ANGLE OF CONTACT OF LENS AND LUNAR MAPS AND PRODUCTS OF COMPOSITION AND ITERATED DIFFERENTIATION

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ABSTRACT. In this paper, we characterize boundedness and compactness of products of composition operators induced by the lens and the lunar maps and iterated differentiation acting between Hardy and weighted Bergman spaces of the unit disk in terms of angle of contact of these maps with the unit circle.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measureon $\mathbb{D}$ (i.e. $A(\mathbb{D}) = 1$) and $H(\mathbb{D})$ the class of all holomorphic functionson $\mathbb{D}$. Recall that the Hardy space $\mathcal{H}^2$ is the space of holomorphic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with the norm defined as

$$||f||_{\mathcal{H}^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 < \infty.$$

For each $\alpha \in (-1, \infty)$, let

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

Then dA_α is a probability measure on $\mathbb{D}$. Let $\mathcal{A}_\alpha^2$ be the weighted Bergman space of holomorphic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with the norm defined as

$$||f||_{\mathcal{A}_\alpha^2}^2 = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) = \sum_{k=0}^{\infty} \frac{k!\Gamma(2+\alpha)}{\Gamma(k+2+\alpha)} |a_k|^2 \asymp \sum_{k=0}^{\infty} \frac{|a_k|^2}{(n+1)^{\alpha+1}} < \infty,$$

where by $a \asymp b$, we mean that there is a positive constant C such that $a/C \le b \le aC$. The limiting case, as $\alpha \to -1$, of these spaces is the Hardy

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space $\mathcal{H}^2$. As in [8], it is natural to denote $\mathcal{A}^2_{-1}$ to be the Hardy space $\mathcal{H}^2$. Thus by considering

$$\|f\|^2_{\mathcal{A}^2_\alpha} \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{\alpha+2}dA(z), \quad \alpha \geq -1, \tag{1}$$

we can give a unified treatment of Hardy and weighted Bergman spaces. The Hardy and Bergman spaces are Hilbert spaces of holomorphic functions on $\mathbb{D}$ with the reproducing kernel K_a^α, namely, for every $a \in \mathbb{D}$:

$$f(a) = \langle f, K_a^\alpha \rangle \text{ for all } f \in \mathcal{A}^2_\alpha \ (\alpha \geq -1).$$

The Bergman kernel and its norm are explicitly given by

$$K_a^\alpha(z) = \left(\frac{1}{1-\bar{a}z}\right)^{\alpha+2} \text{ and } \|K_a^\alpha\|^2_{\mathcal{A}^2_\alpha} = K_a^\alpha(a) = \left(\frac{1}{1-|z|^2}\right)^{\alpha+2} \quad (\alpha \geq -1),$$

respectively. It is well known that $f \in \mathcal{A}^2_\alpha$ if and only if $f^{(n)}(z)(1-|z|^2)^n \in \mathcal{L}^2(dA_\alpha)$ for all $n \in \mathbb{N}$ and

$$\|f\|_{\mathcal{A}^2_\alpha} \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)| + \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p(1-|z|^2)^{2n+\alpha}dA(z)\right)^{1/2}. \tag{2}$$

That is, if $f \in \mathcal{A}^2_\alpha$, then $f^{(n)} \in \mathcal{A}^2_{2n+\alpha}$ and $\|f^{(n)}\|_{\mathcal{A}^2_{2n+\alpha}} \leq C\|f\|_{\mathcal{A}^2_\alpha}$.

For $0 < \gamma < 1$, we consider two self-maps φ_γ and ψ_γ of $\mathbb{D}$ defined respectively, as

$$\varphi_\gamma(z) = 1-(1-z)^\gamma \quad \text{and} \quad \psi_\gamma(z) = \frac{(\lambda(z))^\gamma - 1}{(\lambda(z))^\gamma + 1},$$

where $\lambda(z) = (1+z)/(1-z)$. The maps φ_γ has an angle of contact of $\gamma\pi$ at 1 on $\partial\mathbb{D}$ and is known as the lunar map, whereas the map ψ_γ has an angle of contact of $\gamma\pi$ at 1 and -1 on $\partial\mathbb{D}$ and is known as the lens map.

Let φ be a holomorphic self-map of $\mathbb{D}$. For a non-negative integer n, we define a linear operator D^n_φ as follows:

$$D^n_\varphi f = f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

If $n = 0$, then we have $D^n_\varphi = C_\varphi$, the composition operator induced by φ, defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. By a consequence of Littlewood's subordination principle, we see that every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on Hardy spaces as well as on weighted Bergman spaces. For more about composition operators on Hardy and weighted Bergman spaces, see [1]. MacCluer and Shapiro [4] showed that C_φ is compact on weighted Bergman spaces if and only if φ does not have angular derivative at any point of the unit circle. However, non-existence of the angular derivative is a necessary but not a sufficient condition for compactness of C_φ on Hardy spaces in general. Thus for an arbitrary φ the compactness of C_φ on Hardy spaces is quite different from the compactness of C_φ on weighted Bergman spaces. MacCluer and Shapiro [4] gave a nice example of a holomorphic self-map of $\mathbb{D}$ which induces a compact composition operator on weighted Bergman spaces but does not induce a compact composition operator on Hardy spaces. The non-existence of angular derivative characterizes the compactness of C_φ on $\mathcal{H}^p$ if the inducing map φ is univalent. Thus for univalent holomorphic self-maps φ, C_φ is compact on $\mathcal{H}^p$ if and only if it is compact on $\mathcal{A}^p_\alpha$ $(\alpha > -1)$. Since the lens and the lunar maps are univalent and have no angular derivative, so these maps simultaneously induces compact composition operators on the Hardy spaces as well as on the Bergman spaces whatever the angle of contact.

On the other hand, if $\varphi(z) = z$, then we have that $D^n_\varphi = D^n$. Since $f^{(n)} \in \mathcal{A}^p_{np+\alpha}$ for each $f \in \mathcal{A}^p_\alpha$, we see that $D^n(\mathcal{A}^p_\alpha) \not\subset \mathcal{A}^p_\alpha$. This means that the differentiation operator D^n is unbounded on $\mathcal{A}^p_\alpha$. In general, the differentiation operator is not bounded on the space of analytic functions on $\mathbb{D}$. Thus the product-type operators of composition and differentiation operators acting on analytic function spaces have been studied by some authors. One of the interesting problem on this product-type operators is to investigate the relation between the operator theoretic properties of these operators and the function theoretic properties of the symbol map φ.

For these studies, we can refer to papers [2,3,5–7,9–12]. Hibschweiler and Portnoy [2] characterized the boundedness and compactness of D^1_φ between weighted Bergman spaces in terms of the Carleson-type measures. Recently, S. Stević [9] proved that the boundedness and compactness of the operator D^n_φ on $\mathcal{A}^2_\alpha$ are characterized by the behavior of the generalized Nevanlinna counting function associated with the self-map φ and the orders n of differentiation.

In this paper, we show that the boundedness and compactness of D^n_φ induced by a univalent self-map φ are characterized in terms of the behavior of

$$\frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}},$$

and notice that the angle of contact of the lens and the lunar maps plays a significant role in the compactness of products of composition operators induced by these maps and iterated differentiation acting on Hardy and weighted Bergman spaces.

2. Boundedness and Compactness of D^n_φ

In this section, we characterize boundedness and compactness of D^n_φ acting on Hardy and weighted Bergman spaces.

Lemma 2.1. *Let $-1 \le \alpha < \infty$, n be a non-negative integer and φ be a holomorphic self-map of $\mathbb{D}$ such that $D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ is bounded. Then*

$D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ is compact if and only if for any sequence $\{f_m\}$ in $\mathcal{A}^2_\alpha$ with $\sup_{m\in\mathbb{N}} \|f_m\|_{\mathcal{A}^2_\alpha} = M < \infty$ and which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\lim_{m\to\infty} \|D^n_\varphi f_m\|_{\mathcal{A}^2_\alpha} = 0$.

The proof follows from standard arguments, for example, to those outlined in Proposition 3.11 of [1]. We omit the details.

Theorem 2.1. *Let $\alpha \geq -1$, n be a non-negative integer and φ be a univalent self-map of $\mathbb{D}$. Then*

(i) *$D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}} < \infty; \tag{3}$$

(ii) *$D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}} = 0. \tag{4}$$

Proof. We will prove (ii) only. Let us assume that φ be a univalent self-map of $\mathbb{D}$ and satisfies condition in (4). Let $\{f_m\}$ be a bounded sequence in $\mathcal{A}^2_\alpha$ that converge to zero uniformly on compact subsets of $\mathbb{D}$. In view of Lemma 2.1, our goal is to show that $\|D^n_\varphi f_m\|_{\mathcal{A}^2_\alpha} \to 0$. Without loss of generality we may assume that $\|f_m\|_{\mathcal{A}^2_\alpha} \leq 1$ for all m. Let $\epsilon > 0$ be given. Then by condition (4), there exists an r_0 in $(0,1)$ such that

$$(1-|z|)^{2+\alpha} \leq \epsilon(1-|\varphi(z)|)^{2+\alpha+2n} \quad \text{for } r_0 < |z| < 1. \tag{5}$$

By (1), we have

$$\begin{aligned}
||D^n_\varphi f_m||^2_{\mathcal{A}^2_\alpha} &\asymp |f^{(n)}_m(\varphi(0))|^2 + \int_{\mathbb{D}} |f^{(n+1)}_m(\varphi(z))|^2 |\varphi'(z)|^2 dA_{\alpha+2}(z) \\
&= |f^{(n)}_m(\varphi(0))|^2 \\
&\quad + \Big(\int_{r_0\overline{\mathbb{D}}} + \int_{\mathbb{D}\setminus r_0\overline{\mathbb{D}}}\Big) |f^{(n+1)}_m(\varphi(z))|^2 |\varphi'(z)|^2 dA_{\alpha+2}(z),
\end{aligned} \tag{6}$$

where $r_0\overline{\mathbb{D}} = \{w \in \mathbb{D} : |z| \leq r_0\}$. Since $f^{(n)}_m \circ \varphi \to 0$ uniformly on compact subsets of $\mathbb{D}$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, the first and the second term of (6) converges to zero, so in what follows we simply denote them by $o(1)$. We can estimate the second integral in (6), by successively using inequality (5), replacing the annulus $\mathbb{D}\setminus r_0\overline{\mathbb{D}}$ by $\mathbb{D}$ and changing the variable $w = \varphi(z)$. Using these steps $||D^n_\varphi f_m||^2_{\mathcal{A}^2_\alpha}$ can be estimated as

$$\begin{aligned}
||D^n_\varphi f_m||^2_{\mathcal{A}^2_\alpha} &\leq o(1) + \int_{\mathbb{D}\setminus r_0\overline{\mathbb{D}}} |f^{(n+1)}_m(\varphi(z))|^2 |\varphi'(z)|^2 dA_{\alpha+2}(z) \\
&\leq o(1) + \epsilon \int_{\mathbb{D}} |f^{(n+1)}_m(\varphi(z))|^2 (1-|\varphi(z)|)^{2+\alpha+2n} |\varphi'(z)|^2 dA(z)
\end{aligned}$$

$$\leq o(1) + \epsilon \int_{\mathbb{D}} |f_m^{(n+1)}(z)|^2 (1-|z|)^{2+\alpha+2n} dA(z)$$
$$\leq o(1) + \epsilon \|f_m\|^2_{\mathcal{A}^2_\alpha} \leq o(1) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\|D^n_\varphi f_m\|_{\mathcal{A}^2_\alpha} \to 0$, as desired. Conversely, suppose that $D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha$ is compact. For $\lambda \in \mathbb{D}$, we consider the function

$$f_\lambda(z) = \frac{(1-|\lambda|^2)^{(\alpha+2)/2}}{(1-\overline{\lambda}z)^{\alpha+2}}. \tag{7}$$

Then $f_\lambda \in \mathcal{A}^2_\alpha$ and $\|f_\lambda\|_{\mathcal{A}^2_\alpha} = 1$. Moreover, f_λ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $\|(D^n_\varphi)^* f_\lambda\|_{\mathcal{A}^2_\alpha} \to 0$ as $|\lambda| \to 1$. Now

$$\langle f, (D^n_\varphi)^* K^\alpha_\lambda \rangle = \langle D^n_\varphi f, K^\alpha_\lambda \rangle = \left\langle f^{(n)} \circ \varphi, K^\alpha_\lambda \right\rangle = f^{(n)}(\varphi(\lambda)).$$

Also recall (see, Theorem 2.16 in [1]) that for each $\lambda \in \mathbb{D}$ and non-negative integer n, the evaluation of the nth derivative of functions in $\mathcal{A}^2_\alpha$ at λ is a bounded linear functional and

$$f^{(n)}(\lambda) = \left\langle f, (K^\alpha_\lambda)^{(n)} \right\rangle.$$

Thus we have

$$(D^n_\varphi)^* K^\alpha_\lambda = (K^\alpha_{\varphi(\lambda)})^{(n)}.$$

One can easily check that

$$\|(D^n_\varphi)^* f_\lambda\|^2_{\mathcal{A}^2_\alpha} = (1-|\lambda|^2)^{\alpha+2} \|(K^\alpha_{\varphi(\lambda)})^n\|^2_{\mathcal{A}^2_\alpha}. \tag{8}$$

Again

$$\|(K^\alpha_{\varphi(\lambda)})^{(n)}\|^2_{\mathcal{A}^2_\alpha} = \left\langle (K^\alpha_{\varphi(\lambda)})^{(n)}, (K^\alpha_{\varphi(\lambda)})^{(n)} \right\rangle$$
$$= (K^\alpha_{\varphi(\lambda)})^{(2n)}(\varphi(\lambda)) = C(\alpha, n) \frac{|\varphi(\lambda)|^{2n}}{(1-|\varphi(\lambda)|^2)^{\alpha+2+2n}}. \tag{9}$$

Combining (8) and (9), we get

$$\lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-|\varphi(\lambda)|)^{2+\alpha+2n}} |\varphi(\lambda)|^{2n} = 0. \tag{10}$$

It is clear that (4) implies (10). To complete the proof, we claim that (10) implies (4). We distinguish two cases:
Case (a) $|\varphi(\lambda)| \to 1$ as $|\lambda| \to 1$.
Case (b) $|\varphi(\lambda)| \to r_0 < 1$ as $|\lambda| \to 1$.
Now if, Case (a) holds, then we have

$$\lim_{|\lambda|\to 1} |\varphi(\lambda)|^{2n} = 1. \tag{11}$$

Thus from (10) and (11), we have

$$\lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-|\varphi(\lambda)|)^{2+\alpha+2n}} = \lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-|\varphi(\lambda)|)^{2+\alpha+2n}} \lim_{|\lambda|\to 1} |\varphi(\lambda)|^{2n}$$

$$= \lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-|\varphi(\lambda)|)^{2+\alpha+2n}}|\varphi(\lambda)|^{2n} = 0.$$

On the other hand, if as $|\lambda| \to 1$, we have $|\varphi(\lambda)| \to r_0 < 1$, then

$$\lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-|\varphi(\lambda)|)^{2+\alpha+2n}} = \lim_{|\lambda|\to 1} \frac{(1-|\lambda|)^{2+\alpha}}{(1-r_0)^{2+\alpha+2n}} = 0.$$

Combining above inequalities, we get the desired result. □

Note that using the test function in (7) and proceeding as in proof of Theorem 2.1 (ii), we can prove that $D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{2+\alpha}}{(1-|\varphi(z)|^2)^{2+\alpha+2n}}|\varphi(z)|^{2n} < \infty. \tag{12}$$

Since (3) $\Rightarrow$ (12) is trivial, so to complete the proof, we need to prove that (12) $\Rightarrow$ (3). By (12), we have

$$\sup_{1/2<|\varphi(z)|<1} \frac{(1-|z|^2)^{2+\alpha}}{(1-|\varphi(z)|^2)^{2+\alpha+2n}} < \infty. \tag{13}$$

Moreover,

$$\sup_{0\le|\varphi(z)|\le 1/2} \frac{(1-|z|^2)^{2+\alpha}}{(1-|\varphi(z)|^2)^{2+\alpha+2n}} \le 2^{2+\alpha+2n}(1-|z|^2)^{2+\alpha} < \infty. \tag{14}$$

Combining (13) and (14), we get (3).

3. D_φ^n INDUCED BY LENS OR LUNAR MAPS

Lemma 3.1. *Let φ_γ be the lunar map. Then*

$$(1-|z|)^\gamma \le 1-|\varphi_\gamma(z)| \le |1-z|^\gamma.$$

Proof. Using the elementary inequality $|a|-|b| \le |a-b|$, we have

$$1-|\varphi_\gamma(z)| \le |1-(1-(1-z)^\gamma)| = |1-z|^\gamma.$$

Again

$$1-(1-z)^\gamma = \gamma z - \frac{\gamma(\gamma-1)}{2!}z^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}z^3 - \cdots.$$

Therefore,

$$|1-(1-z)^\gamma| \le \gamma|z| + \frac{\gamma(1-\gamma)}{2!}|z|^2 + \frac{\gamma(1-\gamma)(2-\gamma)}{3!}|z|^3 + \cdots.$$

Thus

$$\begin{aligned} 1-|1-(1-z)^\gamma| &\ge 1-\gamma|z| + \frac{\gamma(\gamma-1)}{2!}|z|^2 - \frac{\gamma(\gamma-1)(\gamma-2)}{3!}|z|^3 + \cdots \\ &= (1-|z|)^\gamma. \end{aligned}$$

□

Since the angle of contact of the lens map ψ_γ at 1 is equal to the angle of contact of the lunar map φ_γ at 1 (geometrical shape near 1 is same). So we conclude that if $|z|$ is near to 1, then we also have

$$(1-|z|)^\gamma \le 1-|\psi_\gamma(z)| \le |1-z|^\gamma.$$

Thus we conclude that if $\varphi = \varphi_\gamma$ or ψ_γ and $|z| \to 1$, then

$$(1-|z|)^\gamma \le 1-|\varphi(z)| \le |1-z|^\gamma. \tag{15}$$

Corollary 3.1. *Let $\alpha \ge -1$, $0 < \gamma < 1$, n be a non-negative integer and $\varphi = \varphi_\gamma$ or ψ_γ. Then*

(i) *$D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is bounded if and only if*

$$\gamma \le \frac{\alpha+2}{\alpha+2+2n} \tag{16}$$

(ii) *$D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is compact if and only if*

$$\gamma < \frac{\alpha+2}{\alpha+2+2n} \tag{17}$$

Proof. (i) Assume that $\varphi = \varphi_\gamma$ or ψ_γ, $0 < \gamma < 1$, and (16) holds. Then there exists some r_0, $0 < r_0 < 1$ such that

$$(1-|z|^2)^\gamma \le 1-|\varphi(z)|^2 \le |1-z|^\gamma \tag{18}$$

for $z \in \mathbb{D} \setminus r_0\overline{\mathbb{D}}$. First suppose that (16) holds. Then by (18), we have

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}} \lesssim \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{2+\alpha}}{(1-|z|)^{\gamma(2+\alpha+2n)}} < \infty,$$

and so by Theorem 2.1, $D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is bounded. Conversely, suppose that $D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is bounded. Then

$$\sup_{|z|>r_0} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}} \le \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{2+\alpha+2n}} < \infty. \tag{19}$$

Using second inequality in (18) and taking $z = x \in (r_0, 1)$, (19) implies that

$$\sup_{x\in(r_0,1)} (1-x^2)^{(2+\alpha)-\gamma(2+\alpha+2n)} < \infty,$$

which is possible only if (16) holds.

(ii) Suppose that (17) holds. Then by (18), we have

$$\lim_{|z|\to 1} \frac{(1-|z|)^{2+\alpha}}{(1-|\varphi(z)|)^{\gamma(2+\alpha+2n)}} \lesssim \lim_{|z|\to 1} \frac{(1-|z|)^{2+\alpha}}{(1-|z|)^{\gamma(2+\alpha+2n)}} = 0,$$

and so by Theorem 2.1 (ii), $D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is compact. Conversely, suppose that $D_\varphi^n : \mathcal{A}_\alpha^2 \to \mathcal{A}_\alpha^2$ is compact. Then for every sequence $\{z_m\}$ in $\mathbb{D}$, we have

$$\frac{(1-|z_m|)^{2+\alpha}}{(1-|z_m|)^{\gamma(2+\alpha+2n)}} \lesssim \frac{(1-|z_m|)^{2+\alpha}}{(1-|\varphi(z_m)|)^{2+\alpha+2n}} \to 0 \tag{20}$$

as $|z_m| \to 1$. In particular, if we consider the sequence $z_m = \frac{m}{m+1}$. Then $\{z_m\} \in \mathbb{D}$ and $|z_m| \to 1$ as $m \to \infty$. Thus by (20), we have

$$\lim_{m\to\infty} \left(1 - \frac{m}{m+1}\right)^{(2+\alpha)-\gamma(2+\alpha+2n)} = 0,$$

which is possible only if (17) holds. □

As an application of Corollary 3.1, we see that angle of contact of lens and lunar maps plays a significant role in bouundedness and compactness of D^n_φ between Hardy and Bergman spaces. For $0 < \gamma < 1$, let $\varphi = \varphi_\gamma$ or ψ_γ and $n \geq 1$. Let θ be the angle of contact. Then $\theta = \gamma\pi$, and so

$$D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha \text{ is bounded } \Leftrightarrow \theta \leq \frac{\alpha+2}{\alpha+2+2n}\pi,$$

$$D^n_\varphi : \mathcal{A}^2_\alpha \to \mathcal{A}^2_\alpha \text{ is compact } \Leftrightarrow \theta < \frac{\alpha+2}{\alpha+2+2n}\pi.$$

Taking $n = 1, 2, 3, 4$, $\alpha = -1$ and $\alpha = 0$, respectively, we get the following tables.

n	$D^n_\varphi : \mathcal{H}^2 \to \mathcal{H}^2$ is bounded	$D^n_\varphi : \mathcal{H}^2 \to \mathcal{H}^2$ is compact
1	$\theta \leq 60°$	$\theta < 60°$
2	$\theta \leq 36°$	$\theta < 36°$
3	$\theta \leq (180/7)°$	$\theta < (180/7)°$
4	$\theta \leq 20°$	$\theta < 20°$

n	$D^n_\varphi : \mathcal{A}^2 \to \mathcal{A}^2$ is bounded	$D^n_\varphi : \mathcal{A}^2 \to \mathcal{A}^2$ is compact
1	$\theta \leq 90°$	$\theta < 90°$
2	$\theta \leq 60°$	$\theta < 60°$
3	$\theta \leq 45°$	$\theta < 45°$
4	$\theta \leq 36°$	$\theta < 36°$

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