Vague Lie Superalgebras

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Abstract

We introduce the notion of vague Lie sub-superalgebras (resp. vague ideals) and present some of their properties. We investigate the properties of vague Lie sub-superalgebras and vague ideals under homomorphisms of Lie superalgebras. We introduce the concept of vague bracket product and establish its characterizations. We also introduce the notions of solvable vague ideals and nilpotent vague ideals of Lie superalgebras and present the corresponding theorems parallel to Lie superalgebras.

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1 Introduction

The theory of Lie superalgebras was constructed by V.G. Kac [17] in 1977 as a generalization of the theory of Lie algebras. This theory had played an important role in both mathematics and physics. In particular, Lie superalgebras are important in theoretical physics where they are used to describe the mathematics of supersymmetry. Furthermore, Lie superalgebras had found many applications in computer science such as unimodal polynomials.
The concept of fuzzy set was first initiated by Zadeh [19] in 1965 and since

then, fuzzy set has become an important tool in studying scientific subjects, in particular, it can be applied in a wide variety of disciplines such as computer science, medical science, management science, social science, engineering and so on. There are a number of generalizations of Zadeh's fuzzy set theory so far reported in the literature viz., interval-valued fuzzy set theory, intuitionistic fuzzy set theory, L-fuzzy set theory, probabilistic fuzzy set theory, bipolar fuzzy set theory etc. to list a few. In 1993, Gau and Buehrer [16] introduced the notion of vague set theory as a generalization of Zadeh's fuzzy set theory. Vague sets are higher order fuzzy sets. Application of higher order fuzzy sets makes the solution-procedure more complex, but if the complexity on computation-time, computation-volume or memory-space are not the matter of concern then a better results could be achieved. In the most cases of judgment, evaluation is done by human beings (or by an intelligent agent) where there certainly is a limitation of knowledge or intellectual functionaries. Naturally, every decision-maker hesitates, more or less, on every evaluation activity. For example, to judge whether a patient has cancer or not, a doctor (the decision-maker) will hesitate to give his opinion because a fraction of his evaluation is in favor of truth, another fraction is in favor of falseness and the rest remains undecided to him. This is the fundamental philosophy behind the notion of vague set theory.
Chen [11] considered the notion of fuzzy quotient Lie superalgebras over a field. Akram introduced the notion of cofuzzy Lie superalgebras over a cofuzzy field in [1]. Chen [12, 14] introduced the notions of intuitionistic fuzzy Lie sub-superalgebras and intuitionistic fuzzy ideals and investigated several properties. In this paper, we introduce the notion of vague Lie sub-superalgebras (resp. vague ideals) and present some of their properties. We investigate the properties of vague Lie sub-superalgebras and vague ideals under homomorphisms of Lie superalgebras. We introduce the concept of vague bracket product and establish its characterizations. We also introduce the notions of solvable vague ideals and nilpotent vague ideals of Lie superalgebras and present the corresponding theorems parallel to Lie superalgebras. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are refereed to [2-6, 8-10, 13, 18, 20].

2 Preliminaries

In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1. [17] Suppose that V is a vector space and $V_{\bar{0}}, V_{\bar{1}}$ are its (vector) subspaces. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be the direct sum of the subspaces.

Then V (with this decomposition) is called a $\mathbb{Z}_2$*-graded vector space* if each element v of a $\mathbb{Z}_2$-graded vector space has a unique expression of the form $v = v_{\bar{0}} + v_{\bar{1}}$ $(v_{\bar{0}} \in V_{\bar{0}}, v_{\bar{1}} \in V_{\bar{1}})$. The subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are called the even part and odd part of V, respectively. In particular, if v is an element of either $V_{\bar{0}}$ or $V_{\bar{1}}$, v is said to be homogeneous.

Definition 2.2. [17] A $\mathbb{Z}_2$-graded vector space $\mathbb{L} = \mathbb{L}_{\bar{0}} \oplus \mathbb{L}_{\bar{1}}$ with a Lie bracket

$$[\ ,\ ] : \mathbb{L} \times \mathbb{L} \xrightarrow{\text{bilinear}} \mathbb{L}$$

is called a *Lie superalgebra*, if it satisfies the following conditions:

(1) $[\mathbb{L}_i, \mathbb{L}_j] \subseteq \mathbb{L}_{i+j}$ for $i,\ j \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$,

(2) $[x, y] = -(-1)^{|x||y|}[y, x]$ (antisymmetry),

(3) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[[x, z], y]$(Jacobi identity),

where for any homogeneous element $a \in \mathbb{L}_{\bar{n}}$, $n = 0, 1$. The subspaces $\mathbb{L}_{\bar{0}}$ and $\mathbb{L}_{\bar{1}}$ are called the even and odd parts of $\mathbb{L}$, respectively. Therefore, a Lie algebra is a Lie superalgebra with trivial odd part.

Definition 2.3. [17] If $\varphi : \mathbb{L} \to \acute{\mathbb{L}}$ is a linear map between Lie superalgebras $\mathbb{L} = \mathbb{L}_{\bar{0}} \oplus \mathbb{L}_{\bar{1}}$ and $\acute{\mathbb{L}} = \acute{\mathbb{L}}_{\bar{0}} \oplus \acute{\mathbb{L}}_{\bar{1}}$ such that

(4) $\varphi(\mathbb{L}_i) \subseteq \acute{\mathbb{L}}_i$ $(i \in \mathbb{Z}_2)$ (preserving the grading),

(5) $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (preserving the Lie bracket).

Then φ is called a *homomorphism* of Lie superalgebras.

Throughout this paper, we denote V is a vector space, $\mathbb{L}$ is a Lie superalgebra and k is a field.
Let μ be a *fuzzy set* on V, i.e., a map $\mu : V \to [0, 1]$. Let V be a complete lattice whose minimum and maximum we denote by 0 and 1, respectively. In this paper, we use the notations $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. We give here review of two models that extend the Zadeh's fuzzy set theory: intuitionistic fuzzy set theory and vague set theory.

Definition 2.4. [16] A *vague set* A in the universe of discourse X is a pair (t_A, f_A), where $t_A : X \to [0, 1]$, $f_A : V \to [0, 1]$ are true and false membership functions, respectively such that $t_A(x) + f_A(x) \leq 1$ for all $x \in V$.

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A (based on evidence), and $f_A(x)$ is the lower bound for negation of membership of x in A (based on evidence against).

Therefore, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1 - f_A(x)]$. So, a vague set is a special case of interval valued sets studied by many mathematicians and applied in many branches of mathematics (see for example [3]). Also vague sets have many applications (cf. [1, 8, 10]). The interval $[t_A(x), 1 - f_A(x)]$ is called the *vague value* of x in A, and is denoted by $V_A(x)$. We denote zero vague and unit vague value by $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$, respectively.
It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval valued membership value is assigned to each element of the universe considering the "evidence for x" only, without considering "evidence against x". In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.

Definition 2.5. [7] An *intuitionistic fuzzy set* $B = \{< x, \mu_B, \nu_B > \mid x \in V\}$ in a universe of discourse V is characterized by a membership function, μ_B, and a non-membership function, ν_B, as follows: $\mu_B : V \to [0, 1]$, $\nu_B : V \to [0, 1]$, and $\mu_B(x) + \nu_B(x) \leq 1$ for all $x \in V$.

As we can see that the difference between vague sets and intuitionistic fuzzy sets is due to the definition of membership intervals. We have $[t_A(x), 1 - f_A(x)]$ for x in A but (μ_B, ν_B) for x in B. Here the semantics of μ_B is the same as with t_A and ν_B is the same as with f_A. However, the boundary $1 - f_A$ is able to indicate the possible existence of a data value. This subtle difference gives rise to a simpler but meaningful graphical view of data sets.

Definition 2.6. [16] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be two vague sets. Then we define:

(3) $\overline{A} = (f_A, 1 - t_A)$,

(4) $A \subset B \Leftrightarrow V_A(x) \leq V_B(x)$, i.e., $t_A(x) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x)$,

(5) $A = B \Leftrightarrow V_A(x) = V_B(x)$,

(6) $C = A \cap B \Leftrightarrow V_C(x) = \min(V_A(x), V_B(x))$,

(7) $C = A \cup B \Leftrightarrow V_C(x) = \max(V_A(x), V_B(x))$

for all $x \in V$.

Definition 2.7. A vague set $A = (t_A, f_A)$ in vector space V is called a *vague subspace* of V, if

$$t_A(\alpha x + \beta y) \geq \min\{t_A(x), t_A(y)\}, \ f_A(\alpha x + \beta y) \leq \max\{f_A(x), f_A(y)\}$$

for $\alpha, \beta \in F$ and $x, y \in V$. From the definition, we have $t_A(0) = t_A(x-x) \geq \min\{t_A(x), t_A(x)\} = t_A(x)$, $f_A(0) = f_A(x-x) \leq \max\{f_A(x), f_A(x)\} = f_A(x)$ for any $x \in V$. We always assume that $V_A(0) = [1, 1]$.

Definition 2.8. Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague sets of vector space V. We define the *sum* of A and B by $A + B = (t_{A+B}, f_{A+B})$ where

$$t_{A+B}(x) = \sup_{x=a+b} \min\{t_A(a), t_B(b)\}, f_{A+B}(x) = \inf_{x=a+b} \max\{f_A(a), f_B(b)\}.$$

Definition 2.9. Let $A = (t_A, f_A)$ be a vague set of vector space V' and ϕ be a mapping from vector space V to V'. Then the *inverse image* of A, denoted by $f^{-1}(A) = (t_{\phi^{-1}(A)}, f_{\phi^{-1}(A)})$, is the vague set in V with the membership function given by $t_{\phi^{-1}(A)}(x) = t_A(\phi(x))$, $f_{\phi^{-1}(A)}(x) = f_A(\phi(x))$ for all $x \in V$.

Definition 2.10. Let $A = (t_A, f_A)$ be a vague set of vector space V and ϕ be a mapping from vector space V to V'. Then the *image* of A, denoted by $\phi(A) = (t_{\phi(A)}, f_{\phi(A)})$, is the vague set in V' with membership functions defined by

$$t_{\phi(A)}(y) = \begin{cases} \sup\limits_{x \in \phi^{-1}(y)} \{t_A(x)\} & y \in \phi(V) \\ 0 & y \notin \phi(V), \end{cases}$$

$$f_{\phi(A)}(y) = \begin{cases} \inf\limits_{x \in \phi^{-1}(y)} \{t_A(x)\} & y \in \phi(V) \\ 1 & y \notin \phi(V). \end{cases}$$

We state here some properties of vague subspaces of vector spaces without their proofs.

Lemma 2.11. *$A = (t_A, f_A)$ is a vague subspace of vector space V if and only if t_A and f_A are fuzzy subspaces of V.*

Lemma 2.12. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague subspaces of vector space V. Then $A + B$ is also a vague subspace of V.*

Lemma 2.13. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague subspaces of vector space V. Then $A \cap B$ is also a vague subspace of V.*

Lemma 2.14. *Let $A = (t_A, f_A)$ be a vague subspace of vector space V' and ϕ be a mapping from vector space V to V'. Then the inverse image $\phi^{-1}(A)$ is also a vague subspace of V.*

Lemma 2.15. *Let $A = (t_A, f_A)$ be a vague subspace of vector space V and f be a mapping from vector space V to V'. Then the image $\phi(A)$ is also a vague subspace of V.*

3 Vague Lie Superalgebras

Definition 3.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and let $A_{\bar{0}} = (t_{A_{\bar{0}}}, f_{A_{\bar{0}}})$ and $A_{\bar{1}} = (t_{A_{\bar{1}}}, f_{A_{\bar{1}}})$ be vague vector subspaces of $V_{\bar{0}}$, $V_{\bar{1}}$, respectively. Then we define $A'_{\bar{0}} = (t_{A'_{\bar{0}}}, f_{A'_{\bar{0}}})$ and $A'_{\bar{1}} = (t_{A'_{\bar{1}}}, f_{A'_{\bar{1}}})$ by:

$$t_{A'_{\bar{0}}}(x) = \begin{cases} t_{A_{\bar{0}}}(x) & x \in V_{\bar{0}} \\ 0 & x \notin V_{\bar{0}} \end{cases}, \quad f_{A'_{\bar{0}}}(x) = \begin{cases} f_{A_{\bar{0}}}(x) & x \in V_{\bar{0}} \\ 1 & x \notin V_{\bar{0}}, \end{cases}$$

$$t_{A'_{\bar{1}}}(x) = \begin{cases} t_{A_{\bar{1}}}(x) & x \in V_{\bar{1}} \\ 0 & x \notin V_{\bar{1}} \end{cases}, \quad f_{A'_{\bar{1}}}(x) = \begin{cases} f_{A_{\bar{1}}}(x) & x \in V_{\bar{1}} \\ 1 & x \notin V_{\bar{1}}. \end{cases}$$

$A'_{\bar{0}} = (t_{A'_{\bar{0}}}, f_{A'_{\bar{0}}})$ and $A'_{\bar{1}} = (t_{A'_{\bar{1}}}, f_{A'_{\bar{1}}})$ are the vague vector subspaces of V. Moreover, we have $A'_{\bar{0}} \cap A'_{\bar{1}} = (t_{A'_{\bar{0}} \cap A'_{\bar{1}}}, f_{A'_{\bar{0}} \cap A'_{\bar{1}}})$, where

$$t_{A'_{\bar{0}} \cap A'_{\bar{1}}}(x) = t_{A'_{\bar{0}}}(x) \wedge t_{A'_{\bar{1}}}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases},$$

$$f_{A'_{\bar{0}} \cap A'_{\bar{1}}}(x) = f_{A'_{\bar{0}}}(x) \vee f_{A'_{\bar{1}}}(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}.$$

Thus $A'_{\bar{0}} + A'_{\bar{1}}$ is the direct sum and is denoted by $A_{\bar{0}} \oplus A_{\bar{1}}$. If $A = (t_A, f_A)$ is a vague vector subspace of V and $A = A_{\bar{0}} \oplus A_{\bar{1}}$, then $A = (t_A, f_A)$ is called a $\mathbb{Z}_2$-graded vague vector subspace of V.

Definition 3.2. Let $A = (t_A, f_A)$ be a vague set of $\mathbb{L}$. Then $A = (t_A, f_A)$ is called a vague Lie sub-superalgebra of $\mathbb{L}$, if it satisfies the following conditions:
(1)$A = (t_A, f_A)$ is a $\mathbb{Z}_2$-graded vague vector subspace,
(2) $t_A([x, y]) \geq t_A(x) \wedge t_A(y)$ and $f_A([x, y]) \leq f_A(x) \vee f_A(y)$.
If the condition (2) is replaced by (3) $t_A([x, y]) \geq t_A(x) \vee t_A(y)$ and $f_A([x, y]) \leq f_A(x) \wedge f_A(y)$, then $A = (t_A, f_A)$ is called a vague ideal of $\mathbb{L}$.

Example 3.3. Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$, where $N_{\bar{0}} = \langle e \rangle$, $N_{\bar{1}} = \langle a_1, \cdots, a_n, b_1, \cdots, b_n \rangle$ and $[a_i, b_i] = e, i = 1, 2, \cdots n$, the remaining brackets being zero. Then N is Lie superalgebra [17].

Define $A_{\bar{0}} = (t_{A_{\bar{0}}}, f_{A_{\bar{0}}})$ where $t_{A_{\bar{0}}} : N_{\bar{0}} \to [0, 1]$ by

$$t_{A_{\bar{0}}}(x) = \begin{cases} 0.5 & x \in N_{\bar{0}} \setminus \{0\} \\ 1 & x = 0 \end{cases}, \quad f_{A_{\bar{0}}} : N_{\bar{0}} \to [0, 1] \text{ by}$$

$$f_{A_{\bar{0}}}(x) = \begin{cases} 0.1 & x \in N_{\bar{0}} \setminus \{0\} \\ 0 & x = 0 \end{cases}.$$

Define $A_{\bar{1}} = (t_{A_{\bar{1}}}, f_{A_{\bar{1}}})$ where $t_{A_{\bar{1}}} : N_{\bar{1}} \to [0, 1]$ by

$$t_{A_{\bar{1}}}(x) = \begin{cases} 0.3 & x \in N_{\bar{1}} \setminus \{0\} \\ 1 & x = 0 \end{cases}, \quad f_{A_{\bar{1}}} : N_{\bar{1}} \to [0, 1] \text{ by}$$

$$f_{A_{\bar{1}}}(x) = \begin{cases} 0.2 & x \in N_{\bar{1}} \setminus \{0\} \\ 0 & x = 0 \end{cases}.$$

Define $A = (t_A, f_A)$ by $A = A_{\bar{0}} \oplus A_{\bar{1}}$. Then $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of N.

Definition 3.4. For any $s \in [0, 1]$ and fuzzy subset t of $\mathbb{L}$, the set $U(t_A, s) = \{x \in \mathbb{L} | t_A(x) \geq s\}$ (resp. $L(f_A, s) = \{x \in \mathbb{L} | f_A(x) \leq s\}$) is called an upper (resp. lower) s-level cut of s.

We state the following Theorems without its proof.

Theorem 3.5. *If $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of $\mathbb{L}$, then the sets $U(t_A, s)$ and $L(f_A, s)$ are Lie sub-superalgebras of $\mathbb{L}$ for every $s \in Imt_A \cap Imf_A$.*

Theorem 3.6. *If $A = (t_A, f_A)$ is a vague set of $\mathbb{L}$ such that all non-empty level sets $U(t_A, s)$ and $L(f_A, s)$ are Lie sub-superalgebras of $\mathbb{L}$, then $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of $\mathbb{L}$.*

Theorem 3.7. *If $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague Lie sub-superalgebras of $\mathbb{L}$, then so is $A + B = (t_{A+B}, f_{A+B})$.*

Theorem 3.8. *If $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague Lie sub-superalgebras of $\mathbb{L}$, then so is $A \cap B = (t_{A\cap B}, f_{A\cap B})$.*

Proposition 3.9. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a Lie homomorphism. If $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of $\mathbb{L}'$, then the vague set $\varphi^{-1}(A)$ of $\mathbb{L}$ is also a vague Lie sub-superalgebra.*

Proof. Since φ preserves the grading, $\varphi(x) = \varphi(x_{\bar{0}} + x_{\bar{1}}) = \varphi(x_{\bar{0}}) + \varphi(x_{\bar{1}}) \in \mathbb{L}'_{\bar{0}} \oplus \mathbb{L}'_{\bar{1}}$, for $x = x_{\bar{0}} + x_{\bar{1}} \in \mathbb{L}$. We define $\varphi^{-1}(A)_{\bar{0}} = (t_{\varphi^{-1}(A)_{\bar{0}}}, f_{\varphi^{-1}(A)_{\bar{0}}})$ where $t_{\varphi^{-1}(A)_{\bar{0}}} = \varphi^{-1}(t_{A_{\bar{0}}})$, $f_{\varphi^{-1}(A)_{\bar{0}}} = \varphi^{-1}(f_{A_{\bar{0}}})$ and define $\varphi^{-1}(A)_{\bar{1}} = (t_{\varphi^{-1}(A)_{\bar{1}}},$
$f_{\varphi^{-1}(A)_{\bar{1}}})$ where $t_{\varphi^{-1}(A)_{\bar{1}}} = \varphi^{-1}(t_{A_{\bar{1}}})$, $f_{\varphi^{-1}(A)_{\bar{1}}} = \varphi^{-1}(f_{A_{\bar{1}}})$. By Lemma 2.14, we have that they are vague subspaces of $\mathbb{L}_{\bar{0}}, \mathbb{L}_{\bar{1}}$, respectively.
Then we define $\varphi^{-1}(A)'_{\bar{0}} = (t_{\varphi^{-1}(A)'_{\bar{0}}}, f_{\varphi^{-1}(A)'_{\bar{0}}})$, where $t_{\varphi^{-1}(A)'_{\bar{0}}} = \varphi^{-1}(t_{A'_{\bar{0}}})$, $f_{\varphi^{-1}(A)'_{\bar{0}}} = \varphi^{-1}(f_{A'_{\bar{0}}})$, and $\varphi^{-1}(A)'_{\bar{1}} = (t_{\varphi^{-1}(A)'_{\bar{1}}}, f_{\varphi^{-1}(A)'_{\bar{1}}})$, where $t_{\varphi^{-1}(A)'_{\bar{1}}} = \varphi^{-1}(t_{A'_{\bar{1}}})$, $f_{\varphi^{-1}(A)'_{\bar{1}}} = \varphi^{-1}(f_{A'_{\bar{1}}})$.

Clearly,
$$t_{\varphi^{-1}(A)'_{\bar{0}}}(x) = \begin{cases} t_{\varphi^{-1}(A)_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 0 & x \notin \mathbb{L}_{\bar{0}} \end{cases},$$
$$f_{\varphi^{-1}(A)'_{\bar{0}}}(x) = \begin{cases} f_{\varphi^{-1}(A)'_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 1 & x \notin \mathbb{L}_{\bar{0}} \end{cases},$$
and $t_{\varphi^{-1}(A)'_{\bar{1}}}(x) = \begin{cases} t_{\varphi^{-1}(A)'_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 0 & x \notin \mathbb{L}_{\bar{1}} \end{cases}$,
$f_{\varphi^{-1}(A)'_{\bar{1}}}(x) = \begin{cases} f_{\varphi^{-1}(A)'_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 1 & x \notin \mathbb{L}_{\bar{1}} \end{cases}$ .These show that $\varphi^{-1}(A)'_{\bar{0}}$ and

$\varphi^{-1}(A)'_{\bar{1}}$ are the extensions of $\varphi^{-1}(A)_{\bar{0}}$ and $\varphi^{-1}(A)_{\bar{1}}$.
For $0 \neq x \in \mathbb{L}$, we have

$$\begin{aligned} t_{\varphi^{-1}(A)'_{\bar{0}}}(x) \wedge t_{\varphi^{-1}(A)'_{\bar{1}}}(x) &= \varphi^{-1}(t_{A'_{\bar{0}}})(x) \wedge \varphi^{-1}(t_{A'_{\bar{1}}})(x) \\ &= t_{A'_{\bar{0}}}(\varphi(x)) \wedge t_{A'_{\bar{1}}}(\varphi(x)) = 0. \end{aligned}$$

Let $x \in \mathbb{L}$. We have

$$\begin{aligned} t_{\varphi^{-1}(A)'_{\bar{0}}+\varphi^{-1}(A)'_{\bar{1}}}(x) &= \sup_{x=a+b}\{t_{\varphi^{-1}(A)'_{\bar{0}}}(a) \wedge t_{\varphi^{-1}(A)'_{\bar{1}}}(b)\} \\ &= \sup_{x=a+b}\{\varphi^{-1}(t_{A'_{\bar{0}}})(a) \wedge \varphi^{-1}(t_{A'_{\bar{1}}})(b)\} \\ &= \sup_{x=a+b}\{t_{A'_{\bar{0}}}(\varphi(a)) \wedge t_{A'_{\bar{1}}}(\varphi(b))\} \\ &= \sup_{\varphi(x)=\varphi(a)+\varphi(b)}\{t_{A'_{\bar{0}}}(\varphi(a)) \wedge t_{A'_{\bar{1}}}(\varphi(b))\} \\ &= t_{A'_{\bar{0}}+A'_{\bar{1}}}(\varphi(x)) = t_A(\varphi(x)) = t_{\varphi^{-1}(A)}(x). \end{aligned}$$

In a similar way we can verify the analogous property of false membership function. So $\varphi^{-1}(A) = \varphi^{-1}(A)_{\bar{0}} \oplus \varphi^{-1}(A)_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vague vector subspace of $\mathbb{L}$. Let $x, y \in \mathbb{L}$. Then
$t_{\varphi^{-1}(A)}([x,y]) = t_A(\varphi([x,y])) = t_A([\varphi(x),\varphi(y)]) \geq t_A(\varphi(x)) \wedge t_A(\varphi(y)) = t_{\varphi^{-1}(A)}(x) \wedge t_{\varphi^{-1}(A)}(y)$, and $f_{\varphi^{-1}(A)}([x,y]) = f_A(\varphi([x,y])) = f_A([\varphi(x),\varphi(y)]) \leq f_A(\varphi(x)) \vee f_A(\varphi(y)) = f_{\varphi^{-1}(A)}(x) \vee f_{\varphi^{-1}(A)}(y)$, thus $\varphi^{-1}(A)$ is a vague Lie sub-superalgebra. □

Proposition 3.10. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a Lie homomorphism. If $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of $\mathbb{L}$, then the vague fuzzy set $\varphi(A)$ is a vague Lie sub-superalgebra of $\mathbb{L}'$.*

Proof. Since $A = (t_A, f_A)$ is a vague Lie sub-superalgebra of $\mathbb{L}$, we have $A = A_{\bar{0}} \oplus A_{\bar{1}}$ where $A_{\bar{0}} = (t_{A_{\bar{0}}}, f_{A_{\bar{0}}}), A_{\bar{1}} = (t_{A_{\bar{1}}}, f_{A_{\bar{1}}})$ are vague vector subspaces of $\mathbb{L}_{\bar{0}}, \mathbb{L}_{\bar{1}}$, respectively. We define $\varphi(A)_{\bar{0}} = (t_{\varphi(A)_{\bar{0}}}, f_{\varphi(A)_{\bar{0}}})$ where $t_{\varphi(A)_{\bar{0}}} = \varphi(t_{A_{\bar{0}}})$, $f_{\varphi(A)_{\bar{0}}} = \varphi(f_{A_{\bar{0}}})$, $\varphi(A)_{\bar{1}} = (t_{\varphi(A)_{\bar{1}}}, f_{\varphi(A)_{\bar{1}}})$ where $t_{\varphi(A)_{\bar{1}}} = \varphi(t_{A_{\bar{1}}})$, $f_{\varphi(A)_{\bar{1}}} = \varphi(f_{A_{\bar{1}}})$. By Lemma 2.15, $\varphi(A)_{\bar{0}}$ and $\varphi(A)_{\bar{1}}$ are vague subspaces of $\mathbb{L}_{\bar{0}}, \mathbb{L}_{\bar{1}}$, respectively. And extend them to $\varphi(A)'_{\bar{0}}, \varphi(A)'_{\bar{1}}$, we define $\varphi(A)'_{\bar{0}} = (t_{\varphi(A)'_{\bar{0}}}, f_{\varphi(A)'_{\bar{0}}})$ where $t_{\varphi(A)'_{\bar{0}}} = \varphi(t_{A'_{\bar{0}}}), f_{\varphi(A)'_{\bar{0}}} = \varphi(f_{A'_{\bar{0}}})$ and $\varphi(A)'_{\bar{1}} = (t_{\varphi(A)'_{\bar{1}}}, f_{\varphi(A)'_{\bar{1}}})$ where $t_{\varphi(A)'_{\bar{1}}} = \varphi(t_{A'_{\bar{1}}}), f_{\varphi(A)'_{\bar{1}}} = \varphi(f_{A'_{\bar{1}}})$. Clearly,

$$t_{\varphi(A)'_{\bar{0}}}(x) = \begin{cases} t_{\varphi(A)_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 0 & x \notin \mathbb{L}_{\bar{0}} \end{cases}, f_{\varphi(A)'_{\bar{0}}}(x) = \begin{cases} f_{\varphi(A)_{\bar{0}}}(x) & x \in \mathbb{L}_{\bar{0}} \\ 1 & x \notin \mathbb{L}_{\bar{0}} \end{cases},$$

$$t_{\varphi(A)'_{\bar{1}}}(x) = \begin{cases} t_{\varphi(A)_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 0 & x \notin \mathbb{L}_{\bar{1}} \end{cases}, f_{\varphi(A)'_{\bar{1}}}(x) = \begin{cases} f_{\varphi(A)_{\bar{1}}}(x) & x \in \mathbb{L}_{\bar{1}} \\ 1 & x \notin \mathbb{L}_{\bar{1}} \end{cases}.$$

For $0 \neq x \in \mathbb{L}'$, then

$$\begin{aligned} t_{\varphi(A)'_{\bar{0}}}(x) \wedge t_{\varphi(A)'_{\bar{1}}}(x) &= \varphi(t_{A'_{\bar{0}}})(x) \wedge \varphi(t_{A'_{\bar{1}}})(x) \\ &= \sup_{x=\varphi(a)} \{t_{A'_{\bar{0}}}(a)\} \wedge \sup_{x=\varphi(a)} \{t_{A'_{\bar{1}}}(a)\} \\ &= \sup_{x=\varphi(a)} \{t_{A'_{\bar{0}}}(a) \wedge t_{A'_{\bar{1}}}(a)\} = 0. \end{aligned}$$

Let $y \in \mathbb{L}'$. We have

$$\begin{aligned} t_{\varphi(A)'_{\bar{0}}+\varphi(A)'_{\bar{1}}}(y) &= \sup_{y=a+b} \{t_{\varphi(A)'_{\bar{0}}}(a) \wedge t_{\varphi(A)'_{\bar{1}}}(b)\} \\ &= \sup_{y=a+b} \{\varphi(t_{A'_{\bar{0}}})(a) \wedge \varphi(t_{A'_{\bar{1}}})(b)\} \\ &= \sup_{y=a+b} \{ \sup_{a=\varphi(m)} \{t_{A'_{\bar{0}}}(m)\} \wedge \sup_{b=\varphi(n)} \{t_{A'_{\bar{1}}}(n)\}\} \\ &= \sup_{y=\varphi(x)} \{ \sup_{x=m+n} \{t_{A'_{\bar{0}}}(m) \wedge t_{A'_{\bar{1}}}(n)\}\} \\ &= \sup_{y=\varphi(x)} \{(t_{A'_{\bar{0}}+A'_{\bar{1}}})(x)\} = \sup_{y=\varphi(x)} \{t_A(x)\} = t_{\varphi(A)}(y). \end{aligned}$$

In a similar way we can verify the analogous properties of false membership function. So $\varphi(A) = \varphi(A)_{\bar{0}} \oplus \varphi(A)_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vague vector subspace.

Let $x, y \in \mathbb{L}'$. It is enough to show $t_{\varphi(A)}([x,y]) \geq t_{\varphi(A)}(x) \wedge t_{\varphi(A)}(y)$ and $f_{\varphi(A)}([x,y]) \leq f_{\varphi(A)}(x) \vee f_{\varphi(A)}(y)$. Suppose that $t_{\varphi(A)}([x,y]) < t_{\varphi(A)}(x) \wedge t_{\varphi(A)}(y)$, we have $t_{\varphi(A)}([x,y]) < t_{\varphi(A)}(x)$ and $t_{\varphi(A)}([x,y]) < t_{\varphi(A)}(y)$. We choose a number $t \in [0,1]$ such that $t_{\varphi(A)}([x,y]) < t < t_{\varphi(A)}(x)$ and $t_{\varphi(A)}([x,y]) < t < t_{\varphi(A)}(y)$. Then there exist $a \in \varphi^{-1}(x), b \in \varphi^{-1}(y)$ such that $t_A(a) > t, t_A(b) > t$. Since $\varphi([a,b]) = [\varphi(a), \varphi(b)] = [x,y]$, we have $t_{\varphi(A)}([x,y]) = \sup_{[x,y]=\varphi([a,b])} \{t_A([a,b])\} \geq t_A([a,b]) \geq t_A(a) \wedge t_A(b) > t > t_{\varphi(A)}([x,y])$. This is a contradiction. Suppose that $f_{\varphi(A)}([x,y]) > f_{\varphi(A)}(x) \vee f_{\varphi(A)}(y)$, we have $f_{\varphi(A)}([x,y]) > f_{\varphi(A)}(x)$ and $f_{\varphi(A)}([x,y]) > f_{\varphi(A)}(y)$. We choose a number $t \in [0,1]$ such that $\varphi(f)([x,y]) > t > f_{\varphi(A)}(x)$ and $f_{\varphi(A)}([x,y]) > t > f_{\varphi(A)}(y)$. Then there exist $a \in \varphi^{-1}(x), b \in \varphi^{-1}(y)$ such that $f_A(a) < t, f_A(b) < t$. Since $\varphi([a,b]) = [\varphi(a), \varphi(b)] = [x,y]$, we have $f_{\varphi(A)}([x,y]) = \inf_{[x,y]=\varphi([a,b])} \{f_A([a,b])\} \leq f_A([a,b]) \leq f_A(a) \vee f_A(b) < t < f_{\varphi(A)}([x,y])$. This is a contradiction. Therefore, $\varphi(A)$ is a vague Lie sub-superalgebra of $\mathbb{L}'$. □

Proposition 3.11. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a surjective Lie homomorphism. If $A = (t_A, f_A)$ is a vague ideal of $\mathbb{L}$, then the vague set $\varphi(A)$ is a vague ideal of $\mathbb{L}'$.*

Theorem 3.12. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a surjective Lie homomorphism. If $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague ideals of $\mathbb{L}$, then $\varphi(A + B) = \varphi(A) + \varphi(B)$.*

Definition 3.13. For any vague sets $A = (t_A, f_A)$ and $B = (t_B, f_B)$ of $\mathbb{L}$, we define the vague bracket product $[A, B] = (t_{[A,B]}, f_{[A,B]})$ where $t_{[A,B]}(x) = \sup\limits_{x=\sum_{i\in N} \alpha_i[x_i,y_i]} \{\min\limits_{i\in N}\{t_A(x_i) \wedge t_B(y_i)\}\}$ where $\alpha_i \in k$, $x_i, y_i \in \mathbb{L}$,
$t_{[A,B]}(x) = 0$ if x is not expressed as $x = \sum_{i\in N} \alpha_i[x_i, y_i]$
and $f_{[A,B]}(x) = \inf\limits_{x=\sum_{i\in N} \alpha_i[x_i,y_i]} \{\max\limits_{i\in N}\{f_A(x_i) \vee f_B(y_i)\}\}$ where $\alpha_i \in k$, $x_i, y_i \in \mathbb{L}$ $f_{[A,B]}(x) = 0$ if x is not expressed as $x = \sum_{i\in N} \alpha_i[x_i, y_i]$

Lemma 3.14. *Let $A_1 = (t_{A_1}, f_{A_1}), A_2 = (t_{A_2}, f_{A_2}), B_1 = (t_{B_1}, f_{B_1})$ and $B_2 = (t_{B_2}, f_{B_2})$ be vague sets of $\mathbb{L}$ such that $A_1 \subseteq A_2$, $B_1 \subseteq B_2$. Then $[A_1, B_1] \subseteq [A_2, B_2]$. In particular, if $A = (t_A, f_A), B = (t_B, f_B)$ are vague sets of $\mathbb{L}$, then $[A_1, B] \subseteq [A_2, B]$ and $[A, B_1] \subseteq [A, B_2]$.*

Theorem 3.15. *Let $A_1 = (t_{A_1}, f_{A_1}), A_2 = (t_{A_2}, f_{A_2}), B_1 = (t_{B_1}, f_{B_1}), B_2 = (t_{B_2}, f_{B_2})$ and $A = (t_A, f_A), B = (t_B, f_B)$ be any vague vector subspaces of $\mathbb{L}$. Then $[A_1+A_2, B] = [A_1, B]+[A_2, B]$ and $[A, B_1+B_2] = [A, B_1]+[A, B_2]$.*

Theorem 3.16. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague vector subspaces of $\mathbb{L}$. Then for any $\alpha, \beta \in k$, we have $[\alpha A, B] = \alpha[A, B]$ and $[A, \beta B] = \beta[A, B]$.*

The following theorem shows that the vague bracket product [,] remains bilinear.

Theorem 3.17. *Let $A_1 = (t_{A_1}, f_{A_1}), A_2 = (t_{A_2}, f_{A_2}), B_1 = (t_{B_1}, f_{B_1}), B_2 = (t_{B_2}, f_{B_2})$ and $A = (t_A, f_A), B = (t_B, f_B)$ be vague vector subspaces of $\mathbb{L}$. Then for any $\alpha, \beta \in k$, we have*

$$[\alpha A_1 + \beta A_2, B] = \alpha[A_1, B] + \beta[A_2, B]$$

$$[A, \alpha B_1 + \beta B_2] = \alpha[A, B_1] + \beta[A, B_2]$$

Lemma 3.18. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be any two vague vector subspaces of $\mathbb{L}$. Then $[A, B]$ is a vague vector subspace of $\mathbb{L}$.*

Lemma 3.19. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be any two $\mathbb{Z}_2$-graded vague vector subspaces of $\mathbb{L}$. Then*
$[A, B]_{\bar{0}} := [A_{\bar{0}}, B_{\bar{0}}] + [A_{\bar{1}}, B_{\bar{1}}]$ is an vague vector subspace of $\mathbb{L}_{\bar{0}}$,
$[A, B]_{\bar{1}} := [A_{\bar{0}}, B_{\bar{1}}] + [A_{\bar{1}}, B_{\bar{0}}]$ is a vague vector subspace of $\mathbb{L}_{\bar{1}}$ and
$[A, B]$ is a $\mathbb{Z}_2$-graded vague vector subspace of $\mathbb{L}$.

Lemma 3.20. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be any two $\mathbb{Z}_2$-graded vague vector subspaces of $\mathbb{L}$. Then $[A, B] = [B, A]$.*

Theorem 3.21. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be any two vague ideals of $\mathbb{L}$. Then $[A, B]$ is also a vague ideal of $\mathbb{L}$.*

Proof. Since $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague ideals of $\mathbb{L}$, we know that $[A, B]$ is a $\mathbb{Z}_2$-graded vague vector subspace by Lemma 3.20. In order to prove this theorem, we only remain to show that $t_{[A,B]}([x, y]) \geq t_{[A,B]}(x) \vee t_{[A,B]}(y)$ and $f_{[A,B]}([x, y]) \leq f_{[A,B]}(x) \wedge f_{[A,B]}(y)$.

Suppose that $t_{[A,B]}([x, y]) < t_{[A,B]}(x) \vee t_{[A,B]}(y)$, we have $t_{[A,B]}([x, y]) < t_{[A,B]}(x)$ or $t_{[A,B]}([x, y]) < t_{[A,B]}(y)$. Let $t_{[A,B]}([x, y]) < t_{[A,B]}(x)$. Choose a number $t \in [0, 1]$ such that $t_{[A,B]}([x, y]) < t < t_{[A,B]}(x)$, then there exist $x_i, y_i \in \mathbb{L}$ and $\alpha_i \in k$ such that $x = \sum_{i \in N} \alpha_i [x_i, y_i]$ and for all i, $t_A(x_i) > t, t_B(y_i) > t$. Moreover, $t_A(x_i) = (t_{A_{\bar{0}}+A_{\bar{1}}})(x_{i_{\bar{0}}} + x_{i_{\bar{1}}}) = t_{A_{\bar{0}}}(x_{i_{\bar{0}}}) \wedge t_{A_{\bar{1}}}(x_{i_{\bar{1}}}) > t$, then we have $t_{A_{\bar{0}}}(x_{i_{\bar{0}}}) > t, t_{A_{\bar{1}}}(x_{i_{\bar{1}}}) > t$, and $t_B(y_i) = (t_{B_{\bar{0}}+B_{\bar{1}}})(y_{i_{\bar{0}}} + y_{i_{\bar{1}}}) = t_{B_{\bar{0}}}(y_{i_{\bar{0}}}) \wedge t_{B_{\bar{1}}}(y_{i_{\bar{1}}}) > t$, then $t_{B_{\bar{0}}}(y_{i_{\bar{0}}}) > t$, $t_{B_{\bar{1}}}(y_{i_{\bar{1}}}) > t$.

Because $[x, y] = [\sum_{i \in N} \alpha_i [x_i, y_i], y] = \sum_{i \in N} \alpha_i [[x_i, y_i], y]$, and

$$
\begin{aligned}
[[x_i, y_i], y] &= [[x_{i_{\bar{0}}} + x_{i_{\bar{1}}}, y_{i_{\bar{0}}} + y_{i_{\bar{1}}}], y] \\
&= [[x_{i_{\bar{0}}}, y_{i_{\bar{0}}}] + [x_{i_{\bar{1}}}, y_{i_{\bar{1}}}] + [x_{i_{\bar{0}}}, y_{i_{\bar{1}}}] + [x_{i_{\bar{1}}}, y_{i_{\bar{0}}}], y] \\
&= [[x_{i_{\bar{0}}}, y_{i_{\bar{0}}}], y] + [[x_{i_{\bar{1}}}, y_{i_{\bar{1}}}], y] + [[x_{i_{\bar{0}}}, y_{i_{\bar{1}}}], y] + [[x_{i_{\bar{1}}}, y_{i_{\bar{0}}}], y] \\
&= [x_{i_{\bar{0}}}, [y_{i_{\bar{0}}}, y]] - [y_{i_{\bar{0}}}, [x_{i_{\bar{0}}}, y]] + [x_{i_{\bar{1}}}, [y_{i_{\bar{1}}}, y]] + [y_{i_{\bar{1}}}, [x_{i_{\bar{1}}}, y]] \\
&\quad + [x_{i_{\bar{0}}}, [y_{i_{\bar{1}}}, y]] - [y_{i_{\bar{1}}}, [x_{i_{\bar{0}}}, y]] + [x_{i_{\bar{1}}}, [y_{i_{\bar{0}}}, y]] - [y_{i_{\bar{0}}}, [x_{i_{\bar{1}}}, y]] \\
&= [x_i, [y_i, y]] - [y_{i_{\bar{0}}}, [x_{i_{\bar{0}}}, y]] + [y_{i_{\bar{1}}}, [x_{i_{\bar{1}}}, y]] - [y_{i_{\bar{1}}} \\
&, \quad [x_{i_{\bar{0}}}, y]] - [y_{i_{\bar{0}}}, [x_{i_{\bar{1}}}, y]],
\end{aligned}
$$

we get

$$
t_{[A,B]}([x, y]) = t_{[A,B]}(\sum_{i \in N} \alpha_i [[x_i, y_i], y]) \geq t_{[A,B]}([[x_i, y_i], y])
$$

$$
\geq \min \begin{cases}
t_{[A,B]}([x_i, [y_i, y]]) & (1) \\
t_{[A,B]}(-[y_{i_{\bar{0}}}, [x_{i_{\bar{1}}}, y]]) & (2) \\
t_{[A,B]}([y_{i_{\bar{1}}}, [x_{i_{\bar{1}}}, y]]) & (3) \\
t_{[A,B]}(-[y_{i_{\bar{1}}}, [x_{i_{\bar{0}}}, y]]) & (4) \\
t_{[A,B]}(-[y_{i_{\bar{0}}}, [x_{i_{\bar{1}}}, y]]) & (5)
\end{cases}
$$

If (1) is minimum, then we have

$$
\begin{aligned}
t_{[A,B]}([x_i, [y_i, y]]) &\geq t_A(x_i) \wedge t_B([y_i, y]) \\
&\geq t_A(x_i) \wedge (t_B(y_i) \vee t_B(y)) > t;
\end{aligned}
$$

If (2) is minimum, then we have

$$\begin{aligned} t_{[A,B]}(-[y_{i_0},[x_{i_0},y]]) &= t_{[A,B]}([[x_{i_0},y],y_{i_0}]) \\ &\geq t_A([x_{i_0},y]) \wedge t_B(y_{i_0}) \\ &\geq (t_A(x_{i_0}) \vee t_A(y)) \wedge t_B(y_{i_0}) \\ &= (t_{A_0}(x_{i_0}) \vee t_A(y)) \wedge t_{B_0}(y_{i_0}) > t; \end{aligned}$$

If (3) is minimum, then by Lemma 3.20 we have

$$\begin{aligned} t_{[A,B]}([y_{i_1},[x_{i_1},y]]) &= t_{[B,A]}([y_{i_1},[x_{i_1},y]]) \\ &\geq t_B(y_{i_1}) \wedge t_A([x_{i_1},y]) \\ &\geq t_B(y_{i_1}) \wedge (t_A(x_{i_1}) \vee t_A(y)) \\ &= t_{B_1}(y_{i_1}) \wedge (t_{A_1}(x_{i_1}) \vee t_A(y)) > t; \end{aligned}$$

If (4) is minimum, then we have

$$\begin{aligned} t_{[A,B]}(-[y_{i_1},[x_{i_0},y]]) &\geq t_{[A,B]}([y_{i_1},[x_{i_0},y]]) \\ &= t_{[B,A]}([y_{i_1},[x_{i_0},y]]) \\ &\geq t_B(y_{i_1}) \wedge (t_A(x_{i_0}) \vee t_A(y)) \\ &= t_{B_1}(y_{i_1}) \wedge (t_{A_0}(x_{i_0}) \vee t_A(y)) > t; \end{aligned}$$

If (5) is minimum, then the case is similar to (2), we can also get

$$t_{[A,B]}(-[y_{i_0},[x_{i_1},y]]) > t.$$

So we have $t_{[A,B]}([x,y]) > t > t_{[A,B]}([x,y])$, this is a contradiction. We use the similar method to prove the case of $t_{[A,B]}([x,y]) < t_{[A,B]}(y)$.

Also,

$$f_{[A,B]}([x,y]) = f_{[A,B]}(\sum_{i\in N} \alpha_i[[x_i,y_i],y]) \leq f_{[A,B]}([[x_i,y_i],y])$$

$$\leq \max \begin{cases} f_{[A,B]}([x_i,[y_i,y]]) & (1') \\ f_{[A,B]}(-[y_{i_0},[x_{i_1},y]]) & (2') \\ f_{[A,B]}([y_{i_1},[x_{i_1},y]]) & (3') \\ f_{[A,B]}(-[y_{i_1},[x_{i_0},y]]) & (4') \\ f_{[A,B]}(-[y_{i_0},[x_{i_1},y]]) & (5') \end{cases}$$

If (1') is maximum, then we have

$$\begin{aligned} f_{[A,B]}([x_i,[y_i,y]]) &\le f_A(x_i) \vee f_B([y_i,y]) \\ &\le f_A(x_i) \vee (f_B(y_i) \wedge f_B(y)) < t; \end{aligned}$$

If (2') is maximum, then we have

$$\begin{aligned} f_{[A,B]}(-[y_{i_0},[x_{i_0},y]]) &= f_{[A,B]}([[x_{i_0},y],y_{i_0}]) \\ &\le f_A([x_{i_0},y]) \vee f_B(y_{i_0}) \\ &\le (f_A(x_{i_0}) \wedge f_A(y)) \vee f_B(y_{i_0}) \\ &= (f_{A_0}(x_{i_0}) \wedge f_A(y)) \vee f_{B_0}(y_{i_0}) < t; \end{aligned}$$

If (3') is maximum, then by Lemma 3.20 we have

$$\begin{aligned} f_{[A,B]}([y_{i_1},[x_{i_1},y]]) &= f_{[B,A]}([y_{i_1},[x_{i_1},y]]) \\ &\le f_B(y_{i_1}) \vee f_A([x_{i_1},y]) \\ &\le f_B(y_{i_1}) \vee (f_A(x_{i_1}) \wedge f_A(y)) \\ &= f_{B_1}(y_{i_1}) \vee (f_{A_1}(x_{i_1}) \wedge f_A(y)) < t; \end{aligned}$$

If (4') is maximum, then we have

$$\begin{aligned} f_{[A,B]}(-[y_{i_1},[x_{i_0},y]]) &\le f_{[A,B]}([y_{i_1},[x_{i_0},y]]) \\ &= f_{[B,A]}([y_{i_1},[x_{i_0},y]]) \\ &\le f_B(y_{i_1}) \vee (f_A(x_{i_0}) \wedge f_A(y)) \\ &= f_{B_1}(y_{i_1}) \vee (f_{A_0}(x_{i_0}) \wedge f_A(y)) < t; \end{aligned}$$

If (5') is maximum, then the case is similar to (2'), we can also get

$$f_{[A,B]}(-[y_{i_0},[x_{i_1},y]]) < t.$$

So we have $f_{[A,B]}([x,y]) < t < f_{[A,B]}([x,y])$, this is a contradiction. We use the similar method to prove the case of $f_{[A,B]}([x,y]) > f_{[A,B]}(y)$. Hence $[A,B]$ is a vague ideal of $\mathbb{L}$. □

4 Solvable and nilpotent vague Lie ideals

Definition 4.1. Let $A = (t_A, f_A)$ be a vague Lie ideal of $\mathbb{L}$. Define inductively a sequence of vague ideals of $\mathbb{L}$ by $A^{(0)} = A$, $A^{(1)} = [A^{(0)}, A^{(0)}]$,

$A^{(2)} = [A^{(1)}, A^{(1)}], \cdots, A^{(n)} = [A^{(n-1)}, A^{(n-1)}]$, then $A^{(n)}$ is called the nth derived vague Lie ideal of $\mathbb{L}$. In which, $A^{(i+1)} = (t_{A^{(i+1)}}, f_{A^{(i+1)}})$ where

$$t_{A^{(i+1)}}(x) = \sup_{x=\sum\limits_{j\in N}\alpha_j[x_j,y_j]} \{\min_{j\in N}\{t_{A^{(i)}}(x_j) \wedge t_{A^{(i)}}(y_j)\}\}$$

where $\alpha_j \in k$, $x_j, y_j \in \mathbb{L}$

$t_{A^{(i+1)}}(x) = 0$ if x is not expressed as $x = \sum\limits_{j\in N} \alpha_j[x_j, y_j]$,

$$f_{A^{(i+1)}}(x) = \inf_{x=\sum\limits_{j\in N}\alpha_j[x_j,y_j]} \{\max_{j\in N}\{f_{A^{(i)}}(x_j) \vee f_{A^{(i)}}(y_j)\}\}$$

where $\alpha_j \in k$, $x_j, y_j \in \mathbb{L}$

$f_{A^{(i+1)}}(x) = 1$ if x is not expressed as $x = \sum\limits_{j\in N} \alpha_j[x_j, y_j]$

From the definition, we can get $t_{A^{(0)}} \supseteq t_{A^{(1)}} \supseteq t_{A^{(2)}} \supseteq \cdots \supseteq t_{A^{(n)}} \supseteq \cdots$ and $f_{A^{(0)}} \subseteq f_{A^{(1)}} \subseteq f_{A^{(2)}} \subseteq \cdots \subseteq f_{A^{(n)}} \subseteq \cdots$.

Definition 4.2. Let $A^{(n)}$ be as above. Define: $\eta^{(n)} = \sup\{t_{A^{(n)}}(x) : 0 \neq x \in \mathbb{L}\}$ and $\kappa^{(n)} = \inf\{f_{A^{(n)}}(x) : 0 \neq x \in \mathbb{L}\}$. Then it is clear that $\eta^{(0)} \geq \eta^{(1)} \geq \eta^{(2)} \geq \cdots \geq \eta^{(n)} \geq \cdots$ and $\kappa^{(0)} \leq \kappa^{(1)} \leq \kappa^{(1)} \leq \cdots \leq \kappa^{(n)} \leq \cdots$.

Definition 4.3. A vague Lie ideal $A = (t_A, f_A)$ of $\mathbb{L}$ is called a solvable vague Lie ideal, if there is a positive integer n such that $\eta^{(n)} = 0$ and $\kappa^{(n)} = 1$. So it is a solvable vague Lie ideal, then there is positive integer n such that $t_{A^{(n)}} = 1_0$ and $f_{A^{(n)}} = 1_0^c$.

Lemma 4.4. *Let $A = (t_A, f_A)$ be a vague Lie ideal of $\mathbb{L}$. Then $A = (t_A, f_A)$ is a solvable vague ideal if and only if there is a positive integer n such that $t_{A^{(m)}} = 1_0, f_{A^{(m)}} = 1_0^c$ for all $m \geq n$.*

Theorem 4.5. *Homomorphic images of solvable vague Lie ideals are also solvable vague Lie ideals.*

Proof. Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a homomorphism of Lie superalgebra and assume that $A = (t_A, f_A)$ is a vague Lie ideal of $\mathbb{L}$. Let $\varphi(A) = B$, i.e, $t_B = t_{\varphi(A)}, f_B = f_{\varphi(A)}$. We prove $t_{\varphi(A^{(n)})} = t_{B^{(n)}}$ and $f_{\varphi(A^{(n)})} = f_{B^{(n)}}$ by induction on n, where n is any positive integer. Indeed, let $y \in \mathbb{L}'$. Consider

$n = 1$,

$$\begin{aligned} t_{\varphi(A^{(1)})}(y) &= t_{\varphi([A,A])}(y) = \sup_{y=\varphi(x)} \{t_{[A,A]}(x)\} \\ &= \sup_{y=\varphi(x)} \{ \sup_{x=\sum_{i\in N} \alpha_i[x_i,y_i]} \{\min_{i\in N}(t_A(x_i) \wedge t_A(y_i))\}\} \\ &= \sup_{y=\sum_{i\in N} \alpha_i\varphi[x_i,y_i]} \{\min_{i\in N}(t_A(x_i) \wedge t_A(y_i))\} \\ &= \sup_{y=\sum_{i\in N} \alpha_i[a_i,b_i]} \{\min_{i\in N}(t_A(x_i) \wedge t_A(y_i)) : \varphi(x_i) = a_i \\ &\quad , \ \varphi(y_i) = b_i\} \\ &= \sup_{\sum_{i\in N} \alpha_i[a_i,b_i]=y} \{\min_{i\in N}(t_B(a_i) \wedge t_B(b_i))\} \\ &= t_{[B,B]}(y) = t_{B^{(1)}}(y). \end{aligned}$$

In a similar way we can verify the analogous property of false membership function. These prove the case of $n = 1$. Suppose that the case of $n-1$ is true, then $t_{\varphi(A^{(n)})} = t_{\varphi([A^{(n-1)},A^{(n-1)}])} = t_{[\varphi(A^{(n-1)}),\varphi(A^{(n-1)})]}$ $= t_{[B^{(n-1)},B^{(n-1)}]} = t_{B^{(n)}}$ and $f_{\varphi(A^{(n)})} = f_{\varphi([A^{(n-1)},A^{(n-1)}])}$ $= f_{[\varphi(A^{(n-1)}),\varphi(A^{(n-1)})]} = f_{[B^{(n-1)},B^{(n-1)}]} = f_{B^{(n)}}$. Let m be a positive integer such that $t_{A^{(m)}} = 1_0$ and $f_{A^{(m)}} = 1_0^c$. Then for any $0 \neq y \in \mathbb{L}'$, we get $t_{B^{(m)}}(y) = t_{\varphi(A^{(m)})}(y) = \sup_{y=\varphi(x)} \{1_0(x)\} = 0$, $f_{B^{(m)}}(y) = \varphi(f_{A^{(m)}})(y) = \inf_{y=\varphi(x)} \{1_0^c(x)\} = 1$. So $t_{B^{(m)}} = 1_0$ and $f_{B^{(m)}} = 1_0^c$. □

Theorem 4.6. *Let $A = (t_A, f_A)$ be an vague Lie ideal of $\mathbb{L}$ and A/I be a solvable vague Lie ideal of $\mathbb{L}/I$. If $B = (t_B, f_B)$ is a solvable vague Lie ideal of $\mathbb{L}$ and is also a vague ideal of $A = (t_A, f_A)$ such that $B(I) = A(I)$, then $A = (t_A, f_A)$ is solvable.*

Proof. Let φ be the canonical projection from $\mathbb{L}$ to $\mathbb{L}/I$. From the proof of 4.5, we can get $t_{\varphi(A^{(n)})} = t_{(A/I)^{(n)}}$ and $f_{\varphi(A^{(n)})} = f_{(A/I)^{(n)}}$. Since A/I is solvable, there exists n such that $t_{(A/I)^{(n)}} = 1_0$ and $f_{(A/I)^{(n)}} = 1_0^c$.

For $0 \neq \bar{y} \in \mathbb{L}/I$, we have $\sup_{m\in\varphi^{-1}(\bar{y})} \{t_{A^{(n)}}(m)\} = t_{\varphi(A^{(n)})}(\bar{y}) = t_{(A/I)^{(n)}}(\bar{y}) =$ 0 and $\inf_{m\in\varphi^{-1}(\bar{y})} \{f_{A^{(n)}}(m)\} = f_{\varphi(A^{(n)})}(\bar{y}) = f_{(A/I)^{(n)}}(\bar{y}) = 1$. Notice that $m \in \mathbb{L}$ and $m \neq 0$, we get $t_{A^{(n)}}(m) = 0$ and $f_{A^{(n)}}(m) = 1$.

For $\bar{y} = 0$, we have $\sup_{m\in\varphi^{-1}(0)} \{t_{A^{(n)}}(m)\} = t_{\varphi(A^{(n)})}(0) = 1$ and

$\inf_{m\in\varphi^{-1}(0)}\{f_{A^{(n)}}(m)\} = f_{\varphi(A^{(n)})}(0) = 0$. Since $\varphi^{-1}(0) = I$ and $B(I) = A(I)$, we have $t_{B^{(n)}}(I) = t_{A^{(n)}}(I)$ and $f_{B^{(n)}}(I) = f_{A^{(n)}}(I)$. For any $x \in I$, B is solvable, then there exists n such that $t_{B^{(n)}} = 1_0$ and $f_{B^{(n)}} = 1_0^c$, we have $t_{A^{(n)}} = 1_0$ and $f_{A^{(n)}} = 1_0^c$.

Hence for any $x \in \mathbb{L}$, we always have that $t_{A^{(n)}} = 1_0$ and $f_{A^{(n)}} = 1_0^c$, which imply that $A = (t_A, f_A)$ is solvable. □

Lemma 4.7. *Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague Lie ideals of $\mathbb{L}$. Then $(A \oplus B)^{(n)} = A^{(n)} \oplus B^{(n)}$.*

Theorem 4.8. *Direct sum of any solvable vague Lie ideals is also a solvable vague Lie ideal.*

Definition 4.9. Let $A = (t_A, f_A)$ be a vague Lie ideal of $\mathbb{L}$. Define inductively a sequence of vague Lie ideals of $\mathbb{L}$ by $A^0 = A$, $A^1 = [A, A^0]$, $A^2 = [A, A^1], \cdots, A^n = [A, A^{n-1}]\cdots$, which is called the descending central series of a vague Lie ideal $A = (t_A, f_A)$ of $\mathbb{L}$. We get $t_{A^0} \supseteq t_{A^1} \supseteq t_{A^2} \supseteq \cdots \supseteq t_{A^n} \supseteq \cdots$ and $f_{A^0} \subseteq f_{A^1} \subseteq f_{A^2} \subseteq \cdots \subseteq f_{A^n} \subseteq \cdots$,

Example 4.10. Let us take the basis h, e, f of $\mathfrak{sl}(1|1)$ as follows

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Then h is an even element, and e and f are odd element. Their bracket products are as follows: $[e, f] = [f, e] = h$, the other brackets $= 0$. Then $\mathfrak{sl}(1|1)$ is a three-dimensional Lie superalgebra.
Define $A_{\bar{0}} = (t_{A_{\bar{0}}}, f_{A_{\bar{0}}}) : \mathfrak{sl}(1|1)_{\bar{0}} \to [0, 1]$ where

$$t_{A_{\bar{0}}}(x) = \begin{cases} 0.6 & x = h \\ 1 & \text{otherwise} \end{cases}, \quad f_{A_{\bar{0}}}(x) = \begin{cases} 0.4 & x = h \\ 0 & \text{otherwise} \end{cases}$$

Define $A_{\bar{1}} = (t_{A_{\bar{1}}}, f_{A_{\bar{1}}}) : \mathfrak{sl}(1|1)_{\bar{1}} \to [0, 1]$ where

$$t_{A_{\bar{1}}}(x) = \begin{cases} 0.3 & x = e \\ 0.5 & x = f \\ 1 & \text{otherwise} \end{cases}, \quad f_{A_{\bar{1}}}(x) = \begin{cases} 0.7 & x = e \\ 0.5 & x = f \\ 0 & \text{otherwise} \end{cases}$$

Define $A = (t_A, f_A) : \mathfrak{sl}(1|1) \to [0, 1]$ where $t_A(x) = t_{A_{\bar{0}}}(x_{\bar{0}}) \wedge t_{A_{\bar{1}}}(x_{\bar{1}})$ and $f_A(x) = f_{A_{\bar{0}}}(x_{\bar{0}}) \vee f_{A_{\bar{1}}}(x_{\bar{1}})$. Then A is a vague Lie ideal of $\mathfrak{sl}(1|1)$.

Let $A^0 = A$. We define $A^1 = [A, A^0]$, then if $x \in \mathfrak{sl}(1|1)_{\bar{1}}$, x can not be expressed as $x = \sum \alpha_i [x_i, y_i]$, $x_i, y_i \in \mathfrak{sl}(1|1)$ then $t_{A^1}(x) = 0, f_{A^1}(x) = 1$. If $x \in \mathfrak{sl}(1|1)_{\bar{0}}$, $x = \alpha[e, f]$, $\alpha \in k$, then $t_{A^1}(x) = \sup\{t_A(e) \wedge t_{A^0}(f)\} = 0.3$ and $f_{A^1}(x) = \inf\{f_A(e) \vee f_{A^0}(f)\} = 0.7$.

Define $A^2 = [A, A^1]$, we calculate if $x \in \mathfrak{sl}(1|1)_{\bar{1}}$, $t_{A^2}(x) = 0, f_{A^2}(x) = 1$. If $x \in \mathfrak{sl}(1|1)_{\bar{0}}$, $t_{A^2}(x) = \sup\{t_A(e) \wedge t_{A^1}(f)\} = 0$ and $f_{A^1}(x) = \inf\{f_A(e) \vee f_{A^1}(f)\} = 1$. Then we get $\eta^0 \geq \eta^1 \geq \eta^2 = 0$ and $\kappa^0 \leq \kappa^1 \leq \kappa^2 = 1$. So A is a nilpotent vague Lie ideal of $\mathfrak{sl}(1|1)$.

Definition 4.11. For any vague Lie ideal $A = (t_A, f_A)$, define $\eta^n = \sup\{t_{A^n}(x) : 0 \neq x \in \mathbb{L}\}$ and $\kappa^n = \inf\{f_{A^n}(x) : 0 \neq x \in \mathbb{L}\}$, for any positive integer n. The vague Lie ideal is called a nilpotent vague Lie ideal, if there is a positive integer m such that $\eta^m = 0$ and $\kappa^m = 1$, or equivalently, $t_{A^m} = 1_0$ and $f_{A^m} = 1_0^c$.

We state the following Theorems without their proofs.

Theorem 4.12. *Homomorphic images of nilpotent vague Lie ideals are also nilpotent vague Lie ideals. Direct sum of nilpotent vague Lie ideals is also a nilpotent vague Lie ideal.*

Theorem 4.13. *If $A = (t_A, f_A)$ is a nilpotent vague Lie ideal of $\mathbb{L}$, then it is solvable.*

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References

[1] M. Akram, *Co-fuzzy Lie superalgebras over a co-fuzzy field*, World Applied Sciences Journal, **7**(2009) 25-32.

[2] M. Akram and N. O. Alshehri, *Bipolar fuzzy Lie ideals*, Utilitas Mathematica **87** (2012)265-278.

[3] M. Akram, *Generalized fuzzy Lie subalgebras*, Journal of Generalized Lie Theory and Applications **2**(4) (2008)261-268.

[4] M. Akram and K.P. Shum, *Intuitionistic fuzzy Lie algebras*, Southeast Asian Bulletin of Mathematics **31**(2007) 843-855.

[5] M. Akram and N.O. Al-Shehrie, *Fuzzy K-ideals of K-algebras*, Ars Combinatoria **99**(2011)399-413.

[6] M. Akram and K.H. Dar, *Generalized fuzzy K-algebras*, VDM Verlag, 2010.

[7] K.T. Atanassov, *Intuitionistic fuzzy sets: Theory and applications, Studies in fuzziness and soft computing*, Heidelberg, New York, Physica-Verl., 1999.

[8] R. Biswas, *Vague groups*, Internat. J. Comput. Cognition 4(2) (2006) 20-23.

[9] I. Bloch, *Dilation and erosion of spatial vague sets*, Lecture Notes in Artificial Intelligence (2007) 385-393.

[10] I. Bloch, *Geometry of spatial vague sets based on vague numbers and mathematical morphology*, Fuzzy Logic and Applications, Lecture Notes in Computer Science **5571** (2009) 237-245.

[11] W.J. Chen, *Fuzzy quotient Lie superalgebras*, J. Shandong Univ., Nat. Sci. **43**(2008)25-27.

[12] W.J. Chen, *Intuitionistic fuzzy quotient Lie superalgebras*, International Journal of Fuzzy Systems, **12**(4)(2010)330-339.

[13] W.J. Chen and M. Akram, *Interval-valued fuzzy structures on Lie superalgebras*, The Journal of Fuzzy Mathematics **19**(4)(2011)951-968.

[14] W.J. Chen and S. H. Zhang, *Intuitionistic fuzzy Lie sub-superalgebras and intuitionistic fuzzy ideals*, Computers and Mathematics with Applications, **58**(2009) 1645-1661.

[15] L. Corwin, Y. NéEman, and S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)*, Reviews of Modern Physics **47**(1975)573-603.

[16] W.L. Gau and D.J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics **23** (1993) 610-614.

[17] V.G. Kac, *Lie superalgebras*, Advances in Mathematics **26**(1977)8-96.

[18] S.E. Yehia, *The adjoint representation of fuzzy Lie algebras*, Fuzzy Sets and Systems, **119**(2001)409-417.

[19] L.A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965) 338-353.

[20] L.A. Zadeh, *Similarity relations and fuzzy orderings*, Inform. Sci. **3** (1971) 177-200.