WILLMORE LAGRANGIAN SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACE

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ABSTRACT. Let $\varphi : M \to C^n$ be an n-dimensional compact Willmore Lagrangian submanifold in the Complex Euclidean Space C^n. Denote by S and H the square of the length of the second fundamental form and the mean curvature of M. Let ρ be the non-negative function on M defined by $\rho^2 = S - nH^2$, K, Q be the function which assigns to each point of M the infimum of the sectional curvature, Ricci curvature at the point. In this paper, we prove some integral inequalities of Simons' type for n-dimensional compact Willmore Lagrangian submanifolds $\varphi : M \to C^n$ in the Complex Euclidean Space C^n in terms of ρ^2, K, Q, H and give some rigidity and characterization Theorems.

1. INTRODUCTION

Let N^{n+p} be an oriented smooth Riemannian manifold of dimension $n+p$. Let $\varphi : M \to N^{n+p}$ be an n-dimensional compact submanifold of N^{n+p}. Denote by $h_{ij}^{\alpha}, S, \vec{H}$ and H the second fundamental form, the square of the length of the second fundamental form, the mean curvature vector and the mean curvature of M. We define the following non-negative function on M

$$\rho^2 = S - nH^2, \tag{1.1}$$

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which vanishes exactly at the umbilical points of M. The Willmore functional is the following non-negative functional (see[2],[11],[13])

$$W(\varphi) = \int_M (S - nH^2)^{\frac{n}{2}} dv, \tag{1.2}$$

where dv is the volume element of M. From [2],[11] and [13], we know that $W(\varphi)$ is an invariant under Moebius (or conformal) transformations of N^{n+p}. The Willmore submanifold was defined by Li[9]and Hu-Li[7], that is, a submanifold is called a Willmore submanifold if it is a extremal submanifold to the Willmore functional. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces. In [9](also see [11],[6]), Li obtained a Euler-Lagrange equation of Willmore functional in terms of Euclidean geometry, which is very important to the study of rigidity and geometry of Willmore submanifold in N^{n+p}.

Let C^n be the Complex Euclidean Space with complex coordinates $z_1, \cdots, z_n$ and

$$\Omega = \frac{i}{2} \sum_i dz_i \wedge \bar{z}_i, \tag{1.3}$$

its symplectic structure.If $<,>$ denotes the Euclidean metric and J the standard complex structure on C^n, then $\Omega(u,v) = \langle u, v\rangle$ for any vectors u, v in C^n.

Let $\varphi : M \to C^n$ be an immersion of an n-dimensional manifold M. φ is called Lagrangian if $\varphi^*\Omega \equiv 0$. This means that the complex structure J of C^n carries each tangent space of M into its corresponding normal space. The simplest examples of Lagrangian submanifolds of C^n are the totally geodesic ones, i.e. the Lagrangian subspaces of C^n. The second example, known as the Whitney sphere, can be defined as a Lagrangian immersion of the unit sphere S^n, centered at the origin of R^{n+1} in C^n given by (see

[12]) (up to dilatations of C^n)

$$(1.4)\quad \Phi(x_1,\cdots,x_{n+1}) = \frac{1}{1+x_{n+1}^2}(x_1(1+ix_{n+1}),\cdots,x_n(1+ix_{n+1})).$$

Contrary to the well-known fact that a compact manifold can not be immersed into C^n as a minimal submanifold, i.e. critical of the volume functional, in [1] Castro and Urbano discover the following interesting fact:

Proposition 1.1([1]). *For $n=2$, the Whitney sphere defined in (1.4) is a Willmore sphere of C^2.*

Recently,Hu-Li [7] proved that the Whitney spheres defined in (1.4) are Willmore submanifolds of C^n if and only if $n = 2$.Hu-Li [7] also gave another example of Willmore Lagrangian submanifold in C^n that was called Willmore Lagrangian sphere.

Example 1.1([7]). Willmore Lagrangian sphere.

Define the Lagrangian sphere $\Psi : S^n \to C^n$ by

$$\Psi(x_1,\cdots,x_{n+1}) = \frac{2\sqrt{\frac{n-1}{2n}}\,e^{i\beta(x_{n+1})}}{[(1+x_{n+1})^{\sqrt{\frac{2n}{n-1}}}+(1-x_{n+1})^{\sqrt{\frac{2n}{n-1}}}]^{\sqrt{\frac{n-1}{2n}}}}(x_1,\cdots,x_n),$$

where

$$\beta(x_{n+1}) = \sqrt{\frac{2(n-1)}{n}}\arctan\Big(\frac{(1+x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}-(1-x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}}{(1+x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}+(1-x_{n+1})^{\sqrt{\frac{n}{2(n-1)}}}}\Big).$$

Then Ψ is a Lagrangian Willmore submanifold.

We note that in recent years, due to their backgrounds in mathematical physics, special Lagrangian submanifolds have been extensively studied(see[7],[1]and [12]). In [7] Hu-Li obtained the following

Theorem 1.1([7]). *A Lagrangian submanifold $\varphi : M \to C^n$ is Willmore submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

$$\begin{aligned}(1.5)\qquad &\rho^{n-2}\{\sum_{i,j,k,l^*} h_{ij}^{l^*}h_{ik}^{l^*}h_{kj}^{m^*} - \sum_{i,j,l^*} H^{l^*}h_{ij}^{l^*}h_{ij}^{m^*} - \rho^2 H^{m^*}\}\\ &+(n-1)\rho^{n-2}\Delta^{\perp}H^{m^*} + 2(n-1)\sum_i(\rho^{n-2})_{,i}H_{,i}^{m^*}\\ &+(n-1)H^{m^*}\Delta(\rho^{n-2}) - \Box^{m^*}(\rho^{n-2}) = 0,\end{aligned}$$

where $\Delta(\rho^{n-2}) = \sum_i (\rho^{n-2})_{i,i}$, $\Box^{m^*}(\rho^{n-2}) = \sum_{i,j} (\rho^{n-2})_{i,j}(nH^{m^*}\delta_{ij} - h_{ij}^{m^*})$, $\Delta^{\perp} H^{m^*} = \sum_i H^{m^*}_{,ii}$ *and* $(\rho^{n-2})_{i,j}$ *is the Hessian of* ρ^{n-2} *with respect to the induced metric* $dx \cdot dx$, $H^{m^*}_{,i}$ *and* $H^{m^*}_{,ij}$ *are the components of the first and second covariant derivative of the mean curvature vector field* $\vec{H}$.

Remark 1.1. Fix the index m^* with $n+1 \le m^* \le 2n$, define $\Box^{m^*}: M \to R$ by

$$\Box^{m^*} f = \sum_{i,j} (nH^{m^*}\delta_{ij} - h_{ij}^{m^*}) f_{i,j}, \tag{1.6}$$

where f is any smooth function on M. We know that $\Box^{m^*}$ is a self-adjoint operator (see Cheng-Yau [4]). We can see that this operator naturally appears in the Willmore equation (1.5). This operator will play an important role in the proofs of our theorems.

In this paper, by making use of the self-adjoint operator $\Box^{m^*}$, we shall establish some integral inequalities of Simons' type for n-dimensional compact Willmore Lagrangian submanifolds in the Complex Euclidean Space C^n in terms of the scalar curvatures, the sectional curvatures, the Ricci curvatures and the mean curvatures of the submanifolds and give some rigidity and characterization Theorems of such submanifolds .

2. Fundamental formulas of Lagrangian submanifolds

In this section, we review some related facts for Lagrangian submanifolds in C^n by method of moving frames. We will follow the notation in the first section except agreeing with the convention of indices:

$$A, B, C, \cdots = 1, \cdots, n, 1^*, \cdots, n^*, \quad 1^* = n+1, \cdots, n^* = 2n,$$
$$i, j, k, \cdots = 1, \cdots, n.$$

Let $\varphi : M \to C^n$ be an n-dimensional Lagrangian submanifold. We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{1^*} = Je_1, \cdots, e_{n^*} =$

Je_n in C^n, such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M, where J is the complex structure of C^n. Let $\omega_1, \cdots, \omega_{2n}$ is the field of dual frames. Then we have the structure equations of C^n as follows

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}, \tag{2.2}$$

Let θ_A, θ_{AB} be the restriction of ω_A, ω_{AB} to M. Then $\theta_{i^*} = 0$, taking its exterior derivative and making use of (2.1) and the Cartan lemma we get

$$\theta_{ik^*} = \sum_j h_{ij}^{k^*} \theta_j, \quad h_{ij}^{k^*} = h_{ji}^{k^*}, \tag{2.3}$$

from which we define the second fundamental form $II = \sum_{i,j,k^*} h_{ij}^{k^*} \omega_i \otimes \omega_j e_{k^*}$ and the mean curvature vector $\vec{H}$ of $\varphi : M \to C^n$ as follows:

$$S = \sum_{i,j,k^*} (h_{ij}^{k^*})^2, \quad \vec{H} = \sum_{k^*} H^{k^*} e_{k^*}, \quad H^{k^*} = \frac{1}{n} \sum_i h_{ii}^{k^*}, \quad H = |\vec{H}|.$$

Since $\varphi : M \to C^n$ is Lagrangian, we have for any i, j

$$\langle Je_i, e_j \rangle = 0, \quad \langle e_{i^*}, Je_j \rangle = \delta_{ij}. \tag{2.4}$$

Taking exterior derivative of (2.4), we get for any i, j, k

$$h_{ij}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}, \tag{2.5}$$

$$\theta_{i^*j^*} = \theta_{ij}, \tag{2.6}$$

If we denote by R_{ijkl} the Riemannian curvature tensor of M, we get the Gauss equations

$$R_{ijkl} = \sum_{m^*} (h_{ik}^{m^*} h_{jl}^{m^*} - h_{il}^{m^*} h_{jk}^{m^*}), \tag{2.7}$$

$$R_{ik} = n \sum_{m^*} H^{m^*} h_{ik}^{m^*} - \sum_{j,m^*} h_{ij}^{m^*} h_{jk}^{m^*}, \tag{2.8}$$

$$n(n-1)R = n^2 H^2 - S, \tag{2.9}$$

where R is the normalized scalar curvature of M.

The first covariant derivative $\{h^{m^*}_{ijk}\}$ and the second covariant derivative $\{h^{m^*}_{ijkl}\}$ of $h^{m^*}_{ij}$ are defined by

$$\sum_k h^{m^*}_{ijk}\theta_k = dh^{m^*}_{ij} + \sum_k h^{m^*}_{kj}\theta_{ki} + \sum_k h^{m^*}_{ik}\theta_{kj} + \sum_{k^*} h^{k^*}_{ij}\theta_{k^*m^*}, \tag{2.10}$$

$$\sum_l h^{m^*}_{ijkl}\theta_l = dh^{m^*}_{ijk} + \sum_l h^{m^*}_{ljk}\theta_{li} + \sum_l h^{m^*}_{ilk}\theta_{lj} + \sum_l h^{m^*}_{ijl}\theta_{lk} + \sum_{l^*} h^{l^*}_{ijk}\theta_{\beta m^*}. \tag{2.11}$$

The Codazzi equations and the Ricci identities

$$h^{m^*}_{ijk} = h^{m^*}_{ikj}, \tag{2.12}$$

$$h^{m^*}_{ijkl} - h^{m^*}_{ijlk} = \sum_m h^{m^*}_{mj}R_{mikl} + \sum_m h^{m^*}_{im}R_{mjkl} + \sum_{k^*} h^{k^*}_{ij}R_{k^*m^*kl}. \tag{2.13}$$

The Ricci equations are

$$R_{i^*j^*kl} = \sum_m (h^{i^*}_{km}h^{j^*}_{lm} - h^{j^*}_{km}h^{i^*}_{lm}). \tag{2.14}$$

Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum_{m^*} H^{m^*}e_{m^*}$ in the normal bundle $N(M)$ as follows

$$\sum_i H^{m^*}_{,i}\theta_i = dH^{m^*} + \sum_{k^*} H^{k^*}\theta_{k^*m^*}, \tag{2.15}$$

$$\sum_j H^{m^*}_{,ij}\theta_j = dH^{m^*}_{,i} + \sum_j H^{m^*}_{,j}\theta_{ji} + \sum_{k^*} H^{k^*}_{,i}\theta_{k^*m^*}, \tag{2.16}$$

$$\Delta^{\perp}H^{m^*} = \sum_i H^{m^*}_{,ii}, \qquad H^{m^*} = \frac{1}{n}\sum_k h^{m^*}_{kk}. \tag{2.17}$$

Let f be a smooth function on M. The first, second covariant derivatives $f_i, f_{i,j}$ and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{i,j}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.18}$$

For the fix index $m^*(n+1 \leq m^* \leq 2n)$, we introduce an operator $\square^{m^*}$ due to Cheng-Yau [4] by

$$\square^{m^*} f = \sum_{i,j} (nH^{m^*}\delta_{ij} - h_{ij}^{m^*}) f_{i,j}. \tag{2.19}$$

Since M is compact, the operator $\square^{m^*}$ is self-adjoint (see[4]) if and only if

$$\int_M (\square^{m^*} f) g dv = \int_M f(\square^{m^*} g) dv, \tag{2.20}$$

where f and g are any smooth functions on M.

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \mathtt{trace}(A \cdot A^t) = \sum_{i,j} (a_{ij})^2.$$

Clearly, $N(A) = N(T^t AT)$ for any orthogonal matrix T.

We need the following Lemmas due to Chern-Do Carmo-Kobayashi [5], Li [10] and Cheng [3].

Lemma 2.1([5]) *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \leq 2N(A)N(B), \tag{2.21}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by on orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and if

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta), 1 \leq \alpha, \beta \leq 3$$

then at least one of the matrices A_α must be zero.

Lemma 2.2([10]) *Let $\varphi : M \to C^n$ be an n-dimensional $(n \geq 2)$ Lagrangian submanifold. Then we have*

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2, \tag{2.22}$$

where $|\nabla h|^2 = \sum_{i,j,k,m^*} (h_{ijk}^{m^*})^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,m^*} (H_{,i}^{m^*})^2$.

Lemma 2.3([3]) *Let b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^n b_i = 0$ and $\sum_{i=1}^n b_i^2 = B$. Then*

$$\sum_{i=1}^n b_i^4 - \frac{B^2}{n} \le \frac{(n-2)^2}{n(n-1)} B^2. \tag{2.23}$$

Lemma 2.4([3]) *Let a_i and b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = a$. Then*

$$\left|\sum_{i=1}^n a_i b_i^2\right| \le \sqrt{\sum_{i=1}^n b_i^4 - \frac{(\sum_{i=1}^n b_i^2)^2}{n}}\sqrt{a}. \tag{2.24}$$

3. Integral equalities and propositions

In this section we shall obtain some integral equalities of Willmore Lagrangian submanifolds $\varphi : M \to C^n$. We should note that the self-adjoint operator $\Box^{m^*}$, which appears in Euler-Lagrange equation (1.5) naturally, will play an important role in the proof of these integral equalities.

Define tensors

$$\tilde{h}_{ij}^{m^*} = h_{ij}^{m^*} - H^{m^*}\delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{m^*l^*} = \sum_{i,j} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*}, \quad \sigma_{m^*l^*} = \sum_{i,j} h_{ij}^{m^*} h_{ij}^{l^*}. \tag{3.2}$$

Then the $(n \times n)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{1^*}, \cdots, e_{n^*}$. We set

$$\tilde{\sigma}_{m^*l^*} = \tilde{\sigma}_{m^*}\delta_{m^*l^*}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^{m^*} = 0, \ \tilde{\sigma}_{m^*l^*} = \sigma_{m^*l^*} - nH^{m^*}H^{l^*}, \ \rho^2 = \sum_{m^*} \tilde{\sigma}_{m^*} = S - nH^2, \tag{3.4}$$

(3.5) $$\sum_{i,j,k,m^*} h_{kj}^{l^*} h_{ij}^{m^*} h_{ik}^{m^*} = \sum_{i,j,k,m^*} \tilde{h}_{kj}^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{m^*} + 2\sum_{i,j,m^*} H^{m^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*} + H^{l^*}\rho^2 + nH^2 H^{l^*}.$$

From (3.1),(3.4) and (3.5),the Euler-Lagrange equation(1.5) can be rewritten as

Proposition 3.1. *A Lagrangian submanifold $\varphi : M \to C^n$ is Willmore submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

(3.6)
$$\Box^{m^*}(\rho^{n-2}) = (n-1)\rho^{n-2}\Delta^{\perp} H^{m^*} + 2(n-1)\sum_i (\rho^{n-2})_{,i} H^{m^*}_{,i} + (n-1)H^{m^*}\Delta(\rho^{n-2}) + \rho^{n-2}\Big(\sum_{l^*} H^{l^*}\tilde{\sigma}_{m^* l^*} + \sum_{i,j,k,l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{l^*} \tilde{h}_{kj}^{l^*}\Big).$$

Setting $f = nH^{m^*}$ in (2.19), we have

(3.7) $$\Box^{m^*}(nH^{m^*}) = \sum_{i,j} (nH^{m^*}\delta_{ij} - h_{ij}^{m^*})(nH^{m^*})_{,i,j} = \sum_i (nH^{m^*})(nH^{m^*})_{,i,i} - \sum_{i,j} h_{ij}^{m^*}(nH^{m^*})_{,i,j}.$$

We also have

(3.8)
$$\frac{1}{2}\Delta(nH)^2 = \frac{1}{2}\Delta\sum_{m^*}(nH^{m^*})^2 = \frac{1}{2}\sum_{m^*}\Delta(nH^{m^*})^2$$
$$= \frac{1}{2}\sum_{m^*,i}[(nH^{m^*})^2]_{,i,i} = \sum_{m^*,i}[(nH^{m^*})_{,i}]^2 + \sum_{m^*,i}(nH^{m^*})(nH^{m^*})_{,i,i}$$
$$= n^2|\nabla^{\perp}\vec{H}|^2 + \sum_{m^*,i}(nH^{m^*})(nH^{m^*})_{,i,i}.$$

Therefore, from (3.7) and (3.8), we get

(3.9)
$$\sum_{m^*}\Box^{m^*}(nH^{m^*}) = \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{,i,j}$$
$$= \frac{1}{2}\Delta(n(n-1)H^2 - \rho^2 + S) - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{,i,j}$$
$$\Rightarrow \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*} h_{ij}^{m^*}(nH^{m^*})_{,i,j}.$$

On the other hand, we have

(3.10)

$$\begin{aligned}\frac{1}{2}\Delta S=&\sum_{i,j,k,m^*}(h^{m^*}_{ijk})^2+\sum_{i,j,m^*}h^{m^*}_{ij}\Delta h^{m^*}_{ij}\\ =&|\nabla h|^2+\sum_{i,j,m^*}h^{m^*}_{ij}(nH^{m^*})_{,i,j}+\sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk}+h^{m^*}_{li}R_{lkjk})\\ &+\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}.\end{aligned}$$

Putting (3.10) into (3.9), we have

(3.11)

$$\begin{aligned}\sum_{m^*}\Box^{m^*}(nH^{m^*})=&|\nabla h|^2-n^2|\nabla^{\perp}\vec{H}|^2+\frac{1}{2}n(n-1)\Delta H^2-\frac{1}{2}\Delta\rho^2\\ &+\sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk}+h^{m^*}_{li}R_{lkjk})+\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}.\end{aligned}$$

Multiplying (3.11) by ρ^{n-2} and taking integration, using (2.20), we have

(3.12)

$$\begin{aligned}\sum_{m^*}\int_M(nH^{m^*})\Box^{m^*}(\rho^{n-2})dv=&\int_M\rho^{n-2}(|\nabla h|^2-n^2|\nabla^{\perp}\vec{H}|^2)dv\\ &+\frac{1}{2}n(n-1)\int_M\rho^{n-2}\Delta H^2dv-\frac{1}{2}\int_M\rho^{n-2}\Delta\rho^2dv\\ &+\int_M\rho^{n-2}\sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk}+h^{m^*}_{li}R_{lkjk})dv\\ &+\int_M\rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}dv.\end{aligned}$$

Taking the Euler-Lagrange equation (3.6) into (3.12) and making use of the following

$$\begin{aligned}\int_M\rho^{n-2}\sum_{m^*}H^{m^*}\Delta^{\perp}H^{m^*}dv=&\frac{1}{2}\int_M\rho^{n-2}\sum_{m^*}\Delta^{\perp}(H^{m^*})^2dv\\ &-\int_M\rho^{n-2}\sum_{i,m^*}(H^{m^*}_{,i})^2dv\\ =&\frac{1}{2}\int_M\rho^{n-2}\Delta H^2dv-\int_M\rho^{n-2}|\nabla\vec{H}|^2dv,\end{aligned}$$

$$\int_M H^2\Delta(\rho^{n-2})dv = \int_M \sum_{m^*}(H^{m^*})^2\sum_i(\rho^{n-2})_{i,i}dv$$
$$= \sum_{m^*,i}\int_M (H^{m^*})^2(\rho^{n-2})_{i,i}dv$$
$$= -\sum_{m^*,i}\int_M (\rho^{n-2})_i((H^{m^*})^2)_{,i}dv$$
$$= -2\int_M \sum_{m^*} H^{m^*}\sum_i(\rho^{n-2})_i H^{m^*}_{,i}\,dv,$$

$$-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv = -\frac{1}{2}\sum_i\int_M \rho^{n-2}(\rho^2)_{i,i}dv$$
$$= \frac{1}{2}\sum_i\int_M (\rho^2)_i(\rho^{n-2})_i dv = (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv,$$

we have, by a direct calculation, the following

Proposition 3.2 *For any n-dimensional compact Willmore Lagrangian submanifold $\varphi : M \to C^n$, there holds the following integral equality*

$$\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv$$
$$-\int_M \rho^{n-2}\sum_{m^*,l^*} nH^{m^*}(H^{l^*}\tilde{\sigma}_{m^*l^*} + \sum_{i,j,k}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ik}\tilde{h}^{l^*}_{kj})dv$$
$$+\int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l} h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk} + h^{m^*}_{li}R_{lkjk})dv$$
$$+\int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}dv = 0. \tag{3.13}$$

From (2.14) and (3.1) we have

$$\sum_{m^*,l^*}\sum_{i,j,k} h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk} = \sum_{m^*,l^*}\sum_{i,j,k,l} h^{m^*}_{ij}h^{l^*}_{ki}(h^{l^*}_{jl}h^{m^*}_{lk} - h^{l^*}_{kl}h^{m^*}_{lj}) \tag{3.14}$$
$$= -\frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_l h^{l^*}_{jl}h^{m^*}_{lk} - \sum_l h^{m^*}_{jl}h^{l^*}_{lk})^2$$
$$= -\frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_l \tilde{h}^{l^*}_{jl}\tilde{h}^{m^*}_{lk} - \sum_l \tilde{h}^{m^*}_{jl}\tilde{h}^{l^*}_{lk})^2$$
$$= -\frac{1}{2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}),$$

where $\tilde{A}_{m^*} := (\tilde{h}_{ij}^{m^*}) = (h_{ij}^{m^*} - H^{m^*}\delta_{ij})$.

By use of (2.7),(3.2),(3.4),(3.5) and (3.14) we can take a simple and direct calculation that

(3.15)

$$\begin{aligned}
&\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) \\
=& -\sum_{m^*,l^*}\sum_{i,j,k,l} h_{ij}^{m^*}h_{ij}^{l^*}h_{lk}^{m^*}h_{lk}^{l^*} + n\sum_{m^*,l^*}\sum_{i,j,k} H^{l^*}h_{kj}^{l^*}h_{ij}^{m^*}h_{ik}^{m^*} \\
&+ \sum_{m^*,l^*,i,j,k} h_{ji}^{m^*}h_{ik}^{l^*}R_{l^*m^*jk} \\
=& -\sum_{m^*,l^*}\sigma_{m^*l^*}^2 + n\sum_{m^*,l^*}\sum_{i,j,k} H^{l^*}\tilde{h}_{kj}^{l^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{m^*} + 2n\sum_{m^*,l^*}\sum_{i,j} H^{m^*}H^{l^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ij}^{l^*} \\
&+ n\sum_{l^*}(H^{l^*})^2\rho^2 + n^2H^2\sum_{l^*}(H^{l^*})^2 - \frac{1}{2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \\
=& -\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 + nH^2\rho^2 + n\sum_{m^*,l^*}\sum_{i,j,k} H^{l^*}\tilde{h}_{kj}^{l^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{m^*} \\
&- \frac{1}{2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}).
\end{aligned}$$

Putting (3.14) and (3.15) into (3.13), we have the following

Proposition 3.3 *For any n-dimensional compact Willmore Lagrangian submanifold $\varphi : M \to C^n$, there holds the following integral equality*

$$\begin{aligned}
(3.16)\quad &\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\
&+ n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv \\
&- \int_M \rho^{n-2}\sum_{m^*,l^*}(N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) + \tilde{\sigma}_{m^*l^*}^2)dv = 0.
\end{aligned}$$

4. Integral inequalities and theorems

In this section, we shall obtain some rigidity Theorems of n-dimensional Willmore Lagrangian submanifold $\varphi : M \to C^n$ in terms of the function $\rho^2 = S - nH^2$, sectional and Ricci curvature and mean curvature. We should note that the integral equalities (3.13) and (3.16) will play an important role in the proofs of these Theorems. Firstly, we can conclude the following theorem

Theorem 4.1. *Let $\varphi: M \to C^n$ be an $n(n \geq 2)$-dimensional compact Willmore Lagrangian submanifold. Then there holds the follows*

$$\int_M \rho^{n-2}\{\frac{2n(n-1)}{n+2}|\nabla^\perp \vec{H}|^2 - (2-\frac{1}{n})\rho^4\}dv \leq 0. \tag{4.1}$$

In particular, If

$$\rho^2 \leq n\sqrt{\frac{2(n-1)}{(2n-1)(n+2)}}|\nabla^\perp \vec{H}|, \tag{4.2}$$

then M is totally umbilical.

Proof. From lemma2.1, (3.2) and (3.3), we have

$$\begin{aligned} &-\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) - \sum_{m^*,l^*} \tilde{\sigma}^2_{m^*l^*} \\ &\geq -\sum_{m^*}\tilde{\sigma}^2_{m^*} - 2\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = -2(\sum_{m^*}\tilde{\sigma}_{m^*})^2 + \sum_{m^*}\tilde{\sigma}^2_{m^*} \\ &\geq -2\rho^4 + \frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = -(2-\frac{1}{n})\rho^4, \end{aligned} \tag{4.3}$$

where, we used

$$\sum_{m^*}\tilde{\sigma}^2_{m^*} \geq \frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2. \tag{4.4}$$

We also have

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = \sum_{m^*}(H^{m^*})^2\tilde{\sigma}_{m^*} \leq \sum_{m^*}(H^{m^*})^2\sum_{l^*}\tilde{\sigma}_{l^*} = H^2\rho^2. \tag{4.5}$$

By making use of lemma2.2, (3.16),(4.3) and (4.5), we have

(4.6)

$$\begin{aligned} 0 \geq &\int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv \\ &- \int_M \rho^{n-2}(2-\frac{1}{n})\rho^4 dv \geq \int_M \rho^{n-2}\{\frac{2n(n-1)}{n+2}|\nabla^\perp \vec{H}|^2 - (2-\frac{1}{n})\rho^4\}dv. \end{aligned}$$

(i) If $n=2$, from (4.2) and (4.6), we have $|\nabla^\perp \vec{H}|^2 = \frac{3}{2}\rho^4$ on M.If $\rho^2 = 0$ on M, then M is totally umbilical. If $\rho^2 \neq 0$ on M, from $|\nabla^\perp \vec{H}|^2 = \frac{3}{2}\rho^4$

we know that the equality in (4.6) holds. Therefore, we have

(4.7) $$N(\tilde{A}_3\tilde{A}_4 - \tilde{A}_4\tilde{A}_3) = 2N(\tilde{A}_3)N(\tilde{A}_4), \quad 2(\tilde{\sigma}_3^2 + \tilde{\sigma}_4^2) = (\tilde{\sigma}_3 + \tilde{\sigma}_4)^2,$$

that is

(4.8) $$\tilde{\sigma}_3 = \tilde{\sigma}_4.$$

We also have for $m^*, l^* = 3, 4$,

(4.9) $$\sum_{m^*, l^*} H^{m^*} H^{l^*} \tilde{\sigma}_{m^* l^*} = H^2 \rho^2.$$

From lemma 2.1, we know that at most two of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*}), m^* = 3, 4$, are different from zero. If all of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*})$ are zero, which is contradiction with M is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, which is contradiction with (4.8). Therefore, we may assume that

$$\tilde{A}_3 = \lambda \tilde{A}, \quad \tilde{A}_4 = \mu \tilde{B}, \quad \lambda, \mu \neq 0,$$

where $\tilde{A}$ and $\tilde{B}$ are defined in lemma 2.1.

From (4.9), we have

$$\lambda^2 (H^3)^2 + \mu^2 (H^4)^2 = (\lambda^2 + \mu^2)((H^3)^2 + (H^4)^2).$$

Since $\lambda, \mu \neq 0$, we infer that $H^3 = H^4 = 0$, that is, $\vec{H} = 0$, i.e, M is a minimal Lagrangian submanifold in C^2 .This is contradiction with the well known fact that there are no compact minimal submanifolds in the complex Euclidean space.

(ii) If $n > 2$, from (4.2) and (4.6), we have $\rho = 0$ on M, that is, M is totally umbilical, or $\frac{2n(n-1)}{n+2}|\nabla^\perp \vec{H}|^2 = (2 - \frac{1}{n})\rho^4$. In the latter case, if $\rho^2 = 0$ on M, we have M is totally umbilical. If $\rho^2 \neq 0$ on M, we know that the equality in (4.6) holds. Therefore, we have

(4.10) $$N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) = 2N(\tilde{A}_{m^*})N(\tilde{A}_{l^*}), \quad m^* \neq l^*,$$
$$n \sum_{m^*} \tilde{\sigma}_{m^*}^2 = (\sum_{m^*} \tilde{\sigma}_{m^*})^2,$$

that is

(4.11)
$$\tilde{\sigma}_{n+1} = \cdots = \tilde{\sigma}_{2n}.$$

We also have

(4.12)
$$\sum_{m^*,l^*} H^{m^*} H^{l^*} \tilde{\sigma}_{m^* l^*} = H^2 \rho^2.$$

From lemma 2.1, we know that at most two of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*}), m^* = n+1, \cdots, 2n$, are different from zero. If all of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*})$ are zero, which is contradiction with M is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, which is contradiction with (4.11). Therefore, we may assume that

$$\tilde{A}_{n+1} = \lambda \tilde{A}, \quad \tilde{A}_{n+2} = \mu \tilde{B}, \quad \lambda, \mu \neq 0,$$
$$\tilde{A}_{m^*} = 0, \qquad m^* \geq n+3,$$

where $\tilde{A}$ and $\tilde{B}$ are defined in lemma 2.1.

From (4.12), we have

$$\lambda^2 (H^{n+1})^2 + \mu^2 (H^{n+2})^2 = (\lambda^2 + \mu^2) \sum_{m^*} (H^{m^*})^2.$$

Since $\lambda, \mu \neq 0$, we infer that $H^{m^*} = 0, n+1 \leq m^* \leq 2n$, that is, $\vec{H} = 0$, i.e, M is a minimal Lagrangian submanifold in C^n. This is contradiction with the well known fact that there are no compact minimal submanifolds in the complex Euclidean space. Therefore, we complete the proof of Theorem 4.1.

From (3.13),(3.14),(3.15) and (3.16), we know that for any real number a, the following integral equality holds

(4.13)
$$\int_M \rho^{n-2} (|\nabla h|^2 - n |\nabla^{\perp} \vec{H}|^2) dv + (n-2) \int_M \rho^{n-2} |\nabla \rho|^2 dv$$
$$+ n \int_M \rho^{n-2} (H^2 \rho^2 - \sum_{m^*,l^*} H^{m^*} H^{l^*} \tilde{\sigma}_{m^* l^*}) dv - (a+1) n \int_M H^2 \rho^n dv$$
$$+ (1+a) \int_M \rho^{n-2} \sum_{m^*} \sum_{i,j,k,l} h_{ij}^{m^*} (h_{kl}^{m^*} R_{lijk} + h_{li}^{m^*} R_{lkjk}) dv$$

$$
-(1+a)n\int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{l^*}\tilde{h}_{kj}^{l^*}dv + a\int_M \rho^{n-2}\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 dv
$$
$$
-\frac{1-a}{2}\int_M \rho^{n-2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})dv = 0.
$$

Now, denote by K the function which assigns to each point of M the infimum of the sectional curvature at that point, we have the following

Theorem 4.2 *Let $\varphi : M \to C^n$ be an $n(n \geq 2)$-dimensional compact Willmore Lagrangian submanifold. Then there holds the follows*

$$
\int_M \rho^n\{K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\}dv \leq 0. \tag{4.14}
$$

In particular, if

$$
K \geq \frac{n-2}{\sqrt{n(n-1)}}H\rho + H^2, \tag{4.15}
$$

then M is totally umbilical.

Proof. For a fixed $m^*, n+1 \leq m^* \leq 2n$, we can take a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^{m^*} = \lambda_i^{m^*}\delta_{ij}$. Then, $\tilde{h}_{ij}^{m^*} = \mu_i^{m^*}\delta_{ij}$ with $\mu_i^{m^*} = \lambda_i^{m^*} - H^{m^*}$, $\sum_i \mu_i^{m^*} = 0$. Thus

(4.16)

$$
\begin{aligned}
\sum_{m^*,i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) =& \frac{1}{2}\sum_{m^*,i,j}(\lambda_i^{m^*} - \lambda_j^{m^*})^2 R_{ijij} \\
=& \frac{1}{2}\sum_{m^*,i,j}(\mu_i^{m^*} - \mu_j^{m^*})^2 R_{ijij} \geq nK\rho^2,
\end{aligned}
$$

and the equality in (4.16) holds if and only if $R_{ijij} = K$ for any $i \neq j$.

Let $\sum_i(\tilde{h}_{ii}^{l^*})^2 = \tau_{l^*}$. Then $\tau_{l^*} \leq \sum_{i,j}(\tilde{h}_{ij}^{l^*})^2 = \tilde{\sigma}_{l^*}$. Since $\sum_i \tilde{h}_{ii}^{l^*} = 0$, $\sum_i \mu_i^{m^*} = 0$ and $\sum_i(\mu_i^{m^*})^2 = \tilde{\sigma}_{m^*}$. We have from lemma 2.3 and lemma 2.4

$$
\begin{aligned}
\sum_{m^*,l^*}\sum_{i,j,k} H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{kj}^{l^*}\tilde{h}_{ik}^{l^*} =& \sum_{l^*,m^*}\sum_{i,j,k} H^{l^*}\tilde{h}_{ij}^{l^*}\tilde{h}_{kj}^{m^*}\tilde{h}_{ik}^{m^*} \\
=& \sum_{m^*,l^*} H^{l^*}\sum_i \tilde{h}_{ii}^{l^*}(\mu_i^{m^*})^2 \leq \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*,l^*}|H^{l^*}|\tilde{\sigma}_{m^*}\sqrt{\tau_{l^*}} \\
\leq& \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*}\tilde{\sigma}_{m^*}\sum_{l^*}|H^{l^*}|\sqrt{\tilde{\sigma}_{l^*}}
\end{aligned} \tag{4.17}
$$

$$\leq \frac{n-2}{\sqrt{n(n-1)}}\rho^2\sqrt{\sum_{l^*}(H^{l^*})^2\sum_{l^*}\tilde{\sigma}_{l^*}} = \frac{n-2}{\sqrt{n(n-1)}}H\rho^3.$$

From (3.3), we get

$$\sum_{m^*,l^*}\tilde{\sigma}^2_{m^*l^*} = \sum_{m^*}\tilde{\sigma}^2_{m^*} \geq \frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = \frac{1}{n}\rho^4. \tag{4.18}$$

From lemma 2.1, (3.2) and (3.3), we have

(4.19)

$$\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \leq 2\sum_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = 2(\sum_{m^*}\tilde{\sigma}_{m^*})^2 - 2\sum_{m^*}\tilde{\sigma}^2_{m^*}$$
$$\leq 2\rho^4 - 2\frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = 2\frac{n-1}{n}\rho^4.$$

Therefore, from (4.5),(4.13), lemma 2.2, (4.16)–(4.19), we obtain for $0 \leq a \leq 1$

(4.20)

$$0 \geq \int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv$$
$$+ n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})dv - (1+a)n\int_M H^2\rho^n dv$$
$$+ (1+a)\int_M \rho^{n-2}nK\rho^2 dv - (1+a)n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3 dv$$
$$+ a\int_M \rho^{n-2}\frac{1}{n}\rho^4 dv - (1-a)\int_M \rho^{n-2}\frac{n-1}{n}\rho^4 dv$$
$$\geq (1+a)n\int_M \rho^n(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2)dv$$
$$+ [\frac{a}{n} - (1-a)\frac{n-1}{n}]\int_M \rho^{n+2}dv.$$

Putting $a = \frac{n-1}{n}$, we have

$$0 \geq \int_M \rho^n\{K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\}dv. \tag{4.21}$$

From (4.15) and (4.21), we have $\rho = 0$, that is M is totally umbilical, or $K = \frac{n-2}{\sqrt{n(n-1)}}H\rho + H^2$. In the latter case, if $\rho^2 = 0$ on M, we have M is totally umbilical. If $\rho^2 \neq 0$ on M, then we have the equality in (4.21) holds. Therefore, we know that the equalities in (4.20) hold. Therefore the equalities in lemma 2.1,(4.18) and (4.19) hold. Since we know that M is not totally umbilical, we have (4.10)-(4.12) hold. By making use of the same assertion as in the proof of Theorem 4.1, we infer that M is a minimal Lagrangian submanifold in C^n. This is contradiction with the well known fact that there are no compact minimal submanifolds in the complex Euclidean space. Therefore, we complete the proof of Theorem 4.2.

Now, we consider the rigidity of Willmore Lagrangian submanifolds in terms of Ricci curvatures. Denote by Q the function which assigns to each point of M the infimum of the Ricci curvature at that point, we have the following

Lemma 4.1. *For any n-dimensional Lagrangian submanifold in C^n, there holds the follows*

$$\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \leq 4\{(n-2)H\rho + H^2 - Q\}\rho^2 - \frac{4}{n}\rho^4. \tag{4.22}$$

Proof. From Gauss equation (2.8) and (3.1), we have

$$R_{ik} = (n-2)\sum_{m^*} H^{m^*}\tilde{h}_{ik}^{m^*} + (n-1)H^2\delta_{ik} - \sum_{m^*,j}\tilde{h}_{ij}^{m^*}\tilde{h}_{jk}^{m^*}.$$

Thus, we get

$$R_{ii} = (n-2)\sum_{m^*} H^{m^*}h_{ii}^{m^*} + H^2 - \sum_{m^*,j}(\tilde{h}_{ij}^{m^*})^2. \tag{4.23}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{m^*} H^{m^*}h_{ii}^{m^*} \leq \sqrt{\sum_{m^*}(H^{m^*})^2}\sqrt{\sum_{m^*}(h_{ii}^{m^*})^2} \leq H\rho. \tag{4.24}$$

(4.23) and (4.24) infer that

$$Q \leq (n-2)H\rho + H^2 - \sum_{m^*,j}(\tilde{h}_{ij}^{m^*})^2. \tag{4.25}$$

Therefore, we have

$$\sum_{m^*\neq l^*,i}(\tilde{h}_{il}^{m^*})^2 \leq (n-2)H\rho + H^2 - Q - (\tilde{h}_{il}^{m^*})^2. \tag{4.26}$$

From (4.26) and $\tilde{h}_{ij}^{m^*} = \mu_i^{m^*}\delta_{ij}$, it is easy to see

$$\begin{aligned}\sum_{l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) &= \sum_{l^*\neq m^*,i,l} (\tilde{h}_{il}^{l^*})^2(\mu_i^{m^*} - \mu_l^{m^*})^2 \\ &\leq 4 \sum_{l^*\neq m^*,i,l} (\tilde{h}_{il}^{l^*})^2(\mu_l^{m^*})^2 \\ &\leq 4\sum_l \{(n-2)H\rho + H^2 - Q - (\mu_l^{m^*})^2\}(\mu_l^{m^*})^2 \\ &= 4\{(n-2)H\rho + H^2 - Q\}\sum_l (\mu_l^{m^*})^2 - 4\sum_l (\mu_l^{m^*})^4 \\ &\leq 4\{(n-2)H\rho + H^2 - Q\}\sum_l (\mu_l^{m^*})^2 - \frac{4}{n}(\sum_l (\mu_l^{m^*})^2)^2.\end{aligned}$$

Therefore, we know (4.22) holds. This completes the proof of lemma 4.1.

Theorem 4.3. *Let $\varphi : M \to C^n$ be an $n(n \geq 2)$-dimensional compact Willmore Lagrangian submanifold. Then there holds the follows*

$$\int_M \rho^n\{Q - \frac{n-4}{4n}\rho^2 - (n-2)H\rho - H^2\}dv \leq 0. \tag{4.27}$$

In particular, if

$$Q \geq \frac{n-4}{4n}\rho^2 + (n-2)H\rho + H^2, \tag{4.28}$$

then M is totally umbilical.

Proof. From (3.16), lemma 2.2, (3.3), (4.5) and lemma 4.1, we have
(4.29)

$$\begin{aligned}0 &\geq -\int_M \rho^{n-2}\{4((n-2)H\rho + H^2 - Q)\rho^2 - \frac{4}{n}\rho^4\}dv - \int_M \rho^{n-2}\rho^4 dv \\ &= 4\int_M \rho^n\{Q - \frac{n-4}{4n}\rho^2 - (n-2)H\rho - H^2\}dv,\end{aligned}$$

where we used

$$\sum_{m^*,l^*} \tilde{\sigma}_{m^*l^*}^2 = \sum_{m^*} \tilde{\sigma}_{m^*}^2 \leq (\sum_{m^*} \tilde{\sigma}_{m^*})^2 = \rho^4. \tag{4.30}$$

From (4.28) and (4.29), we conclude $\rho = 0$, that is M is totally umbilical, or

$$Q = \frac{n-4}{4n}\rho^2 + (n-2)H\rho + H^2.$$

In the latter case, if $\rho^2 = 0$, then M is totally umbilical; if $\rho^2 \neq 0$, we have the equalities in (4.29) and (4.30) hold. From $\sum_{m^*} \tilde{\sigma}_{m^*}^2 = (\sum_{m^*} \tilde{\sigma}_{m^*})^2$, we have

$\sum_{m^*\neq l^*} \tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = 0$. This implies that $(n-1)$ of the $\tilde{\sigma}_{m^*}$ must be zero. Since $\rho^2 = \sum_{m^*,i,j} (\tilde{h}_{ij}^{m^*})^2 \neq 0$ and $\tilde{\sigma}_{m^*} = \sum_{i,j}(\tilde{h}_{ij}^{m^*})^2$, we infer that $(n-1)$ of the $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*})$ must be zero so that $n = 1$. This is a contradiction for we assume that $n \geq 2$. We complete the proof of Theorem 4.3.

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