Notes on Hamiltonian Graphs and Hamiltonian-Connected Graphs*

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Abstract

Let $G = (V(G), E(G))$ be a graph and $\alpha(G)$ be the independence number of G. For a vertex $v \in V(G)$, $d(v)$ and $N(v)$ represent the degree and the neighborhood of v in G, respectively. In this paper, we prove that if G is a k-connected($k \geqslant 2$) graph of order n, and if $\max\{d(v) : v \in S\} \geqslant n/2$ for every independent set S of G with $|S| = k$ which has two distinct vertices $x, y \in S$ satisfying $1 \leqslant |N(x) \cap N(y)| \leqslant \alpha(G) - 2$, then either G is hamiltonian or else G belongs to one of a family of exceptional graphs. We also give a similar sufficient condition for Hamiltonian-connected graphs.

Key words: Hamiltonian graphs, Hamiltonian-connected graphs, Degree condition.

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1 Introduction and Results

Graphs considered here are simple and connected. For notation and terminology not defined here we refer to [2].

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Let $G = (V, E)$ be a simple graph. We use $V(G)$, $E(G)$, $\delta(G)$ and $\alpha(G)$ to denote its vertex set, edge set, *minimal degree* and *independence number*, respectively. The order of G is $|G| = |V|$ and its size is $e(G) = |E|$. We denote by $d(u, v)$ the *distance* between two vertices u and v, i.e., the length of the shortest path between u and v. If $u \in V(G)$ and H be a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$. Suppose that C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and we use $G - C$ to denote the subgraph of G induced by $V(G) - V(C)$. Let H be a subgraph or vertex subset of G. We use $G[H]$ to denote the subgraph of G induced by H. Let X be a path or a cycle of G, and $\overrightarrow{X}$ denote the set X with a given orientation. If $u, v \in V(X)$, then the subpath on $\overrightarrow{X}$ from u to v is denoted by $u\overrightarrow{X}v$. The same vertices, in reverse order, are given by $v\overleftarrow{X}u$. For $S \subseteq V(X)$, we use S^+ (resp. S^-) to denote the successors (resp. predecessors) of vertices of S on $\overrightarrow{X}$. Let uHv denote a $u - v$ path in which all internal vertices belong to H. If G contains $k (k \geq 2)$ vertex disjoint subgraphs $C_1, \ldots, C_k$, we use $\cup_{i=1}^{k} C_i$ to denote these k subgraphs. Let k_t denote a complete subgraph of order t in G. For two subgraphs G_1 and G_2 in G, we use $G_1 \vee G_2$ to denote the *join* of G_1 and G_2.

A graph G is *Hamiltonian* if it has a spanning cycle, and *Hamiltonian-connected* if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path. The following sufficient conditions to ensure the existence of a Hamiltonian cycle in a simple graph G of order $n \geqslant 3$ are well known.

Theorem 1.1(Dirac, [4]). *If $\delta(G) \geqslant n/2$, then G is Hamiltonian.*

Theorem 1.2(Ore, [7]). *If $d(u) + d(v) \geqslant n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Define the *k-closure of G* to be the graph obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least k, until no such pair remains. Using this definition, Bondy and Chavatal(see [1]) obtained Theorem 1.3.

Theorem 1.3(Bondy and Chavatal, [1]). *A graph G of order n is Hamiltonian if and only if its n-closure is Hamiltonian. A graph G of order n is Hamiltonian-connected if and only if its $(n+1)$-closure is Hamiltonian-connected.*

Fan [5] and Chen et al [3] obtained the following two results, respectively.

Theorem 1.4(Fan, [5]). *If G is a 2-connected graph and if* $\max\{d(u), d(v)\} \geqslant n/2$ *for each pair vertices* $u, v \in V(G)$ *with* $d(u, v) = 2$, *then G is Hamiltonian.*

Theorem 1.5(Chen et al, [3]). *If G is a k-connected($k \geqslant 2$) graph of order n and if* $\max\{d(v), v \in I\} \geqslant n/2$ *for every independent set I of order k, such that I has two distinct vertices x, y with* $d(x, y) = 2$, *then G is Hamiltonian.*

Since $d(u, v) = 2$ if and only if $|N(u) \cap N(v)| \geqslant 1$ for a pair of nonadjacent vertices u, v of G, Zhao et al unified and extended the above theorems and obtained the following result.

Theorem 1.6(Zhao et al, [9]). *If G is a k-connected($k \geqslant 2$) graph of order n and if* $\max\{d(v), v \in I\} \geqslant n/2$ *for every independent set I of order k, such that I has two distinct vertices x, y with* $1 \leqslant |N(x) \cap N(y)| \leqslant \alpha(G) - 1$, *then G is Hamiltonian.*

Recently, Yan et al also considered the Hamiltonian-connected case and got the following result.

Theorem 1.7(Yan et al, [10]). *If G is a k-connected($k \geqslant 3$) graph of order n and if* $\max\{d(v), v \in I\} \geqslant (n + 1)/2$ *for every independent set I of order $k - 1$ such that I has two distinct vertices x, y with* $1 \leqslant |N(x) \cap N(y)| \leqslant \alpha(G)$, *then G is Hamiltonian-connected.*

In this paper, we obtain the following two results which characterize the extreme case and extend Theorems 1.6 and 1.7. Our proof is standard and likewise the method in [9].

Theorem 1.8. *If G is a k-connected($k \geqslant 2$) graph of order n and if* $\max\{d(v), v \in I\} \geqslant n/2$ *for every independent set I of order k, such that I has two distinct vertices x, y with* $1 \leqslant |N(x) \cap N(y)| \leqslant \alpha(G) - 2$, *then either G is Hamiltonian or G is a spanning subgraph of the nonhamiltonian graph* $(\cup_{i=1}^{\alpha(G)} K_{n_i}) \vee K_{\alpha(G)-1}$, *where* $\sum_{i=1}^{\alpha(G)} n_i + \alpha(G) = n + 1$.

Theorem 1.9. *If G is a k-connected($k \geqslant 3$) graph of order n and if* $\max\{d(v), v \in I\} \geqslant (n+1)/2$ *for every independent set I of order $k-1$, such that I has two distinct vertices x, y with* $1 \leqslant |N(x) \cap N(y)| \leqslant \alpha(G)-1$, *then* *either G is Hamiltonian-connected graph or G is a spanning subgraph of the nonhamiltonian-connected graph $(\cup_{i=1}^{\alpha(G)} K_{n_i}) \vee K_{\alpha(G)}$, where $\sum_{i=1}^{\alpha(G)} n_i + \alpha(G) = n$.*

2 Two basic lemmas

In order to prove Theorems 1.8 and 1.9, we will use the following two lemmas:

Lemma 2.1(Shi et al, [8]). *Let G be a 2-connected graph of order n. Then G contains a cycle passing through all vertices of degree of at least $n/2$.*

Lemma 2.2(Menger, [6]). *A graph G has vertex connectivity c if and only if there exist there exist c internally vertex disjoint paths between any two vertices x, y of G.*

3 Proof of Theorem 1.8

Suppose that G is a nonhamiltonian graph which satisfies the hypotheses of Theorem 1.8. Let B=$\{v \in V(G)|d(v) \geqslant n/2\}$. We consider the graph $G' = (V(G), E(G'))$ with $E(G') = E(G) \cup \{uv \notin E(G)|u, v \in V(B)\}$. By Theorem 1.3, G' is also not a hamiltonian graph. For simplicity, throughout the rest of this section, G always denotes the graph G'.

Note that G is k-connected and $k \geqslant 2$, by Lemma 2.1, we may assume C to be a maximal cycle of G which contains all vertices of B and let H be a component of $G - V(C)$ with $N(H) \cap V(C)$ as large as possible. Let $v_1, v_2 \cdots v_l$ be the elements of $N_C(H)$ occuring on $\overrightarrow{C}$ in consecutive order and let $x_i \in N(v_i) \cap V(H)$ for $i = 1, 2, \cdots, l$. Since G is k-connected and $k \geqslant 2$, for any $i, j \in \{1, 2, \cdots, l\}$ with $i \neq j$, the path $v_i^+ \overrightarrow{C} v_j H v_i \overleftarrow{C} v_j^+$ contains at least one vertex of H and contains all vertices of C, thus by the maximality of C, $v_i^+ v_j^+ \notin E(G)$. It follows from the definition of $N_C^+(H)$ that

$$\text{For any } i \text{ with } 1 \leqslant i \leqslant l,\ \{x_i\} \cup N_C^+(H) \text{ is an independent set.} \qquad (1)$$

Also, since $G[B]$ is a clique, $|N_C^+(H) \cap B| \leqslant 1$. Without loss of generality, we may assume that $d(v_i^+) < n/2$ for $i = 1, 2 \cdots, l-1$, that is $d(x_i, v_i^+) = 2$ for every $i \neq l$. On the other hand, by the choice of H and

Lemma 2.2, $l \geqslant k$. This implies that for $i \in \{1, 2, \cdots, l-1\}$, there exists $V^* \subseteq N_C^+(H)$ with $v_i^+ \in V^*$ and with $|V^*| = k-1$, by (1), $\{x_i\} \cup V^*$ is an independent set of G with $|\{x_i\} \cup V^*| = k$ and $d(x_i, v_i^+) = 2$. Since $\max\{d(v), v \in \{x_i\} \cup V^*\} < n/2$, by the hypothesis of Theorem 1.8, we know that

$$|N(x_i) \cap N(v_i^+)| \geqslant \alpha(G) - 1, \quad i = 1, 2 \cdots, l-1. \tag{2}$$

Note that $N(x_i) \cap N(v_i^+) \subseteq \{v_1, v_2, \cdots, v_l\}$, combine with (2), the following two statements hold.

$$\alpha(G) = l+1, \quad \text{and} \quad N(x_i) \cap N(v_i^+) = \{v_1, v_2, \cdots, v_l\}, \quad i = 1, 2 \cdots, l-1. \tag{3}$$

$$\text{If} \quad d(v_l^+) < n/2, \quad \text{then} \quad N(x_l) \cap N(v_l^+) = \{v_1, v_2, \cdots, v_l\}, \tag{4}$$

For $i \neq j$, let $R = v_i^+ \overrightarrow{C} v_j^+$ and $T = V(C) - R$. We conclude that

$$N_R^-(v_i^+) \cap N_R(v_j^+) = \emptyset. \tag{5}$$

Proof of (5). By contradiction. Suppose that there exists $v \in (N_R^-(v_i^+) \cap N_R(v_j^+))$. By (1), $v \neq v_i^+$ and $v \neq v_j$, then we see that $v_i H v_j \overleftarrow{C} v^+ v_i^+ \overrightarrow{C} v v_j^+ \overrightarrow{C} v_i$ is a cycle which contains all vertices of B and longer than C, and this contradiction proves (5). □

A similar argument proves that the following statement also holds.

$$N_T^+(v_i^+) \cap N_T(v_j^+) = \emptyset. \tag{6}$$

Next we shall give a characterization of G by showing a series of claims. Set $V_0 = V(H)$ and $V_i = v_i^+ \overrightarrow{C} v_{i+1}^-$ for $i = 1, 2 \cdots, l$, (indices taken modulo l).

$$\text{For each } i = 0, 1, \cdots, l, \ G[V_i] \quad \text{is complete.} \tag{7}$$

Proof of (7). Suppose that $G[V_0]$ is not complete, then $G[V_0]$ has a pair of vertices a and b with $d(a, b) = 2$, let $S_{ab} = \{a, b\} \cup N_C^+(H)$, then S_{ab} is an independent set of cardinality $l+1$, contradicting (3). If $|V_i| = 2$ for some $i \neq 0$, then we have nothing to prove. So assume for all $i \neq 0$, $|V_i| \geqslant 3$. We first consider the case when $i \neq l$. According to (3), we see that $v_i^+ v_{i+1} \in E(G)$ for $i \neq l$, applying (5), $v_{i+1}^- \notin N(v_j^+)$ for every $j \neq i$ with $1 \leqslant j \leqslant l$. Thus, by (1) and (3), $v_i^+ v_{i+1}^- \in E(G)$, otherwise $\{x_i, v_{i+1}^-\} \cup N_C^+(H)$ would be an independent set of cardinality $l+2$ since $|V_i| \geqslant 3$, contradicting (3). Continuing the process if $|V_i - \{v_{i+1}^-\}| \geqslant 3$ for $i \in \{1, 2, \cdots, l-1\}$, we shall eventually obtain that

$$V_i - \{v_i^+\} \subset N(v_i^+) \quad \text{for} \quad i = 1, 2, \cdots, l-1. \tag{8}$$

If $G[V_i]$ is not complete for some $i \in \{1, 2, \cdots, l-1\}$, then by (8), there exist two vertices $u, v \in V_i - \{v_i^+\}$ such that $uv \notin E(G)$. It is easy to see that $u^+, v^+ \notin \{u, v\}$. From (8) again, we see that $u^+, v^+ \in N(v_i^+)$, combining with (5), we have

$$(N(u) \cup N(v)) \cap (N_C^+(H) - \{v_i^+\}) = \emptyset. \tag{9}$$

Consequently, by (1) again, $\{u, v, x_1\} \cup (N_C^+(H) - \{v_i^+\})$ is an independent set of cardinality $l+2$, and this contradiction shows that $G[V_i]$ is complete for all $i \neq l$. It remains to prove that $G[V_l]$ is complete. We conclude $v_1^- v_l^+ \in E(G)$. If $d(v_l^+) < n/2$, then as above, $G[V_l]$ is complete, we have completed the proof of (7), and so assume that $d(v_l^+) \geqslant n/2$. If $d(v_1^-) < n/2$, by an symmetry of C, $\{x_1\} \cup N_C^-(H)$ is also a maximum independent set of G, therefore, $V_l - \{v_1^-\} \subset N(v_1^-)$. We just need to consider the case when $d(v_1^-) \geqslant n/2$. By the assumption of G, $\{v_l^+, v_1^-\} \in B$, so $v_1^- v_l^+ \in E(G)$. According to (5), we obtain that $N(v_1^{--}) \cap \{x_i, v_1^+, \cdots, v_{l-1}^+\} = \emptyset$, this implies that $v_l^+ v_1^{--} \in E(G)$ if $|V_l| \geqslant 4$, otherwise by (1), $\{x_i, v_1^{--}\} \cup N_C^+(H)$ would be an independent set of cardinality of $l+2$, which contradicts (3). Continuing the same process, we finally obtain that $V_l - \{v_l^+\} \subset N(v_l^+)$. Use a similar argument as the case $i \neq l$, we see that $G[V_l]$ is complete. Therefore, (7) is verified. □

$$\text{For any } i, j \in \{0, 1, \cdots, l\} \text{ with } i \neq j,\ N(V_i) \cap V_j = \emptyset. \tag{10}$$

Proof of (10). Otherwise, assume that there exist two vertices $u \in V_i$ and $u \in V_j$ with $i \neq j$ such that $uv \in E(G)$. By the choice of H and C, $i, j \neq 0$. Without loss of generality, suppose that $i \neq l$. However, we observe that

$$C' = \begin{cases} uv\overleftarrow{C}v_j^+ v^+ \overrightarrow{C} v_i H v_j \overrightarrow{C} u^- v_i^+ \overrightarrow{C} u & \text{if } v \neq v_{j+1}^- \\ uv\overleftarrow{C} v_{i+1} H v_{j+1} \overrightarrow{C} u^- v_{i+1}^- \overleftarrow{C} u & \text{if } v = v_{j+1}^- \end{cases}$$

is a cycle longer than C, a contradiction. □

We prove the following statement to complete the proof of Theorem 1.8.

There is only one component H of $G - V(C)$, i.e. $V(G) = V(C) \cup V(H)$. (11)

Proof of (11). Suppose for a contradiction, let H' be another component of $G - V(C)$ and $y \in V(H')$. Note that $N(y) \cap N_C^+(H) \neq \emptyset$ and $N(y) \cap N_C^-(H) \neq \emptyset$, otherwise, $\{x_1, y\} \cup N_C^+(H)$ or $\{x_1, y\} \cup N_C^-(H)$ would be an independent set of cardinality of $l + 2$, which contradicts (3). Furthermore, if there exist two vertices $\{v_i^+, v_j^+\} \subseteq N_C^+(H) \cap N(y)$ with $i \neq j$, then $v_i^+ y v_j^+ \overrightarrow{C} v_i H v_j \overleftarrow{C} v_i^+$ is a cycle which contains all vertices of C and longer than C, a contradiction, thus $|N_C^+(H) \cap N(y)| = 1$. Similarly, $|N_C^-(H) \cap N(y)| = 1$. Without loss of generality, assume that $yv_1^+ \in E(G)$ and $yv_i^- \in E(G)$ for some $i \in \{1, 2, \cdots, l\}$. If either $|V_1| \geqslant 2$ or $|V_{i-1}| \geqslant 2$, then by (1) and (10), $\{v_1^{++}, x_1, y\} \cup (N_C^+(H) - \{v_1^+\})$ or $\{v_i^{--}, x_1, y\} \cup (N_C^-(H) - \{v_i^-\})$ would be an independent set of cardinality of $l + 2$, again a contradiction, thus we must have $|V_1| = |V_{i-1}| = 1$. Since G is k-connected graph and $k \geqslant 2$, we may assume that $i \neq 2$. By (3), $v_1^+ y v_i^- \overleftarrow{C} v_2 H v_1 \overleftarrow{C} v_i v_1^+$ is a longer cycle than C, a contradiction, thus (11) is proved. □

Set $|V_i| = n_i$ for $i = 0, 1, \cdots, l$, by combining the statements (7), (10) and (11), it is easy to see that G is a spanning subgraph of the nonhamiltonian graph $(\cup_{i=0}^{l} K_{n_i}) \vee K_l = (\cup_{i=1}^{\alpha(G)} K_{n_i}) \vee K_{\alpha(G)-1}$, the proof of Theorem 1.8 is completed. □

4 Proof of Theorem 1.9

Suppose that G is a nonhamiltonian-connected graph which satisfies the hypothesis of Theorem 1.9. Set B=$\{v \in V(G) | d(v) \geqslant (n+1)/2\}$. We consider the graph $G' = (V(G), E(G'))$ with $E(G') = E(G) \cup \{uv \notin E(G) | u, v \in V(B)\}$. For simplicity, throughout the rest of this section, G always denotes the graph G'.

According to Theorem 1.3, G is also not Hamiltonian-connected. Thus, there exists a pair of vertices u, v of G such that no hamiltonian $u-v$ path in G exists. We claim that G contains a $u-v$ path through all vertices of B. Otherwise, let us suppose, for the moment, the above statement is false. By Theorems 1.6 and 1.3, G is Hamiltonian. Let C' be a hamiltonian cycle in G. If $B \subseteq V(u\overrightarrow{C'}v)$ or $B \subseteq V(v\overrightarrow{C'}u)$, then it is easy to see that G contains a $u-v$ path through all vertices of B, a contradiction. Hence, we may assume that B can be partitioned into B_1 and B_2, such that $B_1 \subseteq V(u\overrightarrow{C'}v)$ and $B_2 \subseteq V(v\overrightarrow{C'}u)$. By our construction of G, each vertex in B_1 is adjacent to every vertex in B_2, and vice versa. Choose $v'_1 \in B_1$ with $v'_1 \neq v$ such that $|V(u\overrightarrow{C'}v'_1)|$ is as large as possible. Subject to this requirement, we further choose $v'_2 \in B_2$ with $v'_2 \neq u$ such that $|V(v'_2\overrightarrow{C'}u)|$ is as small as possible. Then $u\overrightarrow{C'}v'_1v'_2\overleftarrow{C'}v$ is a $u-v$ path through all vertices of B, a contradiction.

From the claim above, let P be a maximal $u-v$ path containing B and let H be a component of $G-V(P)$ with $N(H)\cap V(P)$ as large as possible. Let $v_1, v_2 \cdots v_l$ be the elements of $N_P(H)$ occuring on $\overrightarrow{P}$ in consecutive order and let $x_i \in N(v_i)\cap V(H)$ for $i = 1, 2, \cdots, l$. Since G is k-connected and $k \geqslant 3$, we have

$$\text{for any } x \in V(H),\ \{x\}\cup N_P^+(H) \text{ and } \{x\}\cup N_P^-(H) \text{ are independent sets.} \tag{12}$$

Proof of (12). Since P be a maximal $u-v$ path, for any $x \in V(H)$, $N(x)\cap N_P^+(H) = \emptyset$. If there exists $v_i^+, v_j^+ \in N_P^+(H)$ such that $v_i^+v_j^+ \in E(G)$, we assume that $i < j$, then we see that $u\overrightarrow{P}v_iHv_j\overleftarrow{P}v_i^+v_j^+\overrightarrow{P}v$ is a $u-v$ path contains all vertices of B and longer than P, a contradiction. Thus $\{x\}\cup N_P^+(H)$ is an independent set and $\{x\}\cup N_P^-(H)$ is also one by symmetry. □

Also, since $G[B]$ is a clique, $\left|N_P^+(H)\cap B\right| \leqslant 1$. Without loss of generality, we may assume that $d(v_i^+) < (n+1)/2$ for $i = 1, 2\cdots, l-1$, that is $d(x_i, v_i^+) = 2$ for every $i \neq l$. On the other hand, by the choice of H and Lemma 2.2, $l \geqslant k$. This implies that for $i \in \{1, 2, \cdots, l-1\}$, there exists $V^* \subseteq N_P^+(H)$ with $v_i^+ \in V^*$ and with $|V^*| = k-2$, by (12), $\{x_i\}\cup V^*$ is an independent set of G with $|\{x_i\}\cup V^*| = k-1$ and $d(x_i, v_i^+) = 2$. Note that $\max\{d(v), v \in \{x_i\}\cup V^*\} < (n+1)/2$, by the hypotheses of Theorem 1.9, we know that

$$\left|N(x_i)\cap N(v_i^+)\right| \geqslant \alpha(G), \quad i = 1, 2\cdots, l-1. \tag{13}$$

Since $N(x_i)\cap N(v_i^+) \subseteq \{v_1, v_2, \cdots, v_l\}$, in view of (13), we see that the following three statements must hold.

$$\alpha(G) = l, \quad \text{and} \quad v_1 = u \quad \text{and} \quad v_l = v. \tag{14}$$

$$N(x_i)\cap N(v_i^+) = \{v_1, v_2, \cdots, v_l\}, \quad i = 1, 2\cdots, l-1. \tag{15}$$

$$\text{If} \quad d(v_l^+) < (n+1)/2, \quad \text{then} \quad N(x_l)\cap N(v_l^+) = \{v_1, v_2, \cdots, v_l\}, \tag{16}$$

For $i \neq j$, let $R = v_i^+ \overrightarrow{P} v_j^+$ and $T = V(P) - R$. By using an similar arguments as the proof of Theorem 1.8, we can show the following statements (17) through (20) as given below hold.

$$N_R^-(v_i^+) \cap N_R(v_j^+) = \emptyset \text{ and } N_T^+(v_i^+) \cap N_T(v_j^+) = \emptyset. \quad (17)$$

Put $V_0 = V(H)$ and $V_i = v_i^+ \overrightarrow{P} v_{i+1}^-$ for $i = 1, 2 \cdots, l-1$, we have

$$\text{For each } i = 0, 1, \cdots, l-1, \; G[V_i] \text{ is complete.} \quad (18)$$

$$\text{For } i, j = 0, 1, \cdots, l-1 \text{ with } i \neq j, \; N(V_i) \cap V_j = \emptyset. \quad (19)$$

$$V(G) = V(P) \cup V(H). \quad (20)$$

Set $|V_i| = n_i$ for $i = 0, 1, \cdots, l-1$, combining with (18), (19) and (20), it is easy to see that G is a spanning subgraph of the graph $(\cup_{i=0}^{l-1} K_{n_i}) \vee K_l = (\cup_{i=1}^{\alpha(G)} K_{n_i}) \vee K_{\alpha(G)}$. Obviously, the graph $(\cup_{i=1}^{\alpha(G)} K_{n_i}) \vee K_{\alpha(G)}$ is not hamiltonian-connected, thus we have completed the proof of Theorem 1.9. □

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