Trees with the fourth largest number of maximal independent sets

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Abstract

Let $G = (V, E)$ be a simple undirected graph. An independent set is a subset S of V such that no two vertices in S are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set. In this paper, we study the problem of determining the fourth largest numbers of maximal independent sets among all trees and forests. Extremal graphs achieving these values are also given.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. A subset $I \subseteq V$ is *independent* if there is no edge of G between any two vertices of I. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$. For a vertex $x \in V(G)$, let $\mathrm{MI}_{+x}(G) = \{I \in \mathrm{MI}(G) : x \in I\}$ and $\mathrm{MI}_{-x}(G) = \{I \in \mathrm{MI}(G) : x \notin I\}$. The cardinalities of $\mathrm{MI}_{+x}(G)$ and $\mathrm{MI}_{-x}(G)$ are denoted by $mi_{+x}(G)$ and $mi_{-x}(G)$, respectively.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [8]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [4]. Recently, Jou and Lin [6] investigated the second largest numbers of $mi(G)$ among all trees and forests of order n. Jin and Yan [2] solved the third largest number of $mi(G)$ among all trees of order n.

The purpose of this paper is to determine the fourth largest numbers of maximal independent sets among all trees and forests. Additionally, extremal graphs achieving these values are also given.

2 Preliminary

In this section, we present some notations and preliminary results, which will be helpful to the proof of our main result in next section. For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. If $v \in V(G)$ then the *neighborhood* and *closed neighborhood* of v are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = \{v\} \cup N_G(v)$, respectively. Two distinct vertices u and v are called *duplicated vertices* if $N_G(u) = N_G(v)$. The *degree* of x is the cardinality of $N_G(x)$, and it is denoted by $\deg_G(x)$. A vertex x is called a *leaf* if $\deg_G(x) = 1$. A vertex v of G is a *support vertex* if it is adjacent to a leaf in G. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A component of odd (respectively, even) order is called an *odd* (respectively, *even*) *component*. If a graph G is isomorphic to another graph H, we denote $G = H$. Denote by P_n a *path* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$. We begin with some useful lemmas which are needed in this paper.

Lemma 2.1. ([1, 3]) *If G is a graph in which x is adjacent to exactly one vertex y, then $mi(G) = mi(G - N_G[x]) + mi(G - N_G[y])$.*

Lemma 2.2. ([3]) *If a graph G has duplicated leaves x_1 and x_2, then $mi(G) = mi(G - x_2)$.*

Lemma 2.3. ([3]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1)mi(G_2)$.*

The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.4 and 2.5, respectively.

Theorem 2.4. ([3, 5]) *If T is a tree with $n \geq 1$ vertices, then $mi(T) \leq t_1(n)$, where*

$$t_1(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ *if and only if* $T = T_1(n)$, *where*

$$T_1(n) = \begin{cases} B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even;} \\ B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

where $B(i,j)$ *is the set of batons, which are the graphs obtained from the basic path* P *of* $i \geq 1$ *vertices by attaching* $j \geq 0$ *paths of length two to the endpoints of* P *in all possible ways (see Figure 1).*

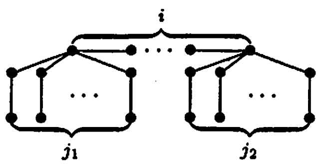


Figure 1: The baton $B(i,j)$ with $j = j_1 + j_2$

Theorem 2.5. ([3, 5]) *If* F *is a forest with* $n \geq 1$ *vertices, then* $mi(F) \leq f_1(n)$, *where*

$$f_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_1(n)$ *if and only if* $F = F_1(n)$, *where*

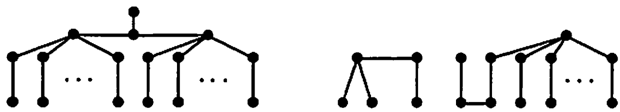
$$F_1(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even;} \\ B(1, \frac{n-1-2s}{2}) \cup sP_2 & \\ \quad \text{for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.6 and 2.7, respectively.

Theorem 2.6. ([6]) *If* T *is a tree with* $n \geq 4$ *vertices having* $T \neq T_1(n)$, *then* $mi(T) \leq t_2(n)$, *where*

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5; \\ 3r^{n-5} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ *if and only if* $T = T_2'(8), T_2''(8), P_{10}$, *or* $T_2(n)$, *where* $T_2(n)$ *and* $T_2'(8)$, $T_2''(8)$ *are shown in Figures 2 and 3, respectively.*



$T_{2e}(n)$, $n \geq 4$ is even $\quad$ $T_2(5)$ $\quad$ $T_{2o}(n)$, $n \geq 7$ is odd

Figure 2: The trees $T_2(n)$

$T_2'(8)$ $\quad$ $T_2''(8)$

Figure 3: The trees $T_2'(8)$ and $T_2''(8)$

Theorem 2.7. ([6]) *If F is a forest with $n \geq 4$ vertices having $F \neq F_1(n)$, then $mi(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5\,; \\ 7r^{n-7}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_2(n)$ if and only if $F = F_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even;} \\ T_2(5) \text{ or } P_4 \cup P_1, & \text{if } n = 5\,; \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.8 and 2.9, respectively.

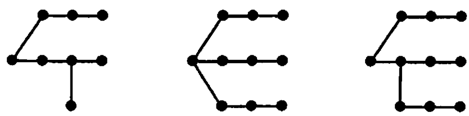
Theorem 2.8. ([2]) *If T is a tree with $n \geq 7$ vertices having $T \neq T_i(n)$, $i = 1, 2$, then $mi(T) \leq t_3(n)$, where*

$$t_3(n) = \begin{cases} 3r^{n-5}, & \text{if } n \geq 7 \text{ is odd;} \\ 7, & \text{if } n = 8\,; \\ 15, & \text{if } n = 10\,; \\ 7r^{n-8} + 2, & \text{if } n \geq 12 \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_3(n)$ if and only if $T = T_3(8), T_3'(10), T_3''(10)$, or $T_3(n)$, where $T_3(n)$ and $T_3(8), T_3'(10), T_3''(10)$ are shown in Figures 4 and 5, respectively.

$T_{3e}(n)$, $n \geq 12$ is even $\qquad$ $T_{3o}(n)$, $n \geq 7$ is odd

Figure 4: The trees $T_3(n)$



$T_3(8)$ $\qquad$ $T_3'(10)$ $\qquad$ $T_3''(10)$

Figure 5: The trees $T_3(8), T_3'(10)$ and $T_3''(10)$

Theorem 2.9. ([7]) *If F is a forest with $n \geq 8$ vertices having $F \neq F_i(n)$, $i = 1, 2$, then $mi(F) \leq f_3(n)$, where*

$$f_3(n) = \begin{cases} 5r^{n-6}, & \textit{if } n \geq 8 \textit{ is even;} \\ 13r^{n-9}, & \textit{if } n \geq 9 \textit{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_3(n)$ if and only if $F = F_3(n)$, where

$$F_3(n) = \begin{cases} T_1(6) \cup \frac{n-6}{2} P_2, & \textit{if } n \geq 8 \textit{ is even;} \\ T_2(9) \cup \frac{n-9}{2} P_2, & \textit{if } n \geq 9 \textit{ is odd.} \end{cases}$$

3 Main results

In this section, we determine the fourth largest values of $mi(G)$ among all trees and forests of order $n \geq 10$, respectively. Moreover, the extremal graphs achieving these values are also determined.

Lemma 3.1. *If F is a forest of even order $n \geq 10$ having $F \neq F_i(n)$, $i = 1, 2, 3$, then $mi(F) \leq 9r^{n-8}$. Furthermore, the equality holds if and only if $F = 2P_4 \cup \frac{n-8}{2} P_2$, or $T_1(8) \cup \frac{n-8}{2} P_2$.*

Proof. It is straightforward to check that $mi(2P_4 \cup \frac{n-8}{2} P_2) = mi(T_1(8) \cup \frac{n-8}{2} P_2) = 9r^{n-8}$. Let F be a forest of even order $n \geq 10$ having $F \neq F_i(n)$, $i = 1, 2, 3$, such that $mi(F)$ is as large as possible. Then $mi(F) \geq 9r^{n-8}$. Suppose that there exist two odd components H_1 and H_2 of F, where $|H_i| = n_i$ for $i = 1, 2$. By Lemma 2.3, Theorems 2.4 and 2.5, we have that $9r^{n-8} \leq mi(F) = mi(H_1) \cdot mi(H_2) \cdot mi(F - (V(H_1) \cup V(H_2)) \leq r^{n_1-1} \cdot r^{n_2-1} \cdot r^{n-n_1-n_2} = r^{n-2} < 9r^{n-8}$, which is a contradiction. Hence F

has no odd component. Since $F \neq F_1(n)$, there exists an even component H of order $m \geq 4$. We consider the following two cases.
Case 1. $F - V(H) \neq F_1(n-m)$.

By Lemma 2.3, Theorems 2.4 and 2.7, we have that $9r^{n-8} \leq mi(F) = mi(H) \cdot mi(F-V(H)) \leq t_1(m) \cdot f_2(n-m) = (r^{m-2}+1) \cdot 3r^{n-m-4} = 3r^{n-6} + 3r^{n-m-4} \leq 9r^{n-8}$. Furthermore, the equalities holding imply that $m = 4$, $H = P_4$ and $F - V(H) = P_4 \cup \frac{n-8}{2} P_2$. In conclusion, $F = 2P_4 \cup \frac{n-8}{2} P_2$.
Case 2. $F - V(H) = F_1(n-m)$.

Since $F \neq F_i(n)$ for $i = 1, 2, 3$, by Lemma 2.3, Theorems 2.4, 2.5, 2.7 and 2.9, we have that

$$\begin{aligned} 9r^{n-8} &\leq mi(F) = mi(H) \cdot mi(F - V(H)) \\ &\leq \begin{cases} (t_1(m) - 1) \cdot f_1(n-m) & \text{if } m = 4, 6, \\ t_1(m) \cdot f_1(n-m) & \text{if } m \geq 8, \end{cases} \\ &= \begin{cases} r^{n-2} & \text{if } m = 4, 6, \\ r^{n-2} + r^{n-m} & \text{if } m \geq 8, \end{cases} \\ &\leq 9r^{n-8}. \end{aligned}$$

Furthermore, the equalities holding imply that $m = 8$, $H = T_1(8)$ and $F - V(H) = \frac{n-8}{2} P_2$. In conclusion, $F = T_1(8) \cup \frac{n-8}{2} P_2$. □

Lemma 3.2. *If F is a forest of odd order $n \geq 11$ having $F \neq F_i(n)$, $i = 1, 2, 3$, then $mi(F) \leq 25r^{n-11}$. Furthermore, the equality holds if and only if $F = T_2(11) \cup \frac{n-11}{2} P_2$.*

Proof. It is straightforward to check that $mi(T_2(11) \cup \frac{n-11}{2} P_2) = 25r^{n-11}$. Let F be a forest of odd order $n \geq 11$ having $F \neq F_i(n)$, $i = 1, 2, 3$, such that $mi(F)$ is as large as possible. Then $mi(F) \geq 25r^{n-11}$. Suppose that F has three odd components H_1, H_2 and H_3, where $|H_i| = n_i$ for $i = 1, 2, 3$. By Lemma 2.3, Theorems 2.4 and 2.5, we have that $25r^{n-11} \leq mi(F) = (\prod_{i=1}^{3} mi(H_i)) \cdot mi(F - \cup_{i=1}^{3} V(H_i)) \leq r^{n_1-1} \cdot r^{n_2-1} \cdot r^{n_3-1} \cdot r^{n-(n_1+n_2+n_3)} = r^{n-3} < 25r^{n-11}$, which is a contradiction. Thus we obtain that F has exactly one odd component H of order $m \geq 3$, there are two cases depending on the structure of $F - V(H)$.
Case 1. $F - V(H) \neq F_1(n-m)$.

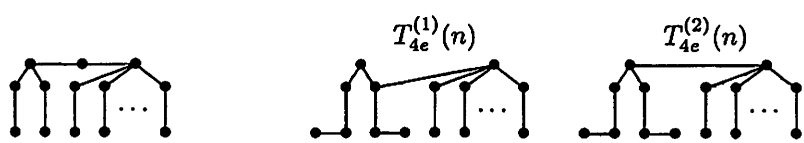
By Lemma 2.3, Theorems 2.4 and 2.7, then we have that $25r^{n-11} \leq mi(F) = mi(H) \cdot mi(F - V(H)) \leq t_1(m) \cdot f_2(n-m) = r^{m-1} \cdot 3r^{n-m-4} = 3r^{n-5} < 25r^{n-11}$, which is a contradiction.
Case 2. $F - V(H) = F_1(n-m)$.

Then $m \geq 5$. Since $F \neq F_i(n)$ for $i = 1, 2, 3$, by Lemma 2.3, Theorems 2.4, 2.5, 2.6, 2.7 and 2.9, we have that

$$
\begin{aligned}
25r^{n-11} &\le mi(F) = mi(H)\cdot mi(F-V(H))\\
&\le \begin{cases} t_2(5)\cdot f_1(n-5) & \text{if } m=5,\\ (t_2(m)-1)\cdot f_1(n-m) & \text{if } m=7,9,\\ t_2(m)\cdot f_1(n-m) & \text{if } m\ge 11,\end{cases}\\
&= \begin{cases} 3r^{n-5} & \text{if } m=5,7,9,\\ 3r^{n-5}+r^{n-m} & \text{if } m\ge 11,\end{cases}\\
&\le 25r^{n-11}.
\end{aligned}
$$

Furthermore, the equalities holding imply that $m = 11$, $H = T_2(11)$ and $F - V(H) = \frac{n-11}{2}P_2$. In conclusion, $F = T_2(11) \cup \frac{n-11}{2}P_2$. □

For an odd integer $n \ge 11$, $T_{4o}(n)$ is the tree obtained from $B(2, \frac{n-7}{2})$ with $j_2 = 0$ by adding a P_5 and a new edge joining the vertex with degree 1 in the basic path of $B(2, \frac{n-7}{2})$ and the vertex of P_5 which is not a support vertex or leaf. For an even integer $n \ge 12$, $T_{4e}(n)$ is the tree obtained from $B(1, \frac{n-8}{2})$ by adding a P_7 and a new edge joining the only vertex in the basic path of $B(1, \frac{n-8}{2})$ and the vertex of P_7 which is not a support vertex or leaf, see Figure 6.



$T_{4o}(n)$, $n \ge 11$ is odd $T_{4e}(n)$, $n \ge 12$ is even

Figure 6: The trees $T_4(n)$

Let T be a tree and x a leaf lying on a longest path P of T, say $P = x, y, z, w, \ldots$ and H the component of $T - N_T[y]$ containing some vertices of P. Since P is a longest path of T, it follows that every component of $T - (N_T[y] \cup V(H))$ is P_1 or P_2. Thus we have that $T - N_T[y] = aP_1 \cup (b-1)P_2 \cup H$, see Figure 7.

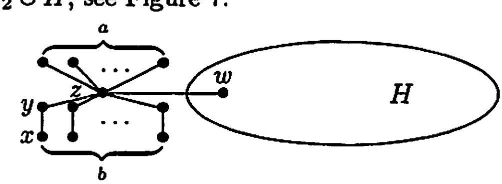


Figure 7: The tree T

Lemma 3.3. *For positive integers* m, p, q, s *and* t, *if* $f(x) = pr^x + qr^{m-x}$ *for* $s \le x \le t$, *then* $f(x)$ *has a maximum value at* $x = s$ *or* t.

Proof. From simple calculation, we have that $f'(x) = (\ln r)(pr^x - qr^{m-x})$ and $f''(x) = (\ln r)^2(pr^x + qr^{m-x})$. Note that $f''(x) > 0$. Hence $f(x)$ yields a maximum value when $x = s$ or t. □

Lemma 3.4. *If* T *is a tree of odd order* $n \ge 11$ *having* $T \ne T_i(n)$, $i = 1,2,3$, *then* $mi(T) \le 5r^{n-7} + 3$. *Furthermore, the equality holds if and only if* $T = T_{4o}(n)$.

Proof. It is straightforward to check that $mi(T_{4o}(n)) = 5r^{n-7} + 3$. Let T be a tree of odd order $n \ge 11$ having $T \ne T_i(n)$, $i = 1,2,3$, such that $mi(T)$ is as large as possible. Then $mi(T) \ge 5r^{n-7} + 3$. Suppose that T has duplicated leaves x_1 and x_2, by Lemma 2.2 and Theorem 2.4, $5r^{n-7}+3 \le mi(T) = mi(T - x_2) \le t_1(n-1) = r^{n-3}+1 < 5r^{n-7}+3$, which is a contradiction. Thus T has no duplicated leaves. Let $P = x, y, z, w, \ldots$ be a longest path of T, we obtain that $a = 0$ or 1 in Figure 7.

Suppose that $a = 1$. Then H is a tree of odd order $n - 2 - 2b \ge 3$. Since $T \ne T_3(n)$, this implies that $T - N_T[x] \ne T_1(n-2)$. By Theorem 2.4, $mi(H) \le t_1(n-2-2b) = r^{n-3-2b}$. Then, by Lemma 2.3, $mi(T - N_T[y]) \le r^{2b-2} \cdot r^{n-3-2b} = r^{n-5}$. By Lemma 2.1 and Theorem 2.6, we have that $3r^{n-7} + 1 = t_2(n-2) \ge mi(T - N_T[x]) = mi(T) - mi(T - N_T[y]) \ge (5r^{n-7} + 3) - r^{n-5} = 3r^{n-7} + 3$, which is a contradiction. Hence we obtain that $a = 0$ and H is a tree of even order $n - 2b - 1$.

Suppose that $b = 1$,then $T - N_T[x]$ and $T - N_T[y]$ are trees, where $T - N_T[x] \ne T_1(n-2)$. By Theorems 2.4 and 2.6, we have that $5r^{n-7} + 3 \le mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \le t_2(n-2) + t_1(n-3) = (3r^{n-7}+1) + (r^{n-5}+1) = 5r^{n-7}+2 < 5r^{n-7}+3$, which is a contradiction. Hence we obtain that $b \ge 2$.

Since $T \ne T_i(n)$ for $i = 1,2,3$, this implies that $|H| = n-2b-1 \ge 6$ and $4 \le 2b \le n-7$. Let $w \in V(H)$ be a neighbor of z. By Theorems 2.4, 2.5 and Lemma 3.3, we have that $5r^{n-7}+3 \le mi(T) = mi_{+z}(T) + mi_{-z}(T) \le mi(H - w) + [(r^{2b} - 1) \cdot mi(H) + 1 \cdot mi(H - N_H[w])] \le r^{n-2b-3} + (r^{2b} - 1)(r^{n-2b-3}+1) + r^{n-2b-3} = r^{n-2b-3} + r^{n-3} - 1 + r^{2b} - r^{n-2b-3} + r^{n-2b-3} = (r^{n-3} - 1) + (r^{2b} + r^{n-2b-3}) \le (r^{n-3} - 1) + (r^4 + r^{n-7}) = 5r^{n-7} + 3$. Furthermore, the equalities holding imply that either $b = 2$, $H = T_1(n-5)$ or $b = (n-7)/2$, $H = T_1(6)$. Note that $H - N_H[w] = \frac{n-2b-3}{2}P_2$. In conclusion, $T = T_{4o}(n)$. □

Lemma 3.5. *If* T *is a tree of even order* $n \ge 12$ *having* $T \ne T_i(n)$, $i = 1,2,3$, *then* $mi(T) \le 7r^{n-8} + 1$. *Furthermore, the equality holds if and only if* $T = T_{4e}(n)$.

Proof. It is straightforward to check that $mi(T_{4e}(n)) = 7r^{n-8}+1$. Let T be a tree of even order $n \geq 12$ having $T \neq T_i(n)$, $i = 1,2,3$ such that $mi(T)$ is as large as possible. By Theorem 2.8, $7r^{n-8}+1 \leq mi(T) \leq t_3(n)-1 = (7r^{n-8}+2)-1 = 7r^{n-8}+1$, hence $mi(T) = 7r^{n-8}+1$. Suppose that T has duplicated leaves x_1 and x_2, then $T' = T - x_2$ is a tree of odd order $n-1$. Since $T \neq T_2(n)$, this implies that $T' \neq T_1(n-1)$. By Theorem 2.6, we have that $7r^{n-8}+1 = mi(T) = mi(T') \leq t_2(n-1) = 3r^{n-6}+1 < 7r^{n-8}$, which is a contradiction. Thus T has no duplicated leaves. Let $P = x, y, z, w, \ldots$ be a longest path of T, we obtain that $a = 0$ or 1 in Figure 7.

Suppose that $a = 1$. Since $T \neq T_1(n)$, this implies that H is a tree of even order $n-2b-2 \geq 4$ and $2 \leq 2b \leq n-6$. By Theorems 2.4, 2.5 and Lemma 3.3, we have that $7r^{n-8}+1 = mi(T) = mi_{+z}(T) + mi_{-z}(T) \leq mi(H-w) + r^{2b} \cdot mi(H) \leq r^{n-2b-4} + r^{2b} \cdot (r^{n-2b-4}+1) = r^{n-2b-4} + r^{n-4} + r^{2b} \leq r^{n-4} + r^{n-6} + 2 = 6r^{n-8}+2 < 7r^{n-8}+1$. This is a contradiction, hence $a = 0$. It follows that $|H| = n-2b-1$ is odd. Since $T \neq T_1(n)$ and $T \neq T_2(n)$, these imply that $H \neq T_1(n-2b-1)$ and $H - w \neq F_1(n-2b-2)$. Thus $mi(H) \leq t_2(n-2b-1)$ and $mi(H-w) \leq f_2(n-2b-2)$. We consider two cases.

Case 1. $b = 1$.

Then $H = T - N_T[y]$ and $|H| = n-3$. By Theorem 2.6, $mi(T - N_T[x]) = mi(T) - mi(T - N_T[y]) \geq (7r^{n-8}+1) - t_2(n-3) = (7r^{n-8}+1) - (3r^{n-8}+1) = r^{n-4} = t_2(n-2)$. By Theorems 2.4 and 2.6, we obtain that $T - N_T[x] = T_1(n-2)$ or $T_2(n-2)$. For the case of $T - N_T[x] = T_1(n-2)$, then $mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) = (7r^{n-8}+1) - (r^{n-4}+1) = 3r^{n-8} = t_3(n-3)$. By Theorem 2.8, then $T - N_T[y] = T_3(n-3)$. In conclusion, $T = T_{4e}^{(1)}(n)$. For the case of $T - N_T[x] = T_2(n-2)$, then $mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) = (7r^{n-8}+1) - r^{n-4} = 3r^{n-8}+1 = t_2(n-3)$. By Theorem 2.6, then $T - N_T[y] = T_2(n-3)$. In conclusion, $T = T_{4e}^{(2)}(n)$.

Case 2. $b \geq 2$.

Note that $|H| = n-2b-1 \geq 5$ is odd. Suppose that $|H| = 5$. Since $T \neq T_1(n)$ and $T \neq T_2(n)$, these imply that $H \neq P_5$. On the other hand, since T has no duplicated leaves, this implies that $H = T_2(5)$ and z is adjacent to one of the duplicated leaves of $T_2(5)$. Then $mi(T) = 3r^{n-6}+2 < 7r^{n-8}+1$, which is a contradiction. Hence $|H| = n-2b-1 \geq 7$ and $4 \leq 2b \leq n-8$.

We claim that $2b = n-8$ and $H = T_2(7)$. Since $b \geq 2$ and $T \neq T_i(n)$ for $i = 1,2$, these imply that $T - N_T[x]$ is a tree having $T - N_T[x] \neq T_i(n-2)$ for $i = 1,2$. By Theorem 2.8, we have that $7r^{n-8}+1 = mi(T) = mi(T - N_T[x]) + mi(T - N_T[y]) \leq t_3(n-2) + r^{2b-2} \cdot mi(H) \leq 7r^{n-10} + 2 + r^{2b-2} \cdot (3r^{n-2b-6}+1) = 7r^{n-10} + 2 + 3r^{n-8} + r^{2b-2} = 13r^{n-10} + 2 + r^{2b-2}$. Thus we obtain that $r^{2b-2} \geq r^{n-10} - 1$ and $2b \geq n-8$. Hence $2b = n-8$, $|H| = 7$ and $mi(H) \leq t_2(7) = 7$. Since $2b = n-8 \geq 4$, this implies that $T - N_T[x] \neq T_3(n-2)$ and $mi(T - N_T[x]) \leq 7r^{n-10} + 1$. Thus $r^{n-10} \cdot mi(H) = mi(T - N_T[y]) = mi(T) - mi(T - N_T[x]) \geq (7r^{n-8} + 1) - (7r^{n-10} + 1) = 7r^{n-10}$, then $mi(H) \geq 7$. Hence $mi(H) = 7$, by Theorem 2.6, $H = P_7$. Since $T \neq T_3(n)$, this implies that w is not a leave of H. If w is a support vertex of H, then $mi(H) = 7r^{n-8} < 7^{n-8} + 1$, which is a contradiction. Hence w is neither a leaf nor a support vertex. In conclusion, $T = T_{4e}^{(1)}(n)$ or $T_{4e}^{(2)}(n)$. □

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