Independent sets in trees

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Abstract

In this paper, we determine the third largest and the fourth largest numbers of independent sets among all trees of order n. Moreover, we determine the k-th largest numbers of independent sets among all forests of order n, where $k \geq 2$. Besides, We characterize those extremal graphs achieving these values.

1 Introduction and preliminary

Given a graph $G = (V(G), E(G))$, a subset $S \subseteq V(G)$ is called *independent set* if no two vertices of S are adjacent in G. The set of all independent sets of a graph G is denoted by $\mathrm{I}(G)$ and its cardinality by $i(G)$. For a vertex $v \in V(G)$, let $\mathrm{I}_{+v}(G) = \{S \in \mathrm{I}(G) : v \in S\}$ and $\mathrm{I}_{-v}(G) = \{S \in \mathrm{I}(G) : v \notin S\}$. Their cardinalities are denoted by $i_{+v}(G)$ and $i_{-v}(G)$, respectively. Note that $i(G) = i_{+v}(G) + i_{-v}(G)$. An empty set is also an independent set in G. The *nth Fibonacci number* f_n is defined by $f_{-1} = 0$, $f_0 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 1$.There are researches on independent sets in graphs from a different point of view. The Fibonacci number of a graph is the number of independent sets and is also known as the *Merrifield-Simmons index.* The concept of the Fibonacci number of a graph was introduced in [4] and discussed in several papers [2, 5]. It is known [4] that the star $K_{1,n-1}$ has the largest number of independent sets and the path

P_n has the smallest number of independent sets among all trees with n vertices. The problems of determining the second largest and the second smallest values of independent sets for a tree T with n vertices and those graphs achieving these values were solved in [1] and [3], respectively.

In this paper, we determine the third largest and the fourth largest numbers of independent sets among all trees of order n. Moreover, we determine the k-th largest numbers of independent sets among all forests of order n, where $k \geq 2$. Besides, We characterize those extremal graphs achieving these values.

We denote by $G = (V(G), E(G))$ a graph of order $n = |G|$. The graph G is called *null* if $|G| = 0$. A connected graph G is called *nontrivial* if $|G| \geq 2$. For a vertex $x \in V(G)$, let $\deg_G(x)$ denote its *degree*. A *leaf* is a vertex of degree 1. For a subset $X \subseteq V(G)$, we define the *neighborhood* $N_G(X)$ of X in G to be the set of all vertices adjacent to vertices in X and the *closed neighborhood* $N_G[X] = N_G(X) \cup X$. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ obtained from G by removing all vertices in A and all edges incident to these vertices. If a graph G is isomorphic to another graph H, we denote $G = H$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. P_n a *path* with n vertices and $K_{1,n-1}$ a *star* with n vertices. The following useful lemmas and theorems which are needed in this paper.

Lemma 1.1. ([1]) *Given a graph $G = (V(G), E(G))$, the following hold.*
(1) *If $x \in V(G)$, then $i(G) = i(G - x) + i(G - N[x])$.*
(2) *If $e \in E(G)$, then $i(G) < i(G - e)$.*

Theorem 1.2. ([4]) *If F is a forest of order $n \geq 1$, then $i(F) \leq 2^n$. The equality holds if and only if $F = nP_1$.*

Theorem 1.3. *If F is a forest of order $n \geq 3$ having $F \neq nP_1$, then $i(F) \leq 3 \cdot 2^{n-2}$. The equality holds if and only if $F = P_2 \cup (n-2)P_1$.*

Proof. Let F be a forest of order $n \geq 3$ having $F \neq nP_1$ such that $i(F)$ is as large as possible. By Lemma 1.1, then $|E(F)| = 1$. So $F = P_2 \cup (n-2)P_1$ and $i(F) = 3 \cdot 2^{n-2}$. □

Theorem 1.4. ([4]) *If T is a tree of order $n \geq 1$, then $i(T) \leq 2^{n-1} + 1$. The equality holds if and only if $T = T^{(1)}(n)$, where $T^{(1)}(n) = K_{1,n-1}$.*

Notice that if T is a tree of order n having $T \neq T^{(1)}(n)$, then $n \geq 4$.

Theorem 1.5. ([1]) *If T is a tree of order $n \geq 4$ having $T \neq T^{(1)}(n)$, then $i(T) \leq 3 \cdot 2^{n-3} + 2$. The equality holds if and only if $T = T^{(2)}(n)$, where $T^{(2)}(n)$ is the graph obtained from a star $K_{1,n-3}$ by adding a path P_2 and a new edge joining the center of $K_{1,n-3}$ and one vertex of P_2 (see Figure 1).*

2 Trees

In this section, we determine the third largest and the fourth largest numbers of independent sets among all trees of order $n \geq 6$.

For $n \geq 6$, we define the following graphs.

- $T^{(3)}(n)$ is the graph obtained from a star $K_{1,n-4}$ by adding a path P_3 and a new edge joining the center of $K_{1,n-4}$ and the center of P_3.
- $T^{(4)}(n)$ is the graph obtained from a star $K_{1,n-4}$ by adding a path P_3 and a new edge joining the centers of $K_{1,n-4}$ and one leaf of P_3.

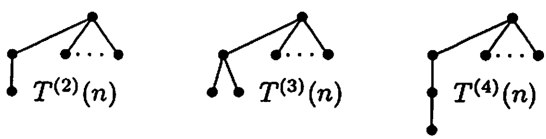

Figure 1: The trees $T^{(2)}(n)$, $T^{(3)}(n)$ and $T^{(4)}(n)$

Lemma 2.1. *For positive integers n, a and b, if $f(x) = 2^x + 2^{n-x}$ for $a \leq x \leq b$, then $f(x)$ has a maximum value when $x = a$ or b.*

Proof. ¿From simple calculation, we have that $f'(x) = (\ln 2)(2^x - 2^{n-x})$ and $f''(x) = (\ln 2)^2(2^x + 2^{n-x})$. Note that $f''(x) > 0$. Hence $f(x)$ yields a maximum value when $x = a$ or b. $\square$

Notice that if T is a tree of order n having $T \neq T^{(1)}(n)$ and $T \neq T^{(2)}(n)$, then $n \geq 6$.

Theorem 2.2. *If T is a tree of order $n \geq 6$ having $T \neq T^{(1)}(n), T^{(2)}(n)$, then $i(T) \leq 5 \cdot 2^{n-4} + 4$. The equality holds if and only if $T = T^{(3)}(n)$.*

Proof. Let T be a tree of order $n \geq 6$ having $T \neq T^{(1)}(n), T^{(2)}(n)$ such that $i(T)$ is as large as possible. Then $i(T) \geq i(T^{(3)}(n)) = 5 \cdot 2^{n-4} + 4$. Let $P : x, y, z, \cdots$ be a longest path of T. Then $T - y = sP_1 \cup T'$, where $1 \leq s \leq n-4$ and T' is a tree of order $n - s - 1$.

Claim. $s \geq 2$.

If $s = 1$, then $T - x$ and $T - N[x]$ are trees. Since $T \neq T^{(1)}(n)$ and $T \neq T^{(2)}(n)$, we have that $T - x \neq K_{1,n-2}$. By Lemma 1.1, Theorem 1.4 and 1.5, we obtain that $5 \cdot 2^{n-4} + 4 \leq i(T) = i(T - x) + i(T - N[x]) \leq (3 \cdot 2^{n-4} + 2) + (2^{n-3} + 1) = 5 \cdot 2^{n-4} + 3$. This is a contradiction, thus $s \geq 2$.

By Lemma 2.1, Theorems 1.2 and 1.4, we have that $5 \cdot 2^{n-4} + 4 \leq i(T) = i(T-y) + i(T-N[y]) \leq 2^s \cdot (2^{n-s-2} + 1) + 2^{n-s-2} = 2^{n-2} + 2^s + 2^{n-s-2} \leq 2^{n-2} + 2^2 + 2^{n-4} = 5 \cdot 2^{n-4} + 4$, where $2 \leq s \leq n-4$. The equalities hold, then $s = 2$ or $n - 4$, i.e., $T - N[y] = (n-4)P_1$ or $2P_1$. Hence $T = T^{(3)}(n)$. $\square$

Theorem 2.3. *If T is a tree of order $n \geq 6$ having $T \neq T^{(1)}(n), T^{(2)}(n), T^{(3)}(n)$, then $i(F) \leq 5 \cdot 2^{n-4} + 3$. The equality holds if and only if $T = T^{(4)}(n)$.*

Proof. Let T be a tree of order $n \geq 6$ having $T \neq T^{(1)}(n), T^{(2)}(n), T^{(3)}(n)$ such that $i(T)$ is as large as possible. Then $i(T^{(4)}(n)) = 5 \cdot 2^{n-4} + 3 \leq i(T) \leq (5 \cdot 2^{n-4} + 4) - 1 = 5 \cdot 2^{n-4} + 3$, so $i(T) = 5 \cdot 2^{n-4} + 3$. Let $P : x, y, z, \cdots$ be a longest path of T. Then $T - y = sP_1 \cup T'$, where $1 \leq s \leq n - 4$ and T' is a tree of order $n - s - 1$.

Claim. $s = 1$ or $s = n - 4$.

Assume that $2 \leq s \leq n - 5$, then $n \geq 7$. Note that $T \neq T^{(3)}(n)$. If $s = 2$, then $T' - N[y] \neq (n-4)P_1$. By Theorems 1.4 and 1.3, $5 \cdot 2^{n-4} + 3 = i(T) = i(T - y) + i(T - N[y]) \leq 2^2(2^{n-4} + 1) + 3 \cdot 2^{n-6} = 19 \cdot 2^{n-6} + 4 < 5 \cdot 2^{n-4} + 3$ for $n \geq 7$. This is a contradiction. If $3 \leq s \leq n - 5$, then $n \geq 8$ and $4 \leq |T'| \leq n - 4$. By Lemma 2.1, Theorems 1.4 and 1.2, $5 \cdot 2^{n-4} + 3 = i(T) = i(T - y) + i(T - N[y]) \leq 2^s(2^{n-s-2} + 1) + 2^{n-s-2} = 2^{n-2} + 2^s + 2^{n-s-2} \leq 2^{n-2} + 2^{n-5} + 2^3 = 9 \cdot 2^{n-5} + 8 < 5 \cdot 2^{n-4} + 3$ for $n \geq 8$. This is a contradiction again. Hence $s = 1$ or $s = n - 4$.

If $s = n - 4$, then $|T'| = 3$. Since $T \neq T^{(3)}(n)$, this follows that $T = T^{(4)}(n)$. If $s = 1$, then $T - x$ and $T - N[x]$ are trees. Since $T \neq T^{(1)}(n)$ and $T \neq T^{(2)}(n)$, these imply that $T - x \neq T^{(1)}(n-1)$. By Theorems 1.4 and 1.5, $i(T - x) \leq 3 \cdot 2^{n-4} + 2$ and $i(T - N[x]) \leq 2^{n-3} + 1$. We obtain that $2^{n-3} + 1 \geq i(T - N[x]) = i(T) - i(T - x) \geq (5 \cdot 2^{n-4} + 3) - (3 \cdot 2^{n-4} + 2) = 2^{n-3} + 1 = i(T^{(1)}(n-2))$. Then $T - N[x] = T^{(1)}(n-2)$. Hence $T = T^{(4)}(n)$. □

3 Forests

In this section, we determine the k-th largest numbers of independent sets among all forests of order $n \geq k \geq 1$.

Lemma 3.1. *Let F be a forest of order $n \geq 4$ having at least two nontrivial components.*

(i) *Then* $i(F) \leq 9 \cdot 2^{n-4}$ *with the equality holding if and only if* $F = 2P_2 \cup (n-4)P_1$.

(ii) *Suppose that* $F \neq 2P_2 \cup (n-4)P_1$, *where* $|F| \geq 5$, *then* $i(F) \leq 15 \cdot 2^{n-5}$ *with the equality holding if and only if* $F = P_3 \cup P_2 \cup (n-5)P_1$.

Proof. (i) Let F be a forest of order $n \geq 4$ having at least two nontrivial components such that $i(F)$ is as large as possible. By Lemma 1.1, then $|E(F)| = 2$, $F = 2P_2 \cup (n-4)P_1$ and $i(F) = 9 \cdot 2^{n-4}$.

(ii) Let $F \neq 2P_2 \cup (n-4)P_1$ be a forest of order $n \geq 5$ having at least two nontrivial components such that $i(F)$ is as large as possible. Then $15 \cdot 2^{n-5} = i(P_3 \cup P_2 \cup (n-5)P_1) \leq i(F)$. If F is a forest of order $n \geq 6$ having at least three nontrivial components, by Lemma 1.1, then $|E(F)| = 3$ and $i(F) = i(3P_2 \cup (n-6)P_1) = 27 \cdot 2^{n-6} < 15 \cdot 2^{n-5}$. This is a contradiction, thus F have two nontrivial components. By Lemma 1.1, then $|E(F)| = 3$, $F = P_3 \cup P_2 \cup (n-5)P_1$ and $i(F) = 15 \cdot 2^{n-5}$. □

Theorem 3.2. *If* F *is a forest of order* $n \geq k \geq 1$ *with the* k*-th largest number of independent sets among all forests of order* n, *then* $i(F) \leq 2^{n-1} + 2^{n-k}$, *and the equality holds if and only if* $F = K_{1,k-1} \cup (n-k)P_1$ *or* $2P_2 \cup (n-4)P_1$ *with* $k = 4$.

Proof. By Theorems 1.2 and 1.3, it's true for $k = 1$ and 2. Assume that F is a forest of order $n \geq 3$ with the third largest number of independent sets among all forests of order n, then $F \neq nP_1, K_{1,1} \cup (n-2)P_1$ and $i(F) \geq i(K_{1,2} \cup (n-3)P_1) = 2^{n-1} + 2^{n-3} > 9 \cdot 2^{n-4}$. By Lemma 3.1, we obtain that F has exactly one nontrivial component. Since $F \neq K_{1,1} \cup (n-2)P_1$, by Lemma 1.1 and Theorem 1.4, $F = K_{1,2} \cup (n-3)P_1$ and $i(F) = 2^{n-1} + 2^{n-3}$. Hence it's true for $k = 3$.

Assume that F is a forest of order $n \geq 4$ with the fourth largest number of independent sets among all forests of order n, then $F \neq nP_1, K_{1,1} \cup (n-2)P_1, K_{1,2} \cup (n-3)P_1$ and $i(F) \geq i(K_{1,3} \cup (n-3)P_1) = 2^{n-1} + 2^{n-4} =$

$9 \cdot 2^{n-4}$. If F have at least two nontrivial components, by Lemma 3.1, then $9 \cdot 2^{n-4} \leq i(F) \leq 9 \cdot 2^{n-4}$. So $i(F) = 9 \cdot 2^{n-4}$ and $F = 2P_2 \cup (n-4)P_1$. If F has exactly one nontrivial component, by Lemma 1.1 and Theorem 1.4, then $9 \cdot 2^{n-4} \leq i(F) \leq i(K_{1,3} \cup (n-4)P_1) = 2^{n-1} + 2^{n-4} = 9 \cdot 2^{n-4}$. Then $i(F) = 2^{n-1} + 2^{n-4}$ and $F = K_{1,3} \cup (n-4)P_1$. Hence it's true for $k = 4$.

Assume that F is a forest of order $n \geq k \geq 5$ with the k-th largest number of independent sets among all forests of order n, then $F \neq 2P_2 \cup (n-4)P_1$ and $i(F) \geq i(K_{1,k-1} \cup (n-k)P_1) = 2^{n-1} + 2^{n-k} > 15 \cdot 2^{n-5}$ for $k \geq 5$. By Lemma 3.1, F has exactly one nontrivial component. By Lemma 1.1 and Theorem 1.4, then $F = K_{1,k-1} \cup (n-k)P_1$ and $i(F) = 2^{n-1} + 2^{n-k}$. Hence it's true for $k \geq 5$. □

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