 50, 12, 16, 45, 46, 58), (54, 0, 40, 71, 75; 64, 66, 102, 6, 63, 68, 8).

$\underline{T(R_4, 197)}$ On the set Z_{197}, fourteen base blocks module 197 :
(49, 0, 40, 74, 86; 13, 14, 42, 41, 40, 30, 6), (62, 0, 26, 57, 71; 69, 67, 77, 72, 75, 2, 100), (39, 0, 21, 97, 98; 100, 99, 6, 2, 4, 8, 5), (69, 0, 30, 58, 81; 86, 94, 111, 140, 84, 83, 109), (87, 0, 15, 56, 59; 38, 31, 75, 14, 5, 10, 1), (92, 0, 35, 60, 68; 36, 164, 107, 25, 131, 8, 3), (36, 0, 24, 85, 91; 55, 59, 84, 28, 27, 62, 80), (17, 0, 22, 75, 88; 54, 53, 69, 36, 51, 50, 71), (32, 0, 48, 90, 95; 74, 75, 10, 13, 12, 79, 19), (29, 0, 10, 89, 93; 70, 68, 77, 20, 22, 66, 75), (7, 0, 20, 63, 72; 28, 77, 2, 1, 78, 76, 93), (50, 0, 19, 73, 84; 98, 43, 112, 169, 40, 97, 114), (70, 0, 27, 64, 82; 92, 7, 114, 152, 91, 90, 116), (38, 0, 16, 94, 96; 86, 88, 13, 4, 9, 10, 11).

$\underline{T(R_4, 211)}$ On the set Z_{211}, fifteen base blocks module 211 :
(15, 0, 32, 81, 99; 46, 45, 38, 71, 50, 51, 64), (61, 0, 33, 90, 91; 3, 14, 48, 52, 35, 23, 45), (77, 0, 7, 50, 52; 86, 90, 87, 139, 175, 84, 88), (4, 0, 47, 78, 102; 79, 76, 5, 25, 24, 74, 30), (60, 0, 19, 87, 93; 91, 68, 14, 16, 45, 75, 84), (40, 0, 38, 89, 92; 102, 11, 20, 29, 40, 25, 22), (44, 0, 23, 85, 95; 98, 78, 109, 1, 97, 95, 10), (26, 0, 41, 76, 104; 94, 95, 132, 7, 8, 93, 133), (9, 0, 37, 79, 101; 71, 63, 14, 35, 67, 64, 98), (11, 0, 34, 80, 100; 59, 55, 91, 42, 40, 33, 48), (5, 0, 88, 75, 105; 106, 107, 173, 5, 103, 102, 6), (48, 0, 21, 86, 94; 74, 72, 3, 6, 44, 87, 53), (39, 0, 27, 83, 97; 105, 28, 126, 1, 104, 103, 5), (36, 0, 29, 82, 98; 41, 7, 49, 61, 39, 25, 45), (103, 0, 25, 84, 96; 150, 101, 114, 3, 100, 90, 17). ■

Theorem 4.6 *There exist* $T(R_4, 14m + 7)$ *for* $m = 2, 3, 6, 7, 10, 11, 14, 15$.

Proof. Below, the vertices x_0 and y_1 are denoted by x and $\overline{y}$ respectively.

$\underline{T(R_4, 35)}$ On the set $(Z_{17} \times Z_2) \bigcup \{\infty\}$, 5 base blocks module $(17, -)$.

$(\infty, 0, \overline{5}, 8, 7; \overline{2}, 6, 3, 4, 1, \overline{15}, \overline{7}), (13, \overline{0}, 6, 1, \overline{7}; \infty, 14, 8, 4, \overline{10}, \overline{6}, \overline{12}),$
$(14, \overline{0}, 10, 7, \overline{2}; 4, \overline{3}, \overline{14}, 5, \overline{13}, \overline{16}, \overline{6}), (\overline{8}, 0, \overline{2}, 6, 2; 4, 8, \overline{11}, 1, \overline{14}, \overline{1}, \overline{10}),$
$(\infty, \overline{0}, \overline{9}, \overline{4}, \overline{3}; 8, 16, \overline{2}, 5, 9, \overline{5}, 3).$

$\underline{T(R_4, 49)}$ On the set $(Z_{24} \times Z_2) \bigcup \{\infty\}$, 7 base blocks module $(24, -)$.

$(5, \overline{0}, 18, \overline{7}, \overline{1}; \overline{9}, \overline{10}, 13, 12, 19, 11, 6), (\infty, 0, \overline{17}, \overline{3}, 2; \overline{12}, \overline{6}, \overline{9}, \overline{11}, \overline{16}, \overline{15}, 13),$
$(\overline{5}, 0, \overline{16}, \overline{20}, 6; \overline{1}, \overline{15}, \overline{9}, \overline{3}, 8, 9, \overline{13}), (\overline{12}(12), \overline{0}, 6, \overline{8}, \overline{3}; \infty(\overline{15}), \overline{6}, \overline{4}, 5, 7, \overline{10}, \overline{5}),$
$(\overline{11}, 0, \overline{8}, 9, 4; \overline{10}, 1, 3, 14, 11, 8, 6), (12(\overline{12}), 0, 11, 8, 1; \infty(\overline{3}), 20, 5, \overline{2}, 4, \overline{8}, \overline{10}),$
$(\infty, \overline{0}, 2, \overline{11}, \overline{2}; 14, 12, \overline{16}, 0, \overline{5}, 3, 9).$

$\underline{T(R_4, 91)}$ On the set $(Z_{45} \times Z_2) \bigcup \{\infty\}$, 13 base blocks module $(45, -)$.

$(\overline{2}, 0, \overline{22}, \overline{0}, 1; \overline{3}, \overline{1}, 15, \overline{23}, 22, 43, 17), (34, \overline{0}, 22, 20, \overline{9}; 6, \overline{10}, 16, \overline{18}, 4, 10, 23),$
$(\overline{6}, 0, \overline{42}, \overline{8}, 14; \overline{10}, \overline{25}, 8, \overline{26}, 7, 31, 25), (31, \overline{0}, 41, 26, \overline{8}; 5, \overline{40}, 29, \overline{2}, 18, 4, 17),$
$(\overline{5}, 0, \overline{43}, \overline{1}, 8; \overline{7}, \overline{24}, 1, \overline{21}, 12, 31, 27), (\infty, \overline{0}, \overline{24}, \overline{18}, \overline{4}; 31, \overline{3}, \overline{38}, \overline{20}, \overline{22}, \overline{19}, \overline{32}),$
$(\infty, 0, 32, 12, 5; \overline{10}, \overline{11}, \overline{6}, \overline{30}, 15, \overline{32}, \overline{18}), (\overline{5}, \overline{0}, \overline{28}, \overline{12}, \overline{2}; \overline{38}, 5, \overline{23}, \overline{6}, \overline{15}, 8, 11),$
$(\overline{7}, 0, \overline{40}, \overline{10}, 22; 32, \overline{9}, 12, \overline{26}, 5, 37, 21), (16, \overline{0}, 32, 28, \overline{13}; 3, \overline{38}, 17, \overline{21}, 37, 8, 5),$
$(\overline{9}, 0, \overline{41}, \overline{3}, 17; \infty, \overline{11}, 15, \overline{16}, 8, 20, 12), (25, \overline{0}, 30, 9, \overline{1}; \overline{3}, \overline{12}, 12, \overline{35}, \overline{10}, 11, 14),$
$(11, 0, 19, 9, 3; \overline{12}, \overline{3}, \overline{25}, \overline{18}, 39, \overline{2}, \overline{33}).$

$T(R_4,105)$ On the set $(Z_{52}\times Z_2)\bigcup\{\infty\}$, 15 base blocks module $(52,-)$.

$(\overline{8},\overline{0},\overline{1},10,0;47,16,20,\overline{51},45,18,9),(\overline{18},0,\overline{51},\overline{21},5;20,19,\overline{5},26,\overline{0},30,23),$
$(\overline{8},0,\overline{27},\overline{40},16;51,26,10,\overline{9},18,4,11),(\overline{10},0,\overline{23},\overline{28},3;10,8,\overline{35},14,20,9,15),$
$(\infty,0,\overline{49},\overline{6},8;\overline{27},3,38,9,12,2,\overline{24}),(\overline{13},0,\overline{12},\overline{22},30;35,13,\overline{30},15,11,36,43),$
$(15,0,32,23,4;\overline{1},\overline{3},\overline{2},\overline{28},\overline{14},\overline{13},\overline{21}),(12,0,31,17,6;\overline{39},\overline{18},\overline{50},\overline{43},\overline{40},\overline{42},\overline{24}),$
$(\overline{45},0,\overline{15},\overline{19},34;24,23,\overline{24},\overline{22},\overline{31},25,\overline{14}),(\overline{47},0,\overline{39},\overline{14},7;\overline{28},\overline{4},4,\overline{3},\overline{50},\overline{29},\overline{16}),$
$(\overline{26}(26),\overline{0},\overline{31},\overline{19},\overline{3};\infty(\overline{29}),\overline{12},\overline{35},\overline{1},\overline{14},\overline{32},\overline{18}),(\overline{11},\overline{0},\overline{35},\overline{20},\overline{6};\overline{1},\overline{9},\overline{15},\overline{12},\overline{2},\overline{3},\overline{28}),$
$(26(\overline{26}),0,\overline{48},\overline{30},13;\infty(\overline{3}),22,\overline{20},34,\overline{15},24,36),(\overline{9},0,\overline{31},\overline{38},2;7,\overline{6},\overline{17},\overline{14},16,3,8),$
$(\infty,\overline{2},\overline{4},51,0;34,32,45,14,25,46,\overline{26}).$

$T(R_4,147)$ On the set $(Z_{73}\times Z_2)\bigcup\{\infty\}$, 21 base blocks module $(73,-)$.

$(13,0,\overline{61},\overline{32},11;15,\overline{30},\overline{40},\overline{64},\overline{19},39,70),(\infty,0,\overline{24},\overline{8},1;\overline{36},\overline{59},9,\overline{44},5,36,28),$
$(20,0,47,32,8;\overline{6},\overline{1},\overline{2},\overline{10},\overline{31},\overline{40},\overline{24}),(30,0,\overline{58},\overline{28},14;13,\overline{29},\overline{39},\overline{71},\overline{66},38,70),$
$(\overline{23},\overline{0},\overline{31},\overline{14},\overline{4};\overline{33},\overline{28},\overline{5},\overline{34},\overline{35},\overline{27},\overline{9}),(33,0,\overline{63},\overline{57},18;12,\overline{36},1,\overline{29},24,52,63),$
$(\overline{5},0,\overline{64},\overline{62},29;\overline{4},\overline{41},68,\overline{38},\overline{68},31,52),(\overline{33},\overline{0},33,2,\overline{8};52,\overline{54},38,\overline{28},16,27,31),$
$(\overline{11},\overline{0},\overline{35},\overline{15},\overline{3};\overline{34},\overline{36},\overline{19},\overline{27},\overline{32},\overline{17},\overline{10}),(\overline{13},0,\overline{51},\overline{46},17;3,\overline{56},5,\overline{49},27,8,24),$
$(\overline{21},\overline{0},13,6,\overline{25};62,\overline{11},35,\overline{13},56,40,24),(\infty,\overline{0},37,1,\overline{1};36,\overline{52},4,\overline{23},\overline{15},48,19),$
$(10,0,\overline{54},\overline{30},5;18,\overline{20},\overline{36},\overline{63},12,33,59),(\overline{53},0,\overline{11},\overline{4},2;\infty,\overline{18},72,\overline{12},26,35,45),$
$(35,0,\overline{59},\overline{41},21;\overline{70},\overline{34},12,\overline{15},\overline{19},1,53).$

$T(R_4,161)$ On the set $(Z_{80}\times Z_2)\bigcup\{\infty\}$, 23 base blocks module $(80,-)$.

$(\overline{31},0,\overline{27},\overline{3},10;\overline{7},\overline{4},26,\overline{24},\overline{33},1,4),(\infty,0,50,35,11;\overline{21},\overline{3},\overline{2},\overline{55},\overline{19},\overline{9},\overline{57}),$
$(33,0,\overline{42},\overline{6},22;\overline{75},\overline{40},7,\overline{3},6,39,14),(\overline{44},0,\overline{19},\overline{2},6;\overline{62},\overline{36},10,\overline{18},\overline{37},7,9),$
$(20,0,\overline{58},\overline{37},19;38,\overline{29},47,\overline{63},43,34,64),(31,0,\overline{48},\overline{7},25;9,\overline{53},3,\overline{3},4,\overline{0},16),$
$(\overline{4},\overline{0},\overline{22},\overline{9},\overline{2};\overline{13},\overline{26},\overline{44},\overline{34},\overline{36},\overline{20},\overline{41}),(\overline{38},\overline{0},17,14,\overline{6};11,\overline{11},3,\overline{15},\overline{36},32,0),$
$(37,0,16,9,4;\overline{63},28,\overline{69},\overline{45},\overline{56},\overline{34},\overline{71}),(36,0,\overline{53},\overline{5},17;\overline{65},\overline{54},55,\overline{38},20,22,7),$
$(\overline{27},\overline{0},64,56,\overline{25};70,\overline{19},33,\overline{7},40,30,11),(\overline{35},0,\overline{34},\overline{4},13;\overline{66},\overline{64},42,\overline{31},5,11,18),$
$(\overline{8},\overline{0},68,42,\overline{16};\overline{43},\overline{22},7,\overline{39},58,72,37),(23,0,\overline{52},\overline{47},2;36,\overline{76},\overline{14},\overline{22},34,32,35),$
$(28,0,\overline{32},\overline{9},34;29,\overline{39},16,\overline{3},7,13,20),(\overline{26},\overline{0},79,58,\overline{29};56,\overline{21},25,\overline{49},16,29,27),$
$(18,0,\overline{26},\overline{8},29;\overline{18},37,45,69,39,36,67),(\overline{35},\overline{0},65,23,\overline{10};18,\overline{13},40,\overline{22},\overline{41},42,5),$
$(\overline{33},\overline{0},51,24,\overline{14};53,\overline{66},36,\overline{79},\overline{48},3,28),(\overline{28},\overline{0},66,34,\overline{19};67,\overline{16},64,\overline{2},31,10,41),$
$(\infty,\overline{0},\overline{46},\overline{15},\overline{3};29,\overline{8},\overline{9},\overline{43},\overline{1},\overline{5},\overline{17}),(40(\overline{40}),0,\overline{11},\overline{0},1;\infty(\overline{43}),\overline{26},24,\overline{12},\overline{25},2,5),$
$(\overline{40}(40),\overline{0},20,6,\overline{1};\infty(\overline{3}),\overline{8},\overline{77},\overline{11},\overline{28},21,36).$

$T(R_4,203)$ On the set $(Z_{101}\times Z_2)\bigcup\{\infty\}$, 29 base blocks module $(101,-)$.

$(\overline{9},0,\overline{80},\overline{1},5;1,\overline{33},3,\overline{49},47,4,57),(\overline{47},\overline{0},79,71,\overline{27};\overline{21},\overline{95},42,\overline{34},14,58,94),$
$(37,0,30,4,1;\overline{52},\overline{79},\overline{54},\overline{22},\overline{84},\overline{66},\overline{57}),(\overline{29},\overline{0},30,84,\overline{1};7,\overline{21},40,\overline{83},\overline{34},11,1),$
$(\overline{85},0,\overline{92},\overline{4},76;\overline{55},\overline{49},7,\overline{48},26,43,32),(\overline{32},\overline{0},39,81,\overline{5};70,\overline{20},38,\overline{88},26,7,13),$
$(50,0,\overline{14},\overline{51},52;15,\overline{0},6,\overline{36},12,22,75),(2,0,65,33,12;\overline{71},\overline{42},\overline{88},\overline{78},\overline{56},\overline{2},\overline{58}),$
$(\overline{90},0,\overline{83},\overline{6},23;\overline{61},\overline{46},11,\overline{44},7,65,87),(\infty,\overline{0},\overline{49},\overline{39},\overline{6};49,\overline{14},\overline{12},\overline{8},\overline{45},\overline{15},\overline{55}),$
$(\overline{45},\overline{0},88,66,\overline{23};44,\overline{93},49,\overline{32},12,50,4),(43,0,\overline{38},\overline{42},15;45,\overline{89},9,\overline{91},19,8,22),$
$(45,0,\overline{34},\overline{45},19;9,\overline{32},8,\overline{41},25,10,28),(\overline{44},\overline{0},73,60,\overline{18};86,\overline{10},32,\overline{29},8,41,55),$
$(\overline{93},0,\overline{81},\overline{8},27;\overline{58},\overline{53},4,\overline{44},49,35,94),(\infty,0,\overline{99},\overline{0},62;\overline{22},\overline{40},48,\overline{51},14,\overline{12},12),$
$(\overline{79},0,\overline{89},\overline{2},16;\overline{9},\overline{48},6,\overline{44},47,40,57),(\overline{98},0,\overline{77},\overline{11},34;\overline{75},\overline{99},20,\overline{36},45,31,64),$

$(\overline{10},0,\overline{74},\overline{3},9;\infty,\overline{19},80,\overline{48},14,46,6),(\overline{50},\overline{0},\overline{63},\overline{42},\overline{17};\overline{16},\overline{93},\overline{52},\overline{22},\overline{36},\overline{25},\overline{35}),$
$(\overline{48},0,\overline{86},\overline{5},18;\overline{80},\overline{47},10,\overline{45},48,39,56),(\overline{96},0,\overline{91},\overline{31},38;\overline{72},\overline{65},30,\overline{58},46,32,5),$
$(\overline{36},\overline{0},46,40,\overline{9};32,\overline{84},\overline{23},\overline{25},42,65,22),(46,0,\overline{44},\overline{52},20;13,\overline{41},5,\overline{57},38,11,32),$
$(40,0,\overline{47},\overline{21},14;\overline{43},\overline{95},16,\overline{68},34,3,19),(41,0,\overline{19},\overline{12},44;17,\overline{20},39,\overline{58},21,2,38),$
$(\overline{40},\overline{0},61,51,\overline{16};10,\overline{12},19,\overline{29},92,30,81),(31,0,35,24,7;\overline{3},\overline{25},\overline{41},\overline{72},\overline{14},\overline{35},\overline{61}),$
$(\overline{48},\overline{0},\overline{34},\overline{15},\overline{3};\overline{4},\overline{1},\overline{2},\overline{6},\overline{43},\overline{38},\overline{63}).$

$\underline{T(R_4,217)}$ On the set $(Z_{108}\times Z_2)\bigcup\{\infty\}$, 31 base blocks module $(108,-)$.
$(\overline{11},\overline{0},53,0,\overline{1};\overline{18},\overline{102},\overline{88},\overline{8},\overline{21},29,23),(\overline{18},\overline{0},51,1,\overline{9};\overline{1},\overline{105},20,\overline{13},\overline{15},28,23),$
$(\overline{30},\overline{0},49,3,\overline{21};90,\overline{52},12,\overline{74},7,\overline{13},25),(14,\overline{0},84,8,\overline{49};6,\overline{66},40,\overline{6},41,10,102),$
$(\overline{27},0,\overline{75},\overline{63},3;20,\overline{84},\overline{51},\overline{71},7,35,78),(\overline{29},\overline{0},50,2,\overline{13};26,\overline{106},5,\overline{12},\overline{5},18,24),$
$(21,0,\overline{91},\overline{3},1;\overline{23},\overline{51},2,\overline{44},\overline{101},53,55),(24,0,\overline{99},\overline{13},7;\overline{96},\overline{60},2,\overline{51},11,52,57),$
$(49,0,14,6,2;\overline{13},\overline{18},\overline{35},\overline{29},\overline{30},\overline{31},\overline{53}),(\infty,0,\overline{20},\overline{17},13;\overline{37},\overline{92},95,\overline{54},28,11,4),$
$(26,0,\overline{96},\overline{62},11;2,\overline{61},\overline{66},\overline{26},\overline{94},51,60),(\overline{16},0,\overline{89},\overline{42},28;31,\overline{2},\overline{58},\overline{19},\overline{92},42,69),$
$(\overline{22},0,\overline{87},\overline{83},40;32,\overline{105},\overline{61},\overline{66},1,38,3),(31,0,42,25,5;\overline{38},\overline{48},\overline{75},\overline{81},\overline{46},\overline{47},\overline{54}),$
$(44,0,70,27,9;\overline{87},20,24,56,52,49,57),(\overline{40},\overline{0},\overline{17},\overline{10},\overline{2};82,68,65,55,45,86,64),$
$(\overline{43},\overline{0},76,7,\overline{46};39,\overline{44},34,\overline{92},41,17,92),(\overline{45},\overline{0},73,6,\overline{42};79,\overline{47},32,\overline{87},33,20,54),$
$(\overline{32},\overline{0},62,5,\overline{35};61,\overline{49},45,\overline{83},91,92,84),(\overline{31},\overline{0},56,4,\overline{27};68,\overline{51},18,\overline{77},83,89,31),$
$(\infty,\overline{0},\overline{80},\overline{39},\overline{16};46,\overline{26},\overline{28},\overline{67},\overline{53},\overline{49},\overline{30}),(33,0,\overline{95},\overline{44},19;\overline{91},\overline{17},\overline{67},\overline{13},\overline{95},46,64),$
$(36,0,\overline{93},\overline{68},23;\overline{86},\overline{66},\overline{95},\overline{97},33,44,66),(43,\overline{0},89,10,\overline{50};31,\overline{68},\overline{23},\overline{9},63,28,104),$
$(\overline{50},0,\overline{37},\overline{31},22;\overline{59},\overline{70},75,\overline{107},30,7,16),(\overline{26},\overline{0},\overline{70},\overline{33},\overline{14};\overline{48},\overline{24},\overline{75},\overline{58},\overline{47},\overline{45},\overline{27}),$
$(30,0,\overline{97},\overline{49},15;36,\overline{65},\overline{68},\overline{15},\overline{93},48,62),(45,0,\overline{88},\overline{64},35;\overline{89},\overline{107},\overline{74},\overline{44},12,39,75),$
$(54(\overline{54}),0,\overline{26},\overline{21},16;\infty(\overline{57}),\overline{99},97,\overline{55},103,\overline{34},21),$
$(\overline{54}(54),\overline{0},75,41,\overline{53};\infty(\overline{3}),\overline{70},92,\overline{14},33,40,88),$
$(\overline{28},0,\overline{84},\overline{48},10;53,\overline{100},\overline{59},\overline{63},35,34,46).$ ■

Theorem 4.7 *There exist* $T(R_4, 14m+s)$ *for* $s = 1, 7$ *and* $m = 26, 27, 30, 31, 46, 47$.

Proof. There exists a $4\text{-}RGDD(3^{4t})$ for $t \geq 2$ and $t \notin \{7, 11, 22, 38, 46, 55, 71, 72\}$ by Lemma 1.3 (2), which consists of $4t-1$ parallel classes $\mathcal{B}_i, 1 \leq i \leq 4t-1$. For $1 \leq r \leq 4t$, adding a new element x_i to each 4-block in $\mathcal{B}_i$ for $1 \leq i \leq r-1$, adding a new element x_0 to each group, and adding a new r-block $\{x_0, x_1, \cdots, x_{r-1}\}$, we can get a $B[\{4, 5, r^*\}, 1; 12t+r]$.

By Theorem 4.1, there exist $T(R_4, 14^4)$ and $T(R_4, 14^5)$. And, by Theorems 4.3, there exist $T(R_4, 14+1)$ and $T(R_4, 21 : 7)$. Therefore, if there exists a $T(R_4, 14r+1)$ then there exists a $T(R_4, 14(12t+r)+1)$ by Theorem 2.4; and if there exists a $T(R_4, 14r+7)$ then there exists a $T(R_4, 14(12t+r)+7)$ by Theorem 2.6.

Now, take $t = 2$ ($r = 2, 3, 6, 7$) and $t = 3$ ($r = 10, 11$), we can get $m = 12t+r = 26, 27, 30, 31, 46, 47$. There exist $T(R_4, 14r+1)$ and $T(R_4, 14r+7)$ for $r = 2, 3, 6, 7, 10, 11$ by Theorems 4.5 and 4.6, so the conclusion holds. ■

Theorem 4.8 *For any integer $t \equiv 0, 1 \bmod 4$, $t \geq 4$ and $0 \leq u \leq t$, if there exists a $T(R_4, 14u+s)$, then there exists a $T(R_4, 14(4t+u)+s)$, where $s = 1$ or 7.*

Proof. By Lemma 1.3(1), for $t \equiv 0, 1 \bmod 4$, $t \geq 4$ and $0 \leq u \leq t$, there is a $\{4,5\}$-$GDD(t^4u^1)$, which implies a $B[\{4, 5, t, u^*\}, 1; 4t+u]$. By Theorem 4.2, there exist $T(R_4, 14^4), T(R_4, 14^5)$ and $T(R_4, 14^t)$. And, by Theorems 4.3 and 4.5, there exist $T(R_4, 14+1)$ and $T(R_4, 21 : 7)$. Therefore, if there exists a $T(R_4, 14u+1)$, then there exists a $T(R_4, 14(4t+u)+1)$ by Theorem 2.4; and if there exists a $T(R_4, 14u+7)$, then there exists a $T(R_4, 14(4t+u)+7)$ by Theorem 2.6. ■

Theorem 4.9 *There exist $T(R_4, 14m+s)$ for $s = 1, 7$ and any positive integer m.*

Proof. Consider the conclusion in the above theorem. For given t and $0 \leq u \leq t$, $4t+u$ runs over the interval $[4t, 5t]$. Let $t = 4r$ or $t = 4r+1$, $r \geq 1$, the interval becomes $[16r, 20r]$ or $[16r+4, 20r+5]$. Solve the following inequalities

$$20r+1 \geq 16r+4 \implies r \geq 1;$$
$$(20r+5)+1 \geq 16(r+1) \implies r \geq 3.$$

Note that $(20r+5, 16r+16) = (25, 32)$ or $(45, 48)$ for $r = 1$ or 2. Thus, the positive integers uncovered by all intervals $\{[16r, 20r] \cup [16r+4, 20r+5]\}_{r \geq 1}$ are

$$m = 4t+u = 1, 2, 3, \cdots, 15, 26, 27, 28, 29, 30, 31, 46, 47.$$

However, there exist $T(R_4, 14m+1)$ and $T(R_4, 14m+7)$ for these values m by Theorems 4.3-4.7. Furthermore, by the nature order $r = 1, 2, 3 \cdots$, for each $m \in [16r, 20r] \cup [16r+4, 20r+5]$, the existence of $T(R_4, 14m+s)$ can be recursively obtained from $T(R_4, 14u+s)$ with $u < m$, by Theorem 4.8, where $s = 1, 7$. ■

5 Conclusions

Theorem 5.1 *For any graph H with five vertices and seven edges, there exists a $T(H, v)$ if and only if $v \equiv 1, 7 \bmod 14$ and $v \geq 15$.*

Proof. By Lemma 1.1, for $1 \leq i \leq 4$, there exists a $T(R_i, v)$ only if $v \equiv 1, 7 \bmod 14$. A $T(R_i, v)$ exists for $v \equiv 1, 7 \bmod 14$ and $1 \leq i \leq 3$ by Theorems 2.1, 2.2 and Theorems 3.1-3.3. A $T(R_4, v)$ exists for $v \equiv 1, 7 \bmod 14$ by Theorem 4.9. ■

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