ON THE DYNAMICS OF A RECURSIVE SEQUENCE

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ABSTRACT

We describe the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{ax_{n-p}}{b + c\prod_{i=0}^{k} x_{n-(2i+1)}}, \quad n = 0, 1, ...,$$

where $k, p \in \mathbb{N}$, parameters a, b, c and initial conditions are nonnegative real numbers.

Keywords: Difference Equation, Globally Asymptotically, Periodicity.

1. INTRODUCTION

The study of the nonlinear rational difference equations is quite challenging and interesting. So, many researchers have studied the behavior of the solution of rational difference equations. For example see Refs. [1-19].

Schinas et al. [5] studied the boundedness, the persistence, the attractivity and stability of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^q}.$$

Hamza et al. [2] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}.$$

We [14] studied the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1} = \frac{Ax_{n-m}}{B + C\prod_{i=0}^{2k+1} x_{n-i}}.$$

Elsayed [9] investigated the qualitative behavior of the solution of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}.$$

Elabbasy et al. [7] investigated some qualitative behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Our aim in this paper is to investigate the dynamics of the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-p}}{b + c\prod_{i=0}^{k} x_{n-(2i+1)}}, \quad n = 0, 1, ... \tag{1.1}$$

where $k, p \in \mathbb{N}$, parameters a, b, c and initial conditions are nonnegative real numbers. Also we obtained some results of some special cases of Eq.(1.1).

2. Preliminaries

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-(k+1)}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. *An equilibrium point for Eq.*(2.1) *is a point* $\overline{x} \in I$ *such that* $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$.

Definition 2. *A sequence* $\{x_n\}_{n=-k}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -k$.

Definition 3. *(i) The equilibrium point* $\overline{x}$ *of Eq.*(2.1) *is locally stable if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ *with* $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$, *we have* $|x_n - \overline{x}| < \varepsilon$ *for all* $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ *of Eq.*(2.1) *is locally asymptotically stable if* $\overline{x}$ *is locally stable solution of Eq.(2.1) and there exists* $\gamma > 0$, *such that for*

all $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$ *with* $|x_{-k} - \overline{x}| + |x_{-(k-1)} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$, *we have* $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ *of Eq.*(2.1) *is global attractor if for all* $x_{-k}, x_{-(k-1)}, ..., x_0 \in I$, *we have* $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ *of Eq.*(2.1) *is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq.*(2.1).

(v) The equilibrium point $\overline{x}$ *of Eq.*(2.1) *is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation associated with Eq.(2.2) *is*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x})\, y_{n-i}, \quad n = 0, 1, ... \tag{2.2}$$

The characteristic equation associated with Eq.(2.2) *is*

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, ..., \overline{x})\, \lambda^{k-i} = 0. \tag{2.3}$$

Theorem 1. [19] *Assume that* f *is a* C^1 *function and let* $\overline{x}$ *be an equilibrium point of Eq.*(2.1). *Then the following statements are true.*

(i) If all roots of Eq.(2.3) *lie in open disk* $|\lambda| < 1$, *then* $\overline{x}$ *is locally asymptotically stable.*

(ii) If at least one root of Eq.(2.3) *has absolute value greater than one, then* $\overline{x}$ *is unstable.*

3. Dynamics of Eq.(1.1)

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters are nonnegative real numbers, the initial conditions are nonnegative real numbers and p, k are nonnegative integers.

The change of variables $x_n = \left(\frac{b}{c}\right)^{\frac{1}{k+1}} y_n$ reduces Eq.(1.1) to the difference equation

$$y_{n+1} = \frac{\gamma y_{n-p}}{1 + \prod_{i=0}^{k} y_{n-(2i+1)}}, \quad n = 0, 1, ... \tag{3.1}$$

where $\gamma = \frac{a}{b} > 0$. We can see that $\overline{y}_1 = 0$ is always an equilibrium point of Eq.(3.1). When $\gamma > 1$, Eq.(3.1) also possesses the unique positive equilibrium $\overline{y}_2 = (\gamma - 1)^{\frac{1}{k+1}}$.

Theorem 2. *The following statements are true:*

(i) If $\gamma < 1$, *then the equilibrium point* $\overline{y}_1 = 0$ *of Eq.*(3.1) *is locally asymptotically stable,*

(ii) If $\gamma > 1$, then the equilibrium points $\overline{y}_1 = 0$ and $\overline{y}_2 = (\gamma - 1)^{\frac{1}{k+1}}$ are unstable.

Proof. The linearized equation associated with Eq.(3.1) about $\overline{y}$ is

$$z_{n+1} + \frac{\gamma \overline{y}^{k+1}}{\left(1+\overline{y}^{k+1}\right)^2}\left(\sum_{i=0}^{k} z_{n-(2i+1)} - z_{n-p}\right) - \frac{\gamma}{\left(1+\overline{y}^{k+1}\right)^2} z_{n-p} = 0.$$

The characteristic equation associated with this equation is

$$\lambda^{2k+2} + \gamma \frac{\overline{y}^{k+1}}{\left(1+\overline{y}^{k+1}\right)^2}\left(\sum_{i=0}^{k} \lambda^{2i} - \lambda^{2k+1-p}\right) - \gamma \frac{1}{\left(1+\overline{y}^{k+1}\right)^2} \lambda^{2k+1-p} = 0.$$

Then the linearized equation of Eq.(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} - \gamma z_{n-p} = 0, \quad n = 0, 1, \ldots$$

The characteristic equation of Eq.(3.1) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{2k+2} - \gamma \lambda^{2k+1-p} = 0.$$

So

$\lambda = 0$ and $\lambda = \sqrt[p+1]{\gamma}$.In view of Theorem 1:

If $\gamma < 1$, then $|\lambda| < 1$ for all roots and the equilibrium point $\overline{y}_1 = 0$ is locally asymptotically stable.

If $\gamma > 1$, it follows that the equilibrium point $\overline{y}_1 = 0$ is unstable.

The linearized equation of E q.(3.1) about the equilibrium point $\overline{y}_2 = (\gamma - 1)^{\frac{1}{k+1}}$ becomes

$$z_{n+1} + \left(1 - \frac{1}{\gamma}\right)\left(\sum_{i=0}^{k} z_{n-(2i+1)} - z_{n-p}\right) - \frac{1}{\gamma} z_{n-p} = 0, \quad n = 0, 1, \ldots$$

The characteristic equation of E q.(3.1) about the equilibrium point $\overline{y}_2 = (\gamma - 1)^{\frac{1}{k+1}}$ is

$$\lambda^{2k+2} + \left(1 - \frac{1}{\gamma}\right)\left(\sum_{i=0}^{k} \lambda^{2i} - \lambda^{2k+1-p}\right) - \frac{1}{\gamma} \lambda^{2k+1-p} = 0.$$

It is clear that this equation has a root in the interval $(-\infty, -1)$. Then the equilibrium point $\overline{y}_2 = (\gamma - 1)^{\frac{1}{k+1}}$ is unstable.

Theorem 3. *Assume that $\gamma < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is globally asymptotically stable.*

□

Proof. Let $\{y_n\}_{n=-(2k+1)}^{\infty}$ be a solution of Eq.(3.1). From Theorem 2 we know that the equilibrium point $\overline{y}_1 = 0$ of Eq.(3.1) is locally asymptotically stable. So it is sufficed to show that

$$\lim_{n\to\infty} y_n = 0.$$

Since

$$y_{n+1} = \frac{\gamma y_{n-p}}{1 + \prod_{i=0}^{k} y_{n-(2i+1)}} \leq \gamma y_{n-p}$$

We obtain

$$y_{n+1} \leq \gamma y_{n-p}.$$

Then it can be written for $s = 0, 1, ...$

$$y_{s(p+1)+1} \leq \gamma^{s+1} y_{-p},$$

$$y_{s(p+1)+2} \leq \gamma^{s+1} y_{-(p-1)},$$

$$...$$

$$y_{s(p+1)+p+1} \leq \gamma^{s+1} y_0.$$

If $\gamma < 1$, then $\lim_{l\to\infty} \gamma^{s+1} = 0$
and

$$\lim_{n\to\infty} y_n = 0.$$

The proof is complete. □

Corollary 1. *Assume that $\gamma = 1$. Then every solution of Eq.(3.1) is bounded.*

Proof. Let $\{y_n\}_{n=-(2k+1)}^{\infty}$ be a solution of *Eq.*(3.1). It follows from *Eq.*(3.1) that

$$y_{n+1} = \frac{y_{n-p}}{1 + \prod_{i=0}^{k} y_{n-(2i+1)}} \leq y_{n-p}.$$

Then in view of the proof of Theorem 3, we have for $s = 0, 1, ...$

$$y_{s(p+1)+1} \leq y_{-p},$$

$$y_{s(p+1)+2} \leq y_{-(p-1)},$$

$$...$$

$$y_{s(p+1)+p+1} \leq y_0.$$

So every solution of *Eq.*(3.1) is bounded from above by
$A = \max\left\{y_{-p}, y_{-(p-1)}, ..., y_0, \right\}$. □

Corollary 2. *Assume that any two consecutive initial conditions y_{-i} ($i = 0, 1, ..., p$) of Eq.(3.1) are zero, then the following statements are true:*

(i) If $\gamma > 1$, then every solution of $Eq.(3.1)$ is unbounded except zero.
(ii) If $\gamma = 1$, then $Eq.(3.1)$ has periodic solutions of period $(p+1)$.

Proof. (i) Let $\{y_n\}_{n=-(2k+1)}^{\infty}$ be a solution of $Eq.(3.1)$. It follows from $Eq.(3.1)$ and our assumption that

$$y_{n+1} = \gamma y_{n-p}.$$

Then we get for $s = 0, 1, ...$

$$y_{s(p+1)+1} = \gamma^{s+1} y_{-p},$$

$$y_{s(p+1)+2} = \gamma^{s+1} y_{-(p-1)},$$

$$\dots$$

$$y_{s(p+1)+p+1} = \gamma^{s+1} y_0.$$

If $\gamma > 1$, then $\lim_{s\to\infty} \gamma^{s+1} = \infty$ and every solution of $Eq.(3.1)$ is unbounded except zero.

(ii) If $\gamma = 1$, from (i) it is obvious that $Eq.(3.1)$ has periodic solutions of period $(p+1)$. □

4. Numerical Results

In this section, we give a few numerical results for some special values of the parameters.

Example 1. *Let* $y_{n+1} = \frac{\gamma y_{n-p}}{1+\prod_{i=0}^{k} y_{n-(2i+1)}}$, $n = 0, 1, ..., 49$ *and* $p = 3, k = 2, \gamma = 0.3, y_{-5} = 6, y_{-4} = 5, y_{-3} = 2, y_{-2} = 1, y_{-1} = 4, y_0 = 3$. *Then we have the following results for Theorem* 3*:*

n	y_n	n	y_n
1	0, 0122448979	33	$7,6158838.10^{-7}$
14	0, 0004825164	41	$6,8542954.10^{-8}$
23	0, 0026557092	46	$3,1657873.10^{-8}$
28	0, 0006211050	50	$9,4973620.10^{-9}$

Example 2. *Let* $y_{n+1} = \frac{\gamma y_{n-p}}{1+\prod_{i=0}^{k} y_{n-(2i+1)}}$, $n = 0, 1, , ..., 49$ *and* $p = 3, k = 2, \gamma = 5, y_{-5} = 7, y_{-4} = 6, y_{-3} = 1, y_{-2} = 3, y_{-1} = 2, y_0 = 4$. *Then we have the following results which show* $\overline{y}_1 = 0$ *is unstable.*

n	y_n	n	y_n
1	0, 333333333	27	43732, 0390
14	0, 002004385	34	$2,40140.10^{-10}$
19	1749, 286550	43	$2,73325.10^{7}$
21	0, 0000081014	50	$6,14759.10^{-16}$

Example 3. *Let* $y_{n+1} = \frac{\gamma y_{n-p}}{1+\prod_{i=0}^{k} y_{n-(2i+1)}}$, $n = 0, 1, ..., 49$ *and* $p = 3$, $\gamma = 1, k = 3, y_{-7} = 1, y_{-6} = 2, y_{-5} = 0.1, y_{-4} = 0.2, y_{-3} = 4, y_{-2} = 5, y_{-1} = 0.3, y_0 = 6$. *Then we have the following results from Corollory* 1*:*

n	y_n	n	y_n
1	3, 571428	25	1, 328736
11	0, 135316	34	0, 017096
17	1, 413484	44	1, 677136
20	1, 685496	50	0, 017040

Example 4. *Let* $y_{n+1} = \frac{\gamma y_{n-p}}{1+\prod_{i=0}^{k} y_{n-(2i+1)}}$, $n = 0, 1, ..., 49$ *and* $p = 3, \gamma = 2, k = 2, y_{-5} = 5, y_{-4} = 3, y_{-3} = 2, y_{-2} = 3, y_{-1} = 0, y_0 = 0$. *Then we have the following results from Corollary* $2-(i)$*:*

n	y_n	n	y_n
3	0	24	0
6	12	34	1536
13	32	39	0
21	128	50	24576

Example 5. *Let* $y_{n+1} = \frac{\gamma y_{n-p}}{1+\prod_{i=0}^{k} y_{n-(2i+1)}}$, $n = 0, 1, ..., 49$ *and* $p = 3, \gamma = 1, k = 2, y_{-5} = 3, y_{-4} = 4, y_{-3} = 0, y_{-2} = 0, y_{-1} = 2, y_0 = 1$. *Then we have the following results from Corollary* $2-(ii)$*:*

n	y_n	n	y_n
1	0	5	0
2	0	6	0
3	2	7	2
4	1	8	1

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