SIGN-IMBALANCE OF ALTERNATING PERMUTATIONS AVOIDING A PATTERN OF LENGTH THREE

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ABSTRACT. For a set $\mathcal{P}$ of permutations, the sign-imbalance of $\mathcal{P}$ is the difference between the numbers of even and odd permutations in $\mathcal{P}$. In this paper we determine the sign-imbalances of two classes of alternating permutations, one is the alternating permutations avoiding a pattern of length three and the other is the alternating permutations of genus 0. The sign-imbalance of the former involves Catalan and Fine numbers, and that of the latter is always ± 1. Meanwhile, we give a simpler proof of Dulucq and Simion's result on the number of alternating permutations of genus 0.

1. INTRODUCTION

Let $\mathfrak{S}_n$ be the symmetric group of all permutations on $[n] = \{1, \ldots, n\}$. A permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is *alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$. These permutations (starting with a descent) are also known as "*down-up*" permutations, while those "*up-down*" permutations (starting with an ascent) are called *reverse alternating*. Let Alt_n denote the set of alternating permutations in $\mathfrak{S}_n$.

For a permutation σ, the *sign* of σ, denoted by $\mathsf{sign}(\sigma)$, is defined by $\mathsf{sign}(\sigma) = (-1)^{\mathsf{inv}(\sigma)}$, where $\mathsf{inv}(\sigma) = |\{(\sigma_i, \sigma_j) : i < j \text{ and } \sigma_i > \sigma_j\}|$ is the number of *inversions* of σ.

For a subset $\mathcal{P} \subseteq \mathfrak{S}_n$, the *sign-imbalance* $\mathcal{I}(\mathcal{P})$ of $\mathcal{P}$ is defined by

$$\mathcal{I}(\mathcal{P}) = \sum_{\sigma \in \mathcal{P}} \mathsf{sign}(\sigma).$$

It is obvious that $\mathcal{I}(\mathfrak{S}_n) = 0$ for $n \geq 2$. Also it is not hard to see that $\mathcal{I}(\mathsf{Alt}_n) = \pm 1$ or 0 (e.g., by the following sign-reversing involution on Alt_n: for a permutation $\sigma \in \mathsf{Alt}_n$, find the least integer i such that $2i - 1$ and $2i$ are not adjacent in σ, then interchange $2i - 1$ and $2i$). In this paper we shall consider two classes of Alt_n, the alternating permutations avoiding each pattern of length three and the alternating permutations of genus 0 (see section 1.2. for definition). Meanwhile, we shall give a simpler proof of

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Dulucq-Simion's result on the number of alternating permutations of genus 0.

1.1. Alternating permutations avoiding a pattern of length three. For a permutation $\omega = \omega_1 \cdots \omega_t \in \mathfrak{S}_t$ $(t \le n)$, we say that σ contains an *ω-pattern* if there are indices $i_1 < i_2 < \cdots < i_t$ such that $\sigma_{i_j} < \sigma_{i_k}$ if and only if $\omega_j < \omega_k$. Otherwise, σ is *ω-avoiding.* For a subset $\mathcal{P} \subseteq \mathfrak{S}_n$, let $\mathcal{P}(\omega)$ denote the set of ω-avoiding permutations in $\mathcal{P}$.

Simion and Schmidt showed in their seminal paper [7] that $|\mathfrak{S}_n(\omega)| = C_n := \frac{1}{n+1}\binom{2n}{n}$, the Catalan number, for each $\omega \in \mathfrak{S}_3$. Later, Mansour [6] proved that the cardinality of $\mathsf{Alt}_n(\omega)$ also coincides with the Catalan number (with indices shifted) for each $\omega \in \mathfrak{S}_3$. As for sign-imbalance, Simion-Schmidt [7] also determined the sign-imbalance of the set of 123-avoiding permutations in $\mathfrak{S}_n$, namely,

$$\mathcal{I}(\mathfrak{S}_n(123)) = \begin{cases} C_{\frac{n-1}{2}} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}$$

The first main result of this paper is to completely determine the sign-imbalances of $\mathsf{Alt}_n(\omega)$ for each $\omega \in \mathfrak{S}_3$. By establishing a bijection between $\mathsf{Alt}_n(132)$ and binary trees, we first determine $\mathcal{I}(\mathsf{Alt}_n(132))$ in Theorems 2.1 by a parity-reversing involution on binary trees. The sign-imbalances $\mathcal{I}(\mathsf{Alt}_n(213))$, $\mathcal{I}(\mathsf{Alt}_n(231))$, and $\mathcal{I}(\mathsf{Alt}_n(312))$ can be derived from $\mathcal{I}(\mathsf{Alt}_n(132))$ and we shall show them in Theorem 3.1-3.3. Moreover, $\mathcal{I}(\mathsf{Alt}_n(123))$ can be derived from Theorem 3.4, and a further refinement of $\mathcal{I}(\mathsf{Alt}_n(321))$ will be discussed in S.-P. Eu, T.-S. Fu, Y.-J. Pan and C.-T. Ting [4].

We make a brief list of $\mathcal{I}(\mathsf{Alt}_n(\omega))$ for each $\omega \in \mathfrak{S}_3$ in the following table. It suggests that the result involves only Catalan numbers C_n and Fine numbers F_n. The Catalan numbers C_n have the generating function $C(x) := \sum_{n\ge0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$, and the Fine numbers $\{F_n\}_{n\ge0} = \{1, 0, 1, 2, 6, 18, \cdots\}$ can be defined by the generating function $F(x) = \sum_{n\ge0} F_n x^n = \frac{1}{x}\frac{1-\sqrt{1-4x}}{3-\sqrt{1-4x}}$. Both are ubiquitous in enumerative combinatorics. We refer readers to Deutsch and Shapiro's survey [2] and Stanley [8] for more information.

$\mathcal{I}(\mathsf{Alt}_n(\sigma_3))\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	
$\mathcal{I}(\mathsf{Alt}_n(123))$	1	1	-1	0	1	-1	0	0	-2	2	0	0	5	Catalan
$\mathcal{I}(\mathsf{Alt}_n(132))$	1	1	-1	0	0	1	-1	2	-2	6	-6	18	-18	Fine
$\mathcal{I}(\mathsf{Alt}_n(213))$	1	1	-1	1	0	2	-1	5	-2	14	-6	42	-18	Catalan, Fine
$\mathcal{I}(\mathsf{Alt}_n(231))$	1	1	-1	0	2	1	-5	-2	14	6	-42	-18	132	Catalan, Fine
$\mathcal{I}(\mathsf{Alt}_n(312))$	1	1	-1	-1	2	2	-5	-5	14	14	-42	-42	132	Catalan
$\mathcal{I}(\mathsf{Alt}_n(321))$	1	1	-1	0	0	-1	1	0	0	2	-2	0	0	Catalan

For example, from the table we have

$$\mathcal{I}(\mathsf{Alt}_{2n-1}(231)) = (-1)^{n-1} F_{n-1} \qquad \text{and} \qquad \mathcal{I}(\mathsf{Alt}_{2n}(231)) = (-1)^n C_n.$$

1.2. **Alternating permutations of genus zero.** A hypermap of size n is an ordered pair of permutations of $\mathfrak{S}_n$ such that the group they generate is transitive on $[n]$. Let $z(\sigma)$ be the number of cycles of a permutation σ. Given a hypermap $(\sigma, \alpha) \in \mathfrak{S}_n \times \mathfrak{S}_n$, the genus g of (σ, α) is defined by the relation

$$z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma) = n + 1 - 2g.$$

Based on this notion, Dulucq-Simion [3] considered the *genus* $g(\alpha)$ of a permutation $\alpha \in \mathfrak{S}_n$ with σ restricted to the n-cycle $2 \ldots n\,1$ (in cycle notation $\sigma = (1\,2\,\ldots\,n)$), namely,

$$z(\alpha) + z(\alpha^{-1} \cdot (1\,2\,\ldots\,n)) = n + 1 - 2g(\alpha).$$

For example, if $\alpha = 2\,3\,1\,5\,4$ (in cycle notation $\alpha = (1\,2\,3)(4\,5)$), then $\alpha^{-1}\sigma = (1\,4)(2)(3)(5)$, and hence $g(\alpha) = 0$. They proved that the number of alternating (resp. reverse alternating) permutations of genus zero coincides with the small (resp. large) Schröder numbers. The *large Schröder numbers* $\{R_n\}_{n\geq 0} = \{1, 2, 6, 22, 90, \ldots\}$ have the ordinary generating function $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$, and the *small Schröder numbers* are defined by $S_0 = 1$ and $S_n = R_n/2$ for $n \geq 1$, i.e. $\{S_n\}_{n\geq 0} = \{1, 1, 3, 11, 45, \ldots\}$.

Let $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$ denote the sets of alternating and reverse alternating permutations of genus zero in $\mathfrak{S}_n$, respectively. By encoding a permutation of genus zero by a word in an alphabet of four letters, Dulucq-Simion [3] obtained a system of grammars for the language formed by the words corresponding to the permutations in $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$ for $n \geq 0$, from which they derived the cardinalities of $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$. Meanwhile, we shall present a simpler proof of this result by means of generating functions (see Theorem 4.4). The second main result of this paper is to determine the sign-imbalance of $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$, namely,

$$\mathcal{I}(\mathcal{D}_{2n}^{(0)}) = \mathcal{I}(\mathcal{D}_{2n+1}^{(0)}) = (-1)^n \qquad \text{and} \qquad \mathcal{I}(\mathcal{U}_n^{(0)}) = 0.$$

1.3. **Organization of the paper.** The rest of this paper is organized as follows. Section 2 is devoted to $\mathcal{I}(\mathsf{Alt}_n(132))$. Section 3 deals with $\mathcal{I}(\mathsf{Alt}_n(\omega))$ for other patterns $\omega \in \mathfrak{S}_3$ except for 321. The sign-imbalance of alternating permutations of genus zero will be discussed in section 4.

Though the result may be solved algebraically, our exposition here is deliberately bijective. The basic strategy is to translate the problem to trees and apply a sign-reversing involution.

2. Sign-imbalance of $\mathrm{Alt}_n(132)$

In this section, we shall determine the sign-imbalance of the set $\mathsf{Alt}_n(132)$ and obtain the following theorem.

Theorem 2.1. *For $n \geq 1$, the following identities hold.*

(i) $\mathcal{I}(Alt_{2n-1}(132)) = F_{n-1}$,
(ii) $\mathcal{I}(Alt_{2n}(132)) = -\mathcal{I}(Alt_{2n-1}(132))$.

To prove Theorem 2.1 we first translate the problem to binary trees. Let $\mathcal{B}_n$ be the set of binary trees with n vertices. It is known that $|\mathcal{B}_n| = C_n$. For a binary tree $T \in \mathcal{B}_n$, the *leftmost path* of T is defined by the maximal sequence $v_1, \ldots, v_k$ of vertices of T such that v_1 is the root and v_i is the left child of v_{i-1}, for $2 \leq i \leq k$. Let $\mathsf{lmpb}(T)$ denote the number of vertices in the leftmost path of T. Note that $\mathsf{lmpb}(T) = 1$ if the root has no left child.

Observe that $|\mathsf{Alt}_{2n-1}(132)| = C_n$ (see [6, Theorem 2.2]). Our approach is to establish a bijection Γ between $\mathsf{Alt}_{2n-1}(132)$ and $\mathcal{B}_n$ such that for a $\sigma \in \mathsf{Alt}_{2n-1}(132)$, the sign of σ is opposite to the parity of $\mathsf{lmpb}(\Gamma(\sigma))$. Then we can determine the sign-imbalance of $\mathsf{Alt}_{2n-1}(132)$ by an involution Ψ on $\mathcal{B}_n$ that reverses the parity of $\mathsf{lmpb}(T)$ if T is a binary tree except for the fixed points of Ψ.

Let Γ be a mapping from $\mathsf{Alt}_{2n-1}(132)$ to $\mathcal{B}_n$ defined inductively as follows. Given a $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in \mathsf{Alt}_{2n-1}(132)$, we factorize σ as $\sigma = \sigma_1 p_1 \cdots p_{n-1}$, where the subword $p_i = \sigma_{2i}\sigma_{2i+1}$ consists of an adjacent pair of elements, for $1 \leq i \leq n-1$. For convenience, denote $p_i = (\sigma_{2i}, \sigma_{2i+1})$ and let $p_0 = (0, \sigma_1)$. We shall associate σ with a binary tree $\Gamma(\sigma) \in \mathcal{B}_n$ with vertices labeled by $p_0, p_1, \ldots, p_{n-1}$ and write $\Gamma(\sigma) = \Gamma(p_0 p_1 \cdots p_{n-1})$. Suppose that p_j $(0 \leq j \leq n-1)$ is the subword which contains the greatest element of σ. Then take p_j as the root of $\Gamma(\sigma)$ and put $\Gamma(p_0 \cdots p_{j-1})$ and $\Gamma(p_{j+1} \cdots p_{n-1})$ as the left subtree and the right subtree, respectively (see Example 2.2).

Example 2.2. Take a permutation $\sigma = 9\,8\,10\,7\,11\,4\,5\,3\,6\,1\,2 \in \mathsf{Alt}_{11}(132)$, and factorize σ as $\sigma = p_0 \cdots p_5 = (0,9)(8,10)(7,11)(4,5)(3,6)(1,2)$. The corresponding tree $\Gamma(\sigma)$, along with the vertex-labeling, is shown in Figure 1.

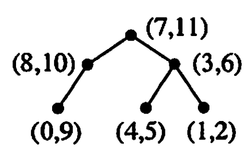

FIGURE 1. The binary tree corresponding to $\sigma = 9\,8\,10\,7\,11\,4\,5\,3\,6\,1\,2$.

Proposition 2.3. *Γ is a bijection between $Alt_{2n-1}(132)$ and $\mathcal{B}_n$ such that a permutation $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in Alt_{2n-1}(132)$ is carried to a tree $\Gamma(\sigma) \in \mathcal{B}_n$ with $lmpb(\Gamma(\sigma)) = 2n - \sigma_1$.*

Proof. To show that Γ is a bijection, it remains to find Γ^{-1}. Given a binary tree $T \in \mathcal{B}_n$, we shall recover the word $\Gamma^{-1}(T)$ by defining inductively a vertex-labeling of T starting with the set $S = \{1, \ldots, 2n-1\}$. Let T_1 and T_2 be the right and left subtrees of the root of T, respectively. If T_1 contains k vertices (possibly empty), then we label T_1 with S_1 which consists of the least $2k$ elements in S, label the root of T by the pair (x, y), where x is the least element and y is the greatest element in $S - S_1$, and label T_2 with $S_2 = S - S_1 - \{x, y\}$. Note that $|S_2|$ is odd, and inductively the leftmost vertex of T will be labeled by a single element (and then we attach 0 to the left).

Observe that whenever $\mathsf{lmpb}(T) = t$ the elements $2n-1, 2n-2, \ldots, 2n-t$ of σ appear in the leftmost path of T accordingly, and the last vertex of this path is labeled by $(0, \sigma_1)$, where $\sigma_1 = 2n - t$. The second assertion follows. □

With Proposition 2.3, the following lemma leads to the fact that the sign of σ is opposite to the parity of $\mathsf{lmpb}(\Gamma(\sigma))$.

Lemma 2.4. *For every $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in Alt_{2n-1}(132)$, the statistic $inv(\sigma)$ has the opposite parity of σ_1.*

Proof. For $1 \le i \le n-1$, let $p_i = (\sigma_{2i}, \sigma_{2i+1})$ be the ith ascent of σ. For each p_i and $2i + 1 < j \le 2n - 1$, we observe that $\sigma_{2i+1} > \sigma_j$ if and only if $\sigma_{2i} > \sigma_j$ since σ is 132-avoiding. It follows that each p_i contributes even number of inversions to $\mathsf{inv}(\sigma)$, and the parity of $\mathsf{inv}(\sigma)$ depends on the number of inversions due to σ_1, which is equal to $\sigma_1 - 1$. □

It is known that the binary trees $T \in \mathcal{B}_n$ can be transformed into plane trees G with n edges such that the degree of root of G equals $\mathsf{lmpb}(T)$ (similar to leftmost child next right sibling, but here we use rightmost child next left sibling, see Figure 2). Moreover, among many other objects, the nth Fine number F_n counts the number of plane trees with n edges where the root is of even degree (e.g., see [2]). Thus F_n counts the number of binary trees $T \in \mathcal{B}_n$ with even $\mathsf{lmpb}(T)$.

For a vertex x of a rooted tree T, let $\tau(x)$ denote the subtree of T rooted at x. Before proceeding the proof of Theorem 2.1, we shall define an involution Ψ on $\mathcal{B}_n$ such that it serves the needs of parity-reversing. Given a $T \in \mathcal{B}_n$, we construct $\Psi(T)$ from T according to $\mathsf{lmpb}(T)$ as follows.

(i) *$lmpb(T)$ is even.* Let u and v be the left and right children of the root of T, respectively. Let x and y be the left and right children of u, respectively. The tree $\Psi(T)$ is constructed from T as follows.

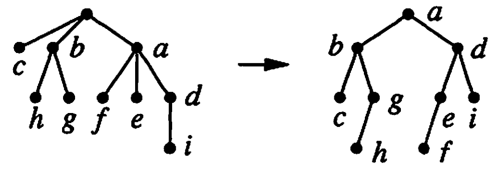

FIGURE 2. A binary-tree representation for plane trees.

Separate the subtrees $\tau(v)$, $\tau(x)$ and $\tau(y)$ from T, and then change u to be the right child of the root. Attach $\tau(y)$ and $\tau(v)$ to u as the left and right subtrees of u, and attach $\tau(x)$ to the root as the left subtree of the root (see Figure 3).

(ii) *lmpb*(T) *is odd.* If the root has no right child, then let $\Psi(T) = T$ (i.e., T is a fixed point), otherwise $\Psi(T)$ is constructed by reversing the operation using in (i).

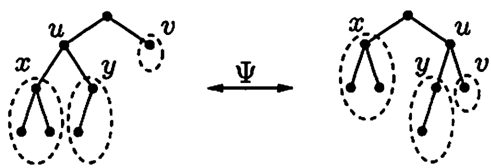

FIGURE 3. An example for the the map Ψ.

Let $\mathcal{F}_n$ denote the set of fixed points of the requested involution Ψ, namely,

$$\mathcal{F}_n = \{T \in \mathcal{B}_n : \mathsf{lmpb}(T) \text{ is odd, and the root has no right child}\}.$$

Lemma 2.5. *For $n \geq 1$, we have*

$$|\mathcal{F}_n| = F_{n-1}.$$

Proof. Given a $T \in \mathcal{F}_n$, it is easy to see that T is in one-to-one correspondence to the tree $T' \in \mathcal{B}_{n-1}$ with even $\mathsf{lmpb}(T')$, where T' is obtained from T with the root vertex removed. The assertion follows. □

We remark that Theorem 2.1.(i) is essentially proved by the following bijective result.

Proposition 2.6. *Ψ is an involution on the set $\mathcal{B}_n$ such that*

(i) *$\mathcal{F}_n$ is the set of fixed points, and*

(ii) *lmpb$(\Psi(T))$ has the opposite parity of lmpb(T) if $T \in \mathcal{B}_n - \mathcal{F}_n$.*

Proof. It is a routine to check that Ψ is an involution on $\mathcal{B}_n$. The assertion (i) is trivial by the definition of Ψ. For a $T \in \mathcal{B}_n - \mathcal{F}_n$, we have $\mathsf{Impb}(\Psi(T)) = \mathsf{Impb}(T) - 1$ if $\mathsf{Impb}(T)$ is even, and $\mathsf{Impb}(\Psi(T)) = \mathsf{Impb}(T) + 1$ if $\mathsf{Impb}(T)$ is odd. The proof is completed. □

Now we are able to evaluate the sign-imbalance of the set $\mathsf{Alt}_n(132)$.

Proof of Theorem 2.1. (i) For the case of odd length, we have

$$\begin{aligned}
\mathcal{I}(\mathsf{Alt}_{2n-1}(132)) &= \sum_{\sigma\in \mathsf{Alt}_{2n-1}(132)} (-1)^{\mathsf{inv}(\sigma)} \\
&= -\sum_{T\in\mathcal{B}_n} (-1)^{\mathsf{Impb}(T)} \quad \text{(by Prop. 2.3 and Lemma 2.4)} \\
&= -\sum_{T\in\mathcal{F}_n} (-1)^{\mathsf{Impb}(T)} \quad \text{(by Prop. 2.6)} \\
&= F_{n-1}.
\end{aligned}$$

(ii) For the case of even length, given a $\sigma = \sigma_1\cdots\sigma_{2n} \in \mathsf{Alt}_{2n}(132)$, observe that $\sigma_{2n} = 1$, otherwise there will be a 132-pattern $(1, \sigma_{2n-1}, \sigma_{2n})$ in σ. There is an immediate bijection between $\mathsf{Alt}_{2n}(132)$ and $\mathsf{Alt}_{2n-1}(132)$ for which σ corresponds to the permutation $\omega = \omega_1\cdots\omega_{2n-1} \in \mathsf{Alt}_{2n-1}(132)$, where $\omega_i = \sigma_i - 1$ for $1 \le i \le 2n-1$. Note that σ has the opposite parity of ω since $\mathsf{inv}(\sigma) = \mathsf{inv}(\omega) + 2n - 1$. Hence $\mathcal{I}(\mathsf{Alt}_{2n}(132)) = -\mathcal{I}(\mathsf{Alt}_{2n-1}(132))$. The proof is completed. □

3. Sign-imbalance of $\mathrm{Alt}_n(\omega)$ for the other patterns ω of length three

In this section, we derive the sign-imbalance of the set $\mathsf{Alt}_n(\omega)$, for $\omega \in \{213, 231, 312, 123\}$.

Theorem 3.1. *For $n \ge 1$, the following identities hold.*

(i) $\mathcal{I}(\mathit{Alt}_{2n}(213)) = -F_{n-1}$,
(ii) $\mathcal{I}(\mathit{Alt}_{2n-1}(213)) = C_n$.

Proof. (i) Under the operations of complement and reverse, there is an immediate bijection between $\mathsf{Alt}_{2n}(213)$ and $\mathsf{Alt}_{2n}(132)$. Namely, to each $\omega = \omega_1\cdots\omega_{2n} \in \mathsf{Alt}_{2n}(213)$ there corresponds a $\sigma = \sigma_1\cdots\sigma_{2n} \in \mathsf{Alt}_{2n}(132)$, where $\sigma_i = 2n+1-\omega_{2n+1-i}$ $(1 \le i \le 2n)$. Moreover, $\mathsf{inv}(\sigma) = \mathsf{inv}(\omega)$ since the pair (σ_i, σ_j) is an inversion of σ if and only if the pair $(\omega_{2n+1-j}, \omega_{2n+1-i})$ is an inversion of ω. Hence $\mathcal{I}(\mathsf{Alt}_{2n}(213)) = \mathcal{I}(\mathsf{Alt}_{2n}(132)) = -F_{n-1}$.

(ii) For every $\sigma = \sigma_1\cdots\sigma_{2n-1} \in \mathsf{Alt}_{2n-1}(213)$, we factorize σ as $\sigma = \sigma_1 p_1\cdots p_{n-1}$, where the subword $p_i = (\sigma_{2i}, \sigma_{2i+1})$ is the ith ascent. For each p_i and $1 \le j \le 2i-1$, observe that $\sigma_j > \sigma_{2i}$ if and only if $\sigma_j > \sigma_{2i+1}$ since σ is 213-avoiding. It follows that each p_i contributes even number of inversions to $\mathsf{inv}(\sigma)$. Hence $\mathsf{inv}(\sigma)$ is even, and $\mathcal{I}(\mathsf{Alt}_{2n-1}(213)) = |\mathsf{Alt}_{2n-1}(213))| = C_n$. □

Theorem 3.2. *For $n \geq 1$, the following identities hold.*

(i) $\mathcal{I}(\mathit{Alt}_{2n-1}(231)) = (-1)^{n-1}F_{n-1}$,
(ii) $\mathcal{I}(\mathit{Alt}_{2n}(231)) = (-1)^{n}C_{n}$.

Proof. (i) Under the operations of reverse, there is an immediate bijection between $\mathsf{Alt}_{2n-1}(231)$ and $\mathsf{Alt}_{2n-1}(132)$. Namely, to each $\omega = \omega_1 \cdots \omega_{2n-1} \in \mathsf{Alt}_{2n-1}(231)$ there corresponds a $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in \mathsf{Alt}_{2n-1}(132)$, where $\sigma_i = \omega_{2n-i}$ $(1 \leq i \leq 2n-1)$. Moreover, $\mathsf{inv}(\sigma) = \binom{2n-1}{2} - \mathsf{inv}(\omega)$ since for $i < j$, $\sigma_i > \sigma_j$ if and only if $\omega_{2n-j} < \omega_{2n-i}$. It follows that $\mathsf{inv}(\sigma)$ has the same parity of $\mathsf{inv}(\omega)$ if n is odd, and has the opposite parity of $\mathsf{inv}(\omega)$ if n is even. Hence $\mathcal{I}(\mathsf{Alt}_{2n-1}(231)) = (-1)^{n-1}\mathcal{I}(\mathsf{Alt}_{2n-1}(132)) = (-1)^{n-1}F_{n-1}$.

(ii) For every $\sigma = \sigma_1 \cdots \sigma_{2n} \in \mathsf{Alt}_{2n}(231)$, we factorize σ as $\sigma = q_1 \cdots q_n$, where the subword $q_i = (\sigma_{2i-1}, \sigma_{2i})$ is the ith descent. For each q_i and $j \leq 2i-2$, we observe that $\sigma_j > \sigma_{2i-1}$ if and only if $\sigma_j > \sigma_{2i}$ since σ is 231-avoiding. It follows that each q_i, along with the inversion of q_i itself, contributes an odd number of inversions to $\mathsf{inv}(\sigma)$. Hence $\mathsf{inv}(\sigma)$ has the same parity of n, and $\mathcal{I}(\mathsf{Alt}_{2n}(231)) = (-1)^{n}|\mathsf{Alt}_{2n}(231)| = (-1)^{n}C_{n}$. □

Theorem 3.3. *For $n \geq 1$, the following identities hold.*

(i) $\mathcal{I}(\mathit{Alt}_{2n-1}(312)) = (-1)^{n-1}C_{n-1}$,
(ii) $\mathcal{I}(\mathit{Alt}_{2n}(312)) = (-1)^{n}C_{n}$.

Proof. (i) Under the operations of reverse, there is an immediate bijection between the two sets $\mathsf{Alt}_{2n-1}(312)$ and $\mathsf{Alt}_{2n-1}(213)$. Namely, to each $\omega = \omega_1 \cdots \omega_{2n-1} \in \mathsf{Alt}_{2n-1}(312)$ there corresponds a $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in \mathsf{Alt}_{2n-1}(213)$, where $\sigma_i = \omega_{2n-i}$ $(1 \leq i \leq 2n-1)$. Moreover, $\mathsf{inv}(\sigma) = \binom{2n-1}{2} - \mathsf{inv}(\omega)$ since for $i < j$, $\sigma_i > \sigma_j$ if and only if $\omega_{2n-j} < \omega_{2n-i}$. It follows that $\mathsf{inv}(\sigma)$ has the same parity of $\mathsf{inv}(\omega)$ if n is odd, and has the opposite parity of $\mathsf{inv}(\omega)$ if n is even. Hence

$$\mathcal{I}(\mathsf{Alt}_{2n-1}(312)) = (-1)^{n-1}\mathcal{I}(\mathsf{Alt}_{2n-1}(213)) = (-1)^{n-1}C_{n-1}.$$

(ii) For every $\sigma = \sigma_1 \cdots \sigma_{2n+1} \in \mathsf{Alt}_{2n+1}(312)$, we observe that the greatest element of σ is $\sigma_{2n+1} = 2n+1$ (otherwise there will be a 312-pattern $(2n+1, \sigma_{2n}, \sigma_{2n+1})$ in σ). Then removing the element σ_{2n+1} from σ results in a member $\sigma' \in \mathsf{Alt}_{2n}(312)$ with $\mathsf{inv}(\sigma') = \mathsf{inv}(\sigma)$. This is a parity-preserving bijection between $\mathsf{Alt}_{2n+1}(312)$ and $\mathsf{Alt}_{2n}(312)$. Hence $\mathcal{I}(\mathsf{Alt}_{2n}(312)) = \mathcal{I}(\mathsf{Alt}_{2n+1}(312))$. The proof is completed. □

To determine $\mathcal{I}(\mathsf{Alt}_{2n-1}(123))$ we first quote the following theorem for the case $\mathcal{I}(\mathsf{Alt}_{2n-1}(321))$.

Theorem 3.4 ([4], Corollary 3.4.). *For $n \geq 1$, the following identities hold.*

(i) $\mathcal{I}(Alt_{2n}(321)) = \begin{cases} (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even}, \end{cases}$

(ii) $\mathcal{I}(Alt_{2n-1}(321)) = -\mathcal{I}(Alt_{2n}(321))$.

□

Theorem 3.5. *The following identities hold.*

(i) *For* $n \geq 1$, $\mathcal{I}(Alt_{2n-1}(123)) = \begin{cases} (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even}. \end{cases}$

(ii) *For* $n \geq 2$, $\mathcal{I}(Alt_{2n}(123)) = -\mathcal{I}(Alt_{2n+1}(123))$.

Proof. (i) Under the operations of reverse, there is an immediate bijection between $\mathsf{Alt}_{2n-1}(123)$ and $\mathsf{Alt}_{2n-1}(321)$. Namely, to each $\omega = \omega_1 \cdots \omega_{2n-1} \in \mathsf{Alt}_{2n-1}(123)$ there corresponds a $\sigma = \sigma_1 \cdots \sigma_{2n-1} \in \mathsf{Alt}_{2n-1}(321)$, where $\sigma_i = \omega_{2n-i}$ $(1 \leq i \leq 2n-1)$. Moreover, $\mathsf{inv}(\sigma) = \binom{2n-1}{2} - \mathsf{inv}(\omega)$ since for $i < j$, $\sigma_i > \sigma_j$ if and only if $\omega_{2n-j} < \omega_{2n-i}$. It follows that $\mathsf{inv}(\sigma)$ has the same parity of $\mathsf{inv}(\omega)$ if n is odd, and has the opposite parity of $\mathsf{inv}(\omega)$ if n is even. By Theorem 3.4, we have $\mathcal{I}(\mathsf{Alt}_{2n-1}(123)) = \mathcal{I}(\mathsf{Alt}_{2n-1}(321)) = (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}$.

(ii) For $n \geq 2$ and for every $\sigma = \sigma_1 \cdots \sigma_{2n+1} \in \mathsf{Alt}_{2n+1}(123)$, we observe that $\sigma_{2n} = 1$, and that $\sigma_{2i-1} > \sigma_{2i+1}$, for $2 \leq i \leq n$ (otherwise there will be 123-patterns in σ). Then removing the element σ_{2n} from σ and subtracting 1 from the other entries leads to a member $\sigma' \in \mathsf{Alt}_{2n}(123)$ with $\mathsf{inv}(\sigma') = \mathsf{inv}(\sigma) - (2n-1)$. This is a parity-reversing bijection between $\mathsf{Alt}_{2n+1}(123)$ and $\mathsf{Alt}_{2n}(123)$. Hence $\mathcal{I}(\mathsf{Alt}_{2n}(123)) = -\mathcal{I}(\mathsf{Alt}_{2n+1}(123))$. The proof is completed. □

4. Sign imbalance of alternating permutations of genus zero

Recall that $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$ denote the set of alternating and reverse alternating permutations of genus zero in $\mathfrak{S}_n$, respectively. In this section, we enumerate $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$ by the method of generating functions for completeness and determine the sign-imbalance of the two sets $\mathcal{D}_n^{(0)}$ and $\mathcal{U}_n^{(0)}$.

We say that an m-cycle of a permutations $\alpha \in \mathfrak{S}_n$ is increasing if its elements are expressible as $i < \alpha(i) < \alpha^2(i) < \cdots < \alpha^{m-1}(i)$. It is known that a permutation of genus zero can be completely characterized as follows, see [1, 5].

Lemma 4.1. *Let* $\alpha \in \mathfrak{S}_n$. *Then* $g(\alpha) = 0$ *if and only if the cycle decomposition of* α *gives a noncrossing partition of* $[n]$, *and each cycle of* α *is increasing.*

The following observation is an immediate consequence of Lemma 4.1.

Proposition 4.2. *For $n \geq 1$, the following identities hold.*

(i) $|\mathcal{D}_{2n}^{(0)}| = |\mathcal{D}_{2n+1}^{(0)}|$,

(ii) $|\mathcal{U}_{2n}^{(0)}| = |\mathcal{U}_{2n+2}^{(0)}|$.

Proof. (i) For every $\alpha = \alpha_1 \cdots \alpha_{2n+1} \in \mathcal{D}_{2n+1}^{(0)}$, we observe that if $\alpha_{2n+1} \neq 2n+1$ then by Lemma 4.1 and the fact $\alpha_{2n} < \alpha_{2n+1}$ there will be a crossing in the cycle decomposition of α. Hence $\alpha_{2n+1} = 2n+1$ and the subword $\alpha_1 \cdots \alpha_{2n} \in \mathcal{D}_{2n}^{(0)}$. This establishes a bijection between the two sets $\mathcal{D}_{2n}^{(0)}$ and $\mathcal{D}_{2n+1}^{(0)}$.

(ii) For every $\alpha = \alpha_1 \cdots \alpha_{2n+2} \in \mathcal{U}_{2n+2}^{(0)}$, by the same argument as in (i) we observe that $\alpha_{2n+2} = 2n+2$ and the subword $\alpha_1 \cdots \alpha_{2n+1} \in \mathcal{U}_{2n+1}^{(0)}$. This establishes a bijection between the two sets $\mathcal{U}_{2n+1}^{(0)}$ and $\mathcal{U}_{2n+2}^{(0)}$. □

We assume $|\mathcal{D}_0^{(0)}| = |\mathcal{U}_1^{(0)}| = 1$, and define the generating functions for $|\mathcal{D}_{2n}^{(0)}|$ and $|\mathcal{U}_{2n+1}^{(0)}|$

$$(1) \qquad S = S(x) = \sum_{i \geq 0} |\mathcal{D}_{2i}^{(0)}| x^i, \qquad R = R(x) = \sum_{i \geq 0} |\mathcal{U}_{2i+1}^{(0)}| x^i.$$

Proposition 4.3. *The following relations hold.*

(i) $R - 1 = 2(S - 1)$,

(ii) $R - 1 = 2xRS$.

Proof. To prove (i), is suffices to show that $|\mathcal{U}_{2n+1}^{(0)}| = 2|\mathcal{D}_{2n}^{(0)}|$, for $n \geq 1$. For every permutation $\alpha = \alpha_1 \cdots \alpha_{2n+1} \in \mathcal{U}_{2n+1}^{(0)}$, by Lemma 4.1 we observe that either $\alpha_1 = 1$ or $\alpha_1 = 2$. Moreover, the elements 1 and 2 are not adjacent in α. Let $\mathcal{U}_{2n+1}^{(0)}$ be partitioned into two sets A_1 and A_2, where $\alpha \in A_1$ if $\alpha_1 = 1$ and $\alpha \in A_2$ if $\alpha_1 = 2$. There is an immediate bijection between A_1 and A_2 by interchanging the elements 1 and 2 of the permutation α. For every $\beta = \beta_1 \cdots \beta_{2n} \in \mathcal{D}_{2n}^{(0)}$, we associate β with a permutation $\alpha = \alpha_1 \cdots \alpha_{2n+1} \in \mathfrak{S}_{2n+1}$, where $\alpha_1 = 1$ and $\alpha_i = \beta_{i-1} + 1$ for $2 \leq i \leq 2n+1$. We observe that $\alpha \in \mathcal{U}_{2n+1}^{(0)}$ and this establishes a bijection between $\mathcal{D}_{2n}^{(0)}$ and A_1. The assertion (i) follows.

(ii) For every $\alpha = \alpha_1 \cdots \alpha_{2n+1} \in A_1$, we have $\alpha_1 = 1$ and $\alpha_{2k+1} = 2$ for some $1 \leq k \leq n$. We factorize α as $\alpha = \alpha_1 \mu \alpha_{2k+1} \nu$, where $\mu = \alpha_2 \cdots \alpha_{2k}$ and $\nu = \alpha_{2k+2} \cdots \alpha_{2n+1}$. By Lemma 4.1, the entries in ν are greater than the entries in μ. We observe that the word ν is a down-up permutation of length $2n-2k$, and that upon normalized ν is in one-to-one correspondence to the member $\beta = \beta_1 \cdots \beta_{2n-2k} \in \mathcal{D}_{2n-2k}^{(0)}$, where $\beta_i = \alpha_{2k+1+i} - 2k - 1$, for $1 \leq i \leq 2(n-k)$. Moreover, the word μ is a down-up permutation of length length $2k-1$, and μ is in one-to-one correspondence to the member $\gamma = \gamma_1 \cdots \gamma_{2k-1} \in \mathcal{U}^{(0)}_{2k-1}$, where $\gamma_j = 2k+2-\alpha_{2k+1-j}$, for $1 \le j \le 2k-1$. The above argument works well for the case $\alpha \in A_2$. This proves the assertion (ii). $\square$

Now, we prove the following theorem.

Theorem 4.4 (Dulucq-Simion). *For $n \ge 1$, the following identities hold.*

(i) $|\mathcal{D}^{(0)}_{2n}| = |\mathcal{D}^{(0)}_{2n+1}| = S_n$,

(ii) $|\mathcal{U}^{(0)}_{2n+1}| = |\mathcal{U}^{(0)}_{2n+2}| = R_n$.

Proof. By Proposition 4.3, we derive that the generating function R satisfies the equation $R = 1 + xR + x^2R^2$. Solving this equation leads to

$$R = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x},$$

which coincides with the generating function for large Schröder numbers. Hence we have $|\mathcal{U}^{(0)}_{2n+1}| = R_n$. Moreover, it follows from the relation $R = 2S - 1$ that $|\mathcal{D}^{(0)}_{2n}| = S_n$. $\square$

Theorem 4.5. *For $n \ge 1$, we have*

(i) $\mathcal{I}(\mathcal{D}^{(0)}_{2n}) = \mathcal{I}(\mathcal{D}^{(0)}_{2n+1}) = (-1)^n$,

(ii) $\mathcal{I}(\mathcal{U}^{(0)}_{n}) = 0$.

Proof. Let $\gamma \in \mathcal{D}^{(0)}_{2n}$ be the permutation with cycle decomposition $\gamma = (1\,2)(3\,4)\cdots(2n-1\;\,2n)$. Note that $\mathsf{sign}(\gamma) = (-1)^n$. We come up with a sign-reversing involution on the set $\mathcal{D}^{(0)}_{2n} - \{\gamma\}$.

For a $\alpha = \alpha_1 \cdots \alpha_{2n} \in \mathcal{D}^{(0)}_{2n} - \{\gamma\}$, find the least integer j such that $(\alpha_{2j-1}\;\alpha_{2j}) \neq (2j\;\;2j-1)$, say $\alpha_{2k} = 2j-1$ for some $k > j$. Then either $\alpha_{2k-1} = 2k-1$ or $\alpha_{2k-1} = 2k$. Moreover, the elements $2k-1$ and $2k$ are not adjacent in α. The request involution is by interchanging the elements $2k-1$ and $2k$ in α. $\square$

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