Generalized Pell numbers and some relations with Fibonacci numbers

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Abstract

In this paper we generalize the companion Pell sequence. We give combinatorial, graph and matrix representations of this sequence. Using these representations we describe some properties of the generalized Pell numbers and the generalized companion Pell numbers. We define the golden Pell matrix for determining the generalized Pell sequences and among other we prove the "generalized Cassini formula" for them. Moreover we give some relations between generalized Pell numbers and the classical Fibonacci numbers.

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1 Introduction

In general we use the standard notation, see [1]. The Fibonacci sequence is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with the initial conditions $F_0 = F_1 = 1$. The Pell sequence is defined by the following recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$. There are some versions of the Pell sequence, one of them is the companion Pell sequence $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with the initial conditions $Q_0 = Q_1 = 1$. The first few Pell and the companion Pell numbers are $0, 1, 2, 5, 12, \ldots$ and $1, 1, 3, 7, 17, \ldots$, respectively. The Pell numbers and the companion Pell numbers are closely related with the Pell equation. If two large integers x and y form a solution of the Pell equation $x^2 - 2y^2 = \pm 1$, then their ratio $\frac{x}{y}$ provides an approximation to $\sqrt{2}$. The sequence of approximations obtained from the Pell equation is $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$, i.e. $\frac{Q_n}{P_n}$.

There are many interesting generalizations of the Pell numbers see for example [4], but a very natural is the concept introduced by I. Włoch in [11], which generalize the Pell numbers in the distance sense. Let $k \geq 2$, $n \geq 0$ be integers. The generalized Pell numbers $P(k, n)$ are defined recursively in the following way

$$P(k,n) = P(k,n-1) + P(k,n-k+1) + P(k,n-k) \text{ for } n \geq k+1, \quad (1)$$

with the initial conditions $P(2,0) = 0$, $P(k,0) = 1$ for $k \geq 3$, $P(k,1) = 1$ for $k \geq 2$ and $P(k,n) = 2n-2$ for $2 \leq n \leq k$. If $k = 2$ and $n \geq 0$, then $P(2,n)$ are the well-known Pell numbers P_n with the initial conditions $P_0 = 0$, $P_1 = 1$.
The following Table includes a few first words of the generalized Pell numbers for special values of k and n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741	13860
$P(3,n)$	1	1	2	4	7	13	24	44	81	149	274	504	927
$P(4,n)$	1	1	2	4	6	9	15	25	40	64	104	169	273
$P(5,n)$	1	1	2	4	6	8	11	17	27	41	60	88	132

Table 1. The generalized Pell numbers $P(k,n)$.

In this paper we give the generalization of the companion Pell numbers in the distance sense. Next we give different representations of the companion Pell numbers and also some identities for them and the generalized Pell numbers. Moreover we give the matrix representations of the generalized Pell sequences and we generalize the Cassini formula for these sequences.

2 Generalizations and some identities

In this section we introduce a generalization of the companion Pell number Q_n.

Let $k \geq 2$, $n \geq 0$ be integers. The generalized companion Pell numbers $Q(k,n)$ are defined by the k-th order linear recurrence relation

$$Q(k,n) = Q(k,n-1) + Q(k,n-k+1) + Q(k,n-k) \text{ for } n \geq k \quad (2)$$

with the initial conditions $Q(k,0) = Q(k,1) = 1$ and $Q(k,n) = 2n-1$ for $2 \leq n \leq k-1$.
We can observe that if $k = 2$, then $Q(2,n) = Q_n$.
We begin with the combinatorial representations of the companion Pell numbers $Q(k,n)$.
Let X be the set of n consecutive integers, $n \geq 0$ and $X^{(2)} = X \times \{1,2\}$. (In particular X is empty if $n = 0$) For fixed integers $k \geq 2$ let $Y \subset X^{(2)}$ be a subset of $X^{(2)}$ such that
(i) $|Y| = t$, $t \geq 0$ and
(ii) for each $(i,j),(u,v) \in Y$ holds $|i-u| + |j-v| \geq k$.

Let $q(k,n,t)$ denote the number of all t-elements subsets Y and let $G(k,n) = \sum_{t\geq 0} q(k,n,t)$.

Theorem 2.1 *Let $n \geq 0$, $k \geq 2$, $t \geq 0$ be integers. Then $q(k,n,0) = 1$, $q(k,n,1) = 2n$ and $q(k,n,t) = 0$ for $n \leq (t-1)(k-1)$. For $t \geq 2$ and for $n > (t-1)(k-1)$ we have the following recurrence relation $q(k,n,t) = q(k,n-1,t) + q(k,n-k+1,t-1) + q(k,n-k,t-1)$.*

PROOF: The initial conditions are obvious. Let $k \geq 2$, $n > (t-1)(k-1)$ and $t \geq 2$. Note that each subset $Y \subset X^{(2)}$ is associated with a binary matrix $M = [a_{ij}] \in M_{2\times n}$ such that

(a) $a_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin Y \\ 1 & \text{if } (i,j) \in Y, \end{cases}$

(b) $\sum_{i=1}^{2} \sum_{j=1}^{n} a_{ij} = t$,

(c) for each $a_{ij} = a_{uv} = 1$ holds $|i-u| + |j-v| \geq k$.

The mapping between subsets Y and such matrices M is bijective, hence, instead of counting t-elements subsets Y we may count such matrices. Let $\mathcal{M}$ denote the set of all matrices M satisfying conditions (a)-(c). Let $\mathcal{M}(k,r,l)$ where $l \leq t$, $r \leq n$ be the family of matrices having exactly l 1's and $a_{1r} = 0$ obtained from matrices of the family $\mathcal{M}$ by deleting columns $r+1,...,n$. Let $\alpha(k,r,l) = |\mathcal{M}(k,r,l)|$. Consider any $M \in \mathcal{M}$. If $a_{1n} = 0$, then $M \in \mathcal{M}(k,n,t)$ and we have $\alpha(k,n,t)$ such matrices. If $a_{1n} = 1$, then $a_{ij} = 0$ for $i = 1$, $j = n-k+1,...,n-1$ and for $i = 2$, $j = n-k+2,...,n$. Hence in this case it suffices to consider the family $\mathcal{M}(k,n-k+1,t-1)$. Thus we obtain $\alpha(k,n-k+1,t-1)$ matrices. Consequently $q(k,n,t) = \alpha(k,n,t) + \alpha(k,n-k+1,t-1)$. Now we calculate the number $\alpha(k,n,t)$. For $a_{1n} = 0$ we have the following possibilities $a_{2n} = 0$ or $a_{2n} = 1$. If $a_{1n} = 0$ and $a_{2n} = 0$, then we get $q(k,n-1,t)$ matrices satisfying conditions (a)-(c). If $a_{1n} = 0$ and $a_{2n} = 1$, then $a_{ij} = 0$ for $i = 1$, $j = n-k+2,...,n$ and for $i = 2$, $j = n-k+1,...,n-1$. Clearly this case gives the same number of possibilities as the case $a_{1n} = 1$ and consequently we have $\alpha(k,n-k+1,t-1)$ matrices satisfying conditions $a_{1n} = 0$ and $a_{2n} = 1$. Finally $\alpha(k,n,t) = q(k,n-1,t) + \alpha(k,n-k+1,t-1)$. Hence from the above considerations we have the following system of recurrence equations

$$\begin{cases} q(k,n,t) = & \alpha(k,n,t) + \alpha(k,n-k+1,t-1) \\ \alpha(k,n,t) = & q(k,n-1,t) + \alpha(k,n-k+1,t-1) \end{cases}$$

By simple calculations we can give the k-th order linear recurrence relation of the form $q(k,n,t) = q(k,n-1,t) + q(k,n-k+1,t-1) + q(k,n-k,t-1)$. Thus the Theorem is proved. □

Theorem 2.2 *Let $k \geq 2$ be integer. If $0 \leq n \leq k-1$, then $G(k,n) = 1+2n$. If $n \geq k$, then $G(k,n) = G(k,n-1) + G(k,n-k+1) + G(k,n-k)$.*

PROOF: If $k \geq 2$, $0 \leq n \leq k-1$, then $G(k,n) = \sum_{t\geq 0} q(k,n,t) = q(k,n,0) + q(k,n,1) = 1+2n$. Assume that $n \geq k$. Then by the Theorem 2.1 we have $G(k,n) = \sum_{t\geq 0} q(k,n,t) = 1+2n+\sum_{t\geq 2} q(k,n-1,t) + \sum_{t\geq 2} q(k,n-k+1,t-1) + \sum_{t\geq 2} q(k,n-k,t-1) = 1+2n+\sum_{t\geq 0} q(k,n-1,t) - 1 - 2(n-1) + \sum_{t\geq 0} q(k,n-k+1,t) - 1 + \sum_{t\geq 0} q(k,n-k,t) - 1 = G(k,n-1) + G(k,n-k+1) + G(k,n-k)$.
Thus the Theorem is proved. □

From (2) and the Theorem 2.2 it follows

Corollary 1 *Let $n \geq 0$, $k \geq 2$ be integers. Then $G(k,n) = Q(k,n+1)$.*

The following table includes a few first words of the generalized companion Pell numbers for special values of k and n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
Q_n	1	1	3	7	17	41	99	239	577	1393	3363	8119	19601
$Q(3,n)$	1	1	3	5	9	17	31	57	105	193	355	653	1201
$Q(4,n)$	1	1	3	5	7	11	19	31	49	79	129	209	337
$Q(5,n)$	1	1	3	5	7	9	13	21	33	49	71	105	159

Table 2. The generalized companion Pell numbers $Q(k,n)$.

Now we give some basic identities for the generalized Pell numbers and the generalized companion Pell numbers.

Theorem 2.3 *Let $k \geq 2$ and $n \geq 0$ be integers. Then*

$$Q(k,n) = P(k,n) + P(k,n-k+1) \textit{ for } n \geq k-1. \tag{3}$$

$$Q(k,n) = P(k,n+1) + P(k,n-k+2) \textit{ for } n \geq k-2. \tag{4}$$

$$P(k,n+1) = \frac{Q(k,n+1)+Q(k,n)}{2}. \tag{5}$$

$$P(k,n) = 2\sum_{i=0}^{k-2} P(k,n-(k-1)-i) + P(k,n-2k+2) \textit{ for } n \geq 2k-2. \tag{6}$$

PROOF: (3) Let $n = k-1$. For $k = 2$ the result is obvious. If $k \geq 3$, then by (1) and (2) we have that $Q(k,k-1) = 2(k-1)-1 = 2k-3$ and $P(k,k-1) + P(k,0) = 2(k-1) - 2 + 1 = 2k-3$.
Assume now that $n \geq k$ and suppose that the equality (3) holds for all

integers $k, ..., n$. We shall prove that (3) is true for integer $n+1$.
Using relations (1), (2) and induction's assumption we obtain that $Q(k,n+1) = Q(k,n)+Q(k,n-k+1)+Q(k,n-k+2) = P(k,n)+P(k,n-k+1)+P(k,n-k+1)+P(k,n-2k+2)+P(k,n-k+2)+P(k,n-2k+3) = P(k,n+1)+P(k,n-k+2)$.

(4) Let $n \geq k-2$. For $k=2$ the result is obvious. Suppose that the equality (4) holds for all integers $k-1, ..., n$. We shall prove that (4) is true for integer $n+1$. Proving analogously as in (3) we obtain that $Q(k,n+1) = Q(k,n)+Q(k,n-k+1)+Q(k,n-k+2) = P(k,n+1)-P(k,n-k+2)+P(k,n-k+2)-P(k,n-k+3)+P(k,n-k+3)-P(k,n-2k+4) = P(k,n+2)-P(k,n-k+3)$.

(5) It easily follows from (3) and (4).

(6) Let $k \geq 2$, $n \geq 2k-2$. If $k=2$, then $P(2,n) = 2P(2,n-1)+P(2,n-2)$ and the identity follows by basic recurrence for the Pell numbers.
Assume now that $k \geq 3$. Then using (1) and some calculations we have

$$\begin{aligned} P(k,n) &= P(k,n-1)+P(k,n-k+1)+P(k,n-k) \\ &= P(k,n-2)+2P(k,n-k)+P(k,n-k-1)+P(k,n-k+1) \\ &= P(k,n-3)+2P(k,n-k)+2P(k,n-k-1)+P(k,n-k-2) \\ &\quad +P(k,n-k+1). \end{aligned}$$

After $(k-1)$ steps we obtain

$$\begin{aligned} P(k,n) &= P(k,n-(k-1))+2P(k,n-k)+2P(k,n-k-1)+... \\ &\quad +2P(k,n-2k+3)+2P(k,n-2k+2)+P(k,n-k+1) \\ &= 2\sum_{i=0}^{k-2} P(k,n-(k-1)-i)+P(k,n-2k+2). \end{aligned}$$

Thus the Theorem is proved. □

From the identity (3) it is easy to see that $\frac{Q(k,n)}{P(k,n)} < 2$, for $k \geq 2$, $n \geq k-1$.

Note that for $k=2$ we obtain the well-known identities for the Pell numbers and for the companion Pell numbers. Namely, $Q_n = P_n + P_{n-1}$, $Q_n = P_{n+1} - P_{n-1}$, $P_{n+1} = \frac{Q_{n+1}+Q_n}{2}$, $P_n = 2P_{n-1}+P_{n-2}$, respectively.

3 A graph representation of the generalized companion Pell numbers

In this section we give a graph interpretation of the generalized companion Pell numbers $Q(k,n)$. We use the standard definition and notation of graph theory, see [2]. It is worth to mention that the graph representation of the generalized Pell numbers $P(k,n)$ was given by I. Włoch in [11].
Let $G = (V(G), E(G))$ be a simple graph. Let $k \geq 2$. A subset $S \subseteq V(G)$ is k-independent if for each $x, y \in S$ the distance between them, is at least k. Moreover the empty set also is k-independent, for each $k \geq 2$. The

number of k-independent sets in G we denote by $NI_k(G)$. Let G, H be two graphs with $V(G) = \{x_1, ..., x_n\}$, $n \geq 1$ and $V(H) = \{y_1, ..., y_m\}$, $m \geq 1$. A Cartesian product of two graphs G, H is a graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(x_i, y_p)(x_j, y_q); (x_i x_j \in E(G)$ and $p = q)$ or $(y_p y_q \in E(H)$ and $i = j)\}$. By $\mathbb{P}_n$, $n \geq 1$ we denote an n-vertex path and by K_n, $n \geq 2$ we mean a complete graph on n-vertices. To give the graph interpretation of the numbers $Q(k, n)$ we use given earlier their combinatorial representation. The set $X^{(2)}$ can be represented as the vertex set of the graph $\mathbb{P}_n \times K_2$. Then Y corresponds to a k-independent set of the graph $\mathbb{P}_n \times K_2$. Thus in the graph terminology the number $Q(k, n+1)$, for $n \geq 0$, $k \geq 2$ is equal to the number of all k-independent sets of the graph $\mathbb{P}_n \times K_2$.
The number of k-independent sets in graphs was studied in many classes of graphs, in particular also in the context of the Fibonacci numbers and their generalizations, see [8], [9], [10].
Consequently it immediately follows:

Fact 3.1 *Let $n \geq 1$, $k \geq 2$ be integers. Then* $NI_k(\mathbb{P}_n \times K_2) = Q(k, n+1)$.

The graph interpretation of the number $Q(k, n)$ can be used for proving some identities:

Theorem 3.2 *Let $n \geq 2k-3$, $k \geq 2$ be integers. Then*

$$Q(k, n+1) = Q(k, n-k+2) + \sum_{i=0}^{k-2} (Q(k, n-(k-1)-i) + Q(k, n-(k-1)+1-i)).$$

PROOF: Let $n \geq 2k-3$, $k \geq 2$ be integers. We use the graph interpretation of the number $Q(k, n+1)$. From Fact 3.1 we obtain that $NI_k(\mathbb{P}_n \times K_2) = Q(k, n+1)$.
Let $V(\mathbb{P}_n) = \{x_1, ..., x_n\}$, $n \geq 1$ and $V(K_2) = \{y_1, y_2\}$ with the numbering vertices from $V(\mathbb{P}_n)$ in the natural fashion. Let $\mathcal{I} = \{(x_i, y_j); i = 1, ..., k-1, j = 1, 2\}$. Assume that S is an arbitrary k-independent set of the graph $\mathbb{P}_n \times K_2$. Let $\mathcal{F}_1 = \{S \subset V(\mathbb{P}_n \times K_2); S \cap \mathcal{I} = \emptyset\}$ and $\mathcal{F}_2 = \{S \subset V(\mathbb{P}_n \times K_2); S \cap \mathcal{I} \neq \emptyset\}$. Clearly $|S \cap \mathcal{I}| \leq 1$ so we consider the following cases:
1. $|S \cap \mathcal{I}| = 0$.
Then it is clear that S is a k-independent set of the graph $(\mathbb{P}_n \times K_2) \backslash \mathcal{I}$ which is isomorphic to $\mathbb{P}_{n-(k-1)} \times K_2$. Hence by Fact 3.1, $|\mathcal{F}_1| = Q(k, n-k+2)$.
2. $|S \cap \mathcal{I}| = 1$.
Without loss of the generality suppose that $(x_1, y_1) \in S$ (the possibility $(x_1, y_2) \in S$ is symmetric, by the definition of Cartesian product of two graphs). Then $(x_p, y_1) \notin S$ for $p = 2, ..., k$ and $(x_q, y_2) \notin S$ for $q = 1, ..., k-1$. This implies that $S = S^* \cup \{(x_1, y_1)\}$, where S^* is a k-independent set of

the graph $(\mathbb{P}_n \times K_2) \setminus (\bigcup_{p=1}^{k} \{(x_p, y_1)\} \cup \bigcup_{q=1}^{k-1} \{(x_q, y_2)\})$ which is isomorphic to the graph $(\mathbb{P}_{n-(k-1)} \times K_2) \setminus \{(x_k, y_1)\}$, where $V(\mathbb{P}_{n-(k-1)}) = \{x_k, ..., x_n\}$.

Let $H(k,n)$ denotes the number of k-independent sets S' the graph $\mathbb{P}_n \times K_2$ such that $(x_1, y_1) \notin S'$ (symmetric possibility $(x_1, y_2) \notin S'$).

Claim 1. $H(k,n) = Q(k,n) + H(k, n-k+1)$.

Assume that $(x_1, y_1) \notin S'$. Then we have two possibilities. If $(x_1, y_2) \notin S'$, then S' is a k-independent set of the graph $\mathbb{P}_n \times K_2 \setminus \{(x_1, y_1), (x_2, y_2)\}$ which is isomorphic to $\mathbb{P}_{n-1} \times K_2$ and by Fact 3.1 we have $Q(k,n)$ such k-independent sets S'. If $(x_1, y_1) \in S'$, then using previous considerations we obtain similarly that there are $H(k, n-k+1)$ sets S'. Consequently the Claim 1 follows.

Claim 2. $Q(k, n+1) = H(k,n) + H(k, n-k+1)$.

The Claim 2 we prove analogously.

Claim 3. $H(k,n) = \frac{1}{2}(Q(k,n) + Q(k,n+1))$.

From Claim 1 and Claim 2 we obtain a system of recurrence equations

$$\begin{cases} H(k,n) &= Q(k,n) + H(k,n-k+1) \\ Q(k,n+1) &= H(k,n) + H(k,n-k+1) \end{cases}$$

and consequently by simple calculations the Claim 3 follows.

From the above it is clear that there are $2H(k, n-k+1)$ possibilities of k-independent sets of $\mathbb{P}_n \times K_2$ such that either $(x_1, y_1) \in S$ or $(x_1, y_2) \in S$. Proving analogously for other vertices from the set $\mathcal{I}$ we obtain that

$$|\mathcal{F}_2| = 2 \sum_{i=0}^{k-2} (H(k, n-(k-1)-i)).$$

Consequently from the above cases we have that

$Q(k, n+1) = Q(k, n-k+2) + 2 \sum_{i=0}^{k-2} H(k, n-(k-1)-i)$ and using Claim 3 we obtain that

$$Q(k,n+1) = Q(k,n-k+2) + \sum_{i=0}^{k-2} (Q(k, n-(k-1)-i) + Q(k, n-(k-1)+1-i)).$$

Thus the Theorem is proved. □

Using the above Theorem and by the identity (5) we obtain

Corollary 2 *Let $k \geq 2$, $n \geq 2k-3$, be integers. Then*

$$Q(k, n+1) = Q(k, n-2k+3) + 2 \sum_{i=0}^{k-2} P(k, n-(k-2)+1-i).$$

4 A matrix representation for the generalized Pell numbers and the generalized companion Pell numbers

In [3] J. Ercolano gives a matrix representation of the Pell sequence.
$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$ where $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.
In this section we give matrix representations for the generalized Pell sequences. We recall that $P(k,n) = P(k,n-1) + P(k,n-k+1) + P(k,n-k)$ with the initial conditions $P(2,0) = 0$, $P(k,0) = 1$ for $k \geq 3$, $P(k,1) = 1$ for $k \geq 2$ and $P(k,n) = 2n-2$ for $2 \leq n \leq k$.
Let $P_k = [p_{ij}]_{k \times k}$. For $1 \leq i \leq k$ an element p_{i1} is equal to the coefficient of $P(k, n-i)$ in the equality (1). Moreover for $j \geq 2$ we have
$p_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$.
Consequently for $k = 2, 3, 4, \ldots$ we obtain matrices:

$$P_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, P_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We will say that the matrix P_k is the generalized Pell matrix.
Let $k \geq 2$ be integer. For a fixed $k \geq 2$ we define the matrix A_k of order k as the matrix of initial conditions

$$A_k = \begin{bmatrix} P(k,2k-2) & P(k,2k-3) & \cdots & P(k,k) & P(k,k-1) \\ P(k,2k-3) & P(k,2k-4) & \cdots & P(k,k-1) & P(k,k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P(k,k) & P(k,k-1) & \cdots & P(k,2) & P(k,1) \\ P(k,k-1) & P(k,k-2) & \cdots & P(k,1) & P(k,0) \end{bmatrix}.$$

Theorem 4.1 *Let $k \geq 2$, $n \geq 1$ be integers. Then* $A_k P_k^n =$

$$\begin{bmatrix} P(k,n+2k-2) & P(k,n+2k-3) & \cdots & P(k,n+k) & P(k,n+k-1) \\ P(k,n+2k-3) & P(k,n+2k-4) & \cdots & P(k,n+k-1) & P(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P(k,n+k) & P(k,n+k-1) & \cdots & P(k,n+2) & P(k,n+1) \\ P(k,n+k-1) & P(k,n+k-2) & \cdots & P(k,n+1) & P(k,n+0) \end{bmatrix}. \quad (7)$$

PROOF: Let $k \geq 2$ be a fixed integer. If $n = 1$, then by simple calculations and by (1) the result immediately follows. Assume that the formula is true for all integers $1, ..., n$. We shall show that Theorem is true for integer $n+1$. Since $A_k P_k^{n+1} = A_k P_k^n P_k$, so by our assumption and from the recurrence form (1) we obtain that $A_k P_k^{n+1} =$

$$\begin{bmatrix} P(k,n+2k-2) & \cdots & P(k,n+k) & P(k,n+k-1) \\ P(k,n+2k-3) & \cdots & P(k,n+k-1) & P(k,n+k-2) \\ \vdots & \ddots & \vdots & \vdots \\ P(k,n+k) & \cdots & P(k,n+2) & P(k,n+1) \\ P(k,n+k-1) & \cdots & P(k,n+1) & P(k,n+0) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} =$$

$$\begin{bmatrix} P(k,n+2k-1) & P(k,n+2k-2) & \cdots & P(k,n+k+1) & P(k,n+k) \\ P(k,n+2k-2) & P(k,n+2k-3) & \cdots & P(k,n+k) & P(k,n+k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P(k,n+k+1) & P(k,n+k) & \cdots & P(k,n+3) & P(k,n+2) \\ P(k,n+k) & P(k,n+k-1) & \cdots & P(k,n+2) & P(k,n+1) \end{bmatrix}.$$

Thus the Theorem is proved. □

Theorem 4.2 *Let $k \geq 2$ be integer. Then*

$$det(P_k) = (-1)^{k+1}. \quad (8)$$

$$det(A_k) = (-1)^{\frac{k(k-1)}{2}}. \quad (9)$$

PROOF: Equality (8) follows from the definition of P_k and basic properties of determinants. To prove (9) auxiliary we define the sequence $P^*(k,n)$ such that

$P^*(k,n) = 0$ for $n = 0, 1, ..., k-2$, $P^*(k,k-1) = 1$ and

$$P^*(k,n) = P^*(k,n-1) + P^*(k,n-k+1) + P^*(k,n-k) \quad \text{for } n \geq k. \quad (10)$$

Let $A^*_{k,m} =$

$$\begin{bmatrix} P^*(k,2k-2+m) & P^*(k,2k-3+m) & \cdots & P^*(k,k+m) & P^*(k,k-1+m) \\ P^*(k,2k-3+m) & P^*(k,2k-4+m) & \cdots & P^*(k,k-1+m) & P^*(k,k-2+m) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P^*(k,k+m) & P^*(k,k-1+m) & \cdots & P^*(k,2+m) & P^*(k,1+m) \\ P^*(k,k-1+m) & P^*(k,k-2+m) & \cdots & P^*(k,1+m) & P^*(k,m) \end{bmatrix}.$$

If $m = 0$, then by the definition of the number $P^*(k,n)$ we have that $A^*_{k,0} = [p^*_{ij}]_{k\times k}$ and $p^*_{ij} = 0$ for $k+1-i < j$, $p^*_{11} = 2$ and $p^*_{ij} = 1$ in otherwise. This immediately gives that

$$A^*_{k,0} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } det(A^*_{k,0}) = (-1)^{\frac{k(k-1)}{2}}.$$

By induction it is easy to prove that $A^*_{k,m} = A^*_{k,0} \cdot P^m_k$, for $1 \leq m \leq k-1$. Moreover by definitions of the matrices A_k, $A^*_{k,m}$ and by (10) we have that $A_k = A^*_{k,0} \cdot P^{2k-4}_k$. Consequently from basic properties of determinants $det(A_k) = (-1)^{\frac{k(k-1)}{2}}$.

Thus the Theorem is proved. □

The sequence $P^*(k,n)$ we will call as the extended generalized Pell sequence.

The next Theorem gives two generalizations of the Cassini formula.

Theorem 4.3 *Let $k \geq 2$ be integer. Then*

$$det(A^*_{k,0}P^n_k) = det(A_kP^n_k) = (-1)^{\frac{(k-1)(k+2n)}{2}} \tag{11}$$

$$det(P^n_k) = (-1)^{n(k+1)} \tag{12}$$

□

Now we give a matrix representation of the generalized companion Pell numbers.

$$Q(k,n) = Q(k,n-1) + Q(k,n-k+1) + Q(k,n-k) \text{ for } n \geq k$$

with the initial conditions $Q(k,0) = Q(k,1) = 1$ and $Q(k,n) = 2n-1$ for $2 \leq n \leq k-1$.

Using the same methods as for the generalized Pell numbers we define the matrix B_k of initial conditions

$$B_k = \begin{bmatrix} Q(k,2k-2) & Q(k,2k-3) & \cdots & Q(k,k) & Q(k,k-1) \\ Q(k,2k-3) & Q(k,2k-4) & \cdots & Q(k,k-1) & Q(k,k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q(k,k) & Q(k,k-1) & \cdots & Q(k,2) & Q(k,1) \\ Q(k,k-1) & Q(k,k-2) & \cdots & Q(k,1) & Q(k,0) \end{bmatrix}.$$

Theorem 4.4 *Let $k \geq 2$, $n \geq 1$ be integers. Then* $B_kP^n_k =$

$$\begin{bmatrix} Q(k,n+2k-2) & Q(k,n+2k-3) & \cdots & Q(k,n+k) & Q(k,n+k-1) \\ Q(k,n+2k-3) & Q(k,n+2k-4) & \cdots & Q(k,n+k-1) & Q(k,n+k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q(k,n+k) & Q(k,n+k-1) & \cdots & Q(k,n+2) & Q(k,n+1) \\ Q(k,n+k-1) & Q(k,n+k-2) & \cdots & Q(k,n+1) & Q(k,n+0) \end{bmatrix}.$$

□

The extended generalized companion Pell sequence is defined as follows

$$Q^*(k,n) = Q^*(k,n-1) + Q^*(k,n-k+1) + Q^*(k,n-k) \text{ for } n \geq k \quad (13)$$

with the initial conditions $Q^*(k,n) = 1$ for $0 \leq n \leq k-1$.
Then

$$B^*_{k,0} = \begin{bmatrix} Q^*(k,2k-2) & Q^*(k,2k-3) & \cdots & 1 \\ Q^*(k,2k-3) & Q^*(k,2k-4) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We can see that $B_k = B^*_{k,0} \cdot P_k^{k-2}$.

Theorem 4.5 *Let $k \geq 2$, $n \geq 1$ be integers. Then*

$$det(B^*_{k,0}) = det(B_k) = -(-1)^{\frac{k(k+1)}{2}} \cdot 2^{k-1} \quad (14)$$

$$det(B^*_{k,0}P_k^n) = det(B_k P_k^n) = -(-1)^{\frac{(k+1)(k+2n)}{2}} \cdot 2^{k-1} \quad (15)$$

□

The equality (15) generalize the Cassini formula for sequences (13) and (2).

5 Some relations between P(k,n) and the Fibonacci numbers

In this section we shall show that for special k the generalized Pell numbers $P(k,n)$ and the generalized companion Pell numbers $Q(k,n)$ have interesting relations with the classical Fibonacci numbers F_n. This section is inspired by results given by E. Kiliç (with D.Tasci, P. Stanica) in [5, 7, 6], where interesting relationships between some generalized Pell numbers and the classical Fibonacci numbers were given.

From the definition of $P(k,n)$ and $Q(k,n)$ it immediately follows that
$P(2,n) = P_n$,
$Q(2,n) = Q_n$,
$Q(3,n) = T_{n+1}$, where T_n is the well known the Tribonacci sequence with the initial conditions $T_0 = T_1 = T_2 = 1$.
$P(3,n) = T^*_{n+1}$, where T^*_n is also named the Tribonacci sequence with the initial conditions $T^*_0 = 0$, $T^*_1 = T^*_2 = 1$.
Now we give relations of $P(4,n)$ and $Q(4,n)$ with the Fibonacci numbers F_n.

Theorem 5.1 *Let n be an integer. Then*

$$P(4,n) - P(4,n-4) = F_n \text{ for } n \geq 4 \quad (16)$$

$$P(4,n) + P(4,n-2) = F_{n+1} \text{ for } n \geq 2 \quad (17)$$

$$\sum_{i=0}^{3} P(4,n-i) = F_{n+2} \text{ for } n \geq 3 \quad (18)$$

$$P(4,2n-1) = F_n^2 \text{ for } n \geq 1 \quad (19)$$

$$P(4,2n) = \sum_{i=0}^{n} F_i^2 = F_n F_{n+1} \text{ for } n \geq 1 \quad (20)$$

$$Q(4,n) - Q(4,n-4) = 2F_{n-1} \text{ for } n \geq 4 \quad (21)$$

$$Q(4,n) + Q(4,n-2) = 2F_n \text{ for } n \geq 2 \quad (22)$$

$$\sum_{i=0}^{3} Q(4,n-i) = 2F_{n+1} \text{ for } n \geq 3 \quad (23)$$

Proof (16) For $n = 4, ..., 7$ the result follows by simple observation. Assume that $n \geq 8$ and $P(4,n) - P(4,n-4) = F_n$. We shall show that $P(4,n+1) - P(4,n-3) = F_{n+1}$. From the definition of $P(k,n)$ and the induction's assumption we obtain $P(4,n+1) - P(4,n-3) = P(4,n) + P(4,n-2) + P(4,n-3) - P(4,n-4) - P(4,n-6) - P(4,n-7) = F_n + F_{n-2} + F_{n-3} = F_n + F_{n-1} = F_{n+1}$ which ends the proof of part (16).

Remaining parts of the Theorem can be proved by analogy.

6 Concluding remarks

We derive relationships between $P(k,n)$, $Q(k,n)$ and F_n only for special $k = 2, 3, 4$ which give known integer sequences. These relations does not hold for an arbitrary $k \geq 2$. Some special representations using the Fibonacci numbers for $k \geq 5$ can be given. However the computing are very large. It is interesting to find a simple rule (if there exists) between the Fibonacci numbers and $P(k,n)$ and $Q(k,n)$ for $k \geq 5$.

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