n-Colour Even Compositions

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Abstract

An n-colour even composition is defined as an n-colour composition with even parts. In this paper we get generating functions, explicit formulas and a recurrence formula for n-colour even compositions.
Key words: n-colour even compositions; generating functions; explicit formulas; recurrence formula.
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1 Introduction

In the classical theory of partitions, compositions were first defined by MacMahon[1] as ordered partitions. For example, there are 5 partitions and 8 compositions of 4. The partitions are 4, 31, 22, 21^2, 1^4 and the compositions are 4, 31, 13, 22, 21^2, 121, 1^22, 1^4.

Agarwal and Andrews[2] defined an n-colour partition as a partition in which a part of size n can come in n different colours. They denoted different colours by subscripts: n_1, n_2, $\cdots$, n_n. Analogous to MacMahon's ordinary compositions Agarwal[3] defined an n-colour composition as an n-colour ordered partition. Thus, for example, there are 21 n-colour compositions of 4,viz., $4_1, 4_2, 4_3, 4_4, 3_11_1, 3_21_1, 3_31_1, 1_13_1, 1_13_2$, $1_13_3, 2_12_1, 2_12_2, 2_22_2, 2_22_1$, $2_11_11_1, 2_21_11_1, 1_12_11_1, 1_11_12_1, 1_12_21_1, 1_11_12_2, 1_11_11_11_1$.

More properties of n-colour compositions were found in[4, 5]. And Narang and Agarwal[6] also defined an n-colour self-inverse composition and gave more properties. In 2010, Guo[7] studied n-colour even self-inverse compositions.

In this paper, we shall study n-colour even compositions.

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Definition 1.1 *An n-colour even composition whose parts are even.*

Thus, for example, there are 2 n-colour even compositions of 2, viz.,$2_1, 2_2$. And there are 8 n-colour even compositions of 4, viz., $4_1, 4_2, 4_3, 4_4, 2_1 2_1, 2_1 2_2, 2_2 2_2, 2_2 2_1$.

In section 2 we shall give generating functions, explicit formulas and a recurrence formula for n-colour even compositions. In addition, we also get the relation between the number of n-colour even compositions of 2ν and the number of n-colour even self-inverse compositions of $4\nu + 2$.

Agarwal[3] proved the following theorem.

Theorem 1.1 ([3]) *Let $C(m, q)$ and $C(q)$ denote the enumerative generating functions for $C(m, \nu)$ and $C(\nu)$, respectively, where $C(m, \nu)$ is the number of n-colour compositions of ν into m parts and $C(\nu)$ is the number of n-colour compositions of ν. Then*

$$C(m, q) = \frac{q^m}{(1-q)^{2m}}, \tag{1}$$

$$C(q) = \frac{q}{1 - 3q + q^2}, \tag{2}$$

$$C(m, \nu) = \binom{\nu + m - 1}{2m - 1}, \tag{3}$$

$$C(\nu) = F_{2\nu}. \tag{4}$$

2 Main results

We denote the number of n-colour even compositions of ν by $C_e(\nu)$ and the number of n-colour even compositions of ν into m parts by $C_e(m, \nu)$, respectively. In this section, we first prove the following theorem.

Theorem 2.1 *Let $C_e(m, q)$ and $C_e(q)$ denote the enumerative generating functions for $C_e(m, \nu)$ and $C_e(\nu)$, respectively. Then*

$$C_e(m, q) = \sum_{\nu=0}^{\infty} C_e(m, \nu) q^\nu = \frac{2^m q^{2m}}{(1-q^2)^{2m}}, \tag{5}$$

$$C_e(q) = \sum_{\nu=0}^{\infty} C_e(\nu) q^\nu = \frac{2q^2}{1 - 4q^2 + q^4}, \tag{6}$$

$$C_e(m, \nu) = 2^m \binom{\frac{\nu}{2} + m - 1}{2m - 1}. \tag{7}$$

where ν is even.

Proof. Be similar to Agarwal's proof in paper [3], we have

$$\begin{aligned} C_e(m,q) &= \sum_{m=1}^{\infty} C_e(m,\nu)q^{\nu} \\ &= (2q^2+4q^4+\cdots+)^m \\ &= \frac{2^m q^{2m}}{(1-q^2)^{2m}}. \end{aligned}$$

This proves (5).
And

$$\begin{aligned} C_e(q) &= \sum_{m=1}^{\infty} C_e(m,q) \\ &= \sum_{m=1}^{\infty} \frac{2^m q^{2m}}{(1-q^2)^{2m}} \\ &= \frac{2q^2}{1-4q^2+q^4}. \end{aligned}$$

Which proves (6).
On equating the coefficients of q^{ν} in (6), we have

$$C_e(m,\nu) = 2^m \binom{\frac{\nu}{2}+m-1}{2m-1}.$$

This proves (7).
We complete the proof of this theorem.
Using (3) and (7) we have the following corollary easily.

Corollary 2.1

$$C_e(m,2\nu) = 2^m C(m,\nu), \tag{8}$$

$$C_e(2\nu) = \sum_{m=1}^{\nu} 2^m C(m,\nu). \tag{9}$$

This corollary has an easy combinatorial proof. Now we give the proof.

Proof. For every n-colour even composition of 2ν with m parts, we replace each $(2t)_{2j-1}$ and $(2t)_{2j}$ by t_j to get an n-colour composition of ν with m parts, and each such composition arises 2^m times. Thus we see (8) is true. So (9) is correct.

We complete the proof.

From the generating function of the number of n-colour even compositions $C_e(q)$, we have the following recurrence formula.

Theorem 2.2 *Let $C_e(2\nu)$ denote the number of n-colour even compositions of 2ν. Then*

$$C_e(2) = 2, C_e(4) = 8 \text{ and } C_e(2\nu) = 4C_e(2\nu-2) - C_e(2\nu-4) \quad for \quad \nu > 2.$$

Although this result is a corollary of the formula (6), we still present two proofs. The following proofs are similar to Narang and Agarwal's proof of Theorem3.1 in paper[6].

First proof. We have

$$\begin{aligned}
C_e(2\nu) &= \sum_{m=1}^{\nu} 2^m \binom{\nu+m-1}{2m-1} \\
&= \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-1} + \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-2} \\
& \quad (\textit{by the binomial identity} \binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}) \\
&= \sum_{m=1}^{\nu-1} 2^m \binom{\nu+m-2}{2m-1} + 2^\nu \binom{\nu+\nu-2}{2\nu-1} + \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-2} \\
&= C_e(2\nu-2) + \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-2}.
\end{aligned}$$

Now we define

$$f_\nu = \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-2}.$$

and

$$\begin{aligned}
f_\nu &= \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2}{2m-2} \\
&= \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2-1}{2m-2} + \sum_{m=1}^{\nu} 2^m \binom{\nu+m-2-1}{2m-3} \\
&= f_{\nu-1} + \sum_{m=1}^{\nu} 2^m \binom{(\nu-1)+(m-1)-1}{2(m-1)-1} \\
&= f_{\nu-1} + 2C_e(2\nu-2).
\end{aligned}$$

So we get $C_e(2\nu) = C_e(2\nu-2) + f_\nu$ and $f_\nu = f_{\nu-1} + 2C_e(2\nu-2)$. Then $C_e(2\nu-2) = C_e(2\nu-4) + f_{\nu-1}$ and so $C_e(2\nu) - C_e(2\nu-2) = C_e(2\nu-2) - C_e(2\nu-4) + f_\nu - f_{\nu-1} = 3C_e(2\nu-2) - C_e(2\nu-4)$.

Thus, $C_e(2\nu) = 4C_e(2\nu-2) - C_e(2\nu-4)$.

We also give another proof of Theorem2.2

Second proof.(*Combinatorial*) To prove that $C_e(2\nu) = 4C_e(2\nu-2) - C_e(2\nu-4)$, we split the n-colour compositions enumerated by $C_e(2\nu) + C_e(2\nu-4)$ into three classes:
(A) enumerated by $C_e(2\nu)$ and having 2_1 or 2_2 on the right.
(B) enumerated by $C_e(2\nu)$ and having h_t on the right, $h > 2$, $1 \le t \le h-2$ (where, h is even) and n-colour even compositions of 2ν of form $(2\nu)_t$, $1 \le t \le 2\nu - 2$.
(C) enumerated by $C_e(2\nu)$ and having h_t on the right, $h > 2, h-1 \le t \le h$, $(2\nu)_{2\nu-1}$, $(2\nu)_{2\nu}$ and those enumerated by $C_e(2\nu-4)$.

We transform the n-colour even compositions in class (A) by deleting 2_1 or 2_2 on the right. This produces n-colour compositions enumerated by $C_e(2\nu-2)$. Conversely, given any n-colour composition enumerated by $C_e(2\nu-2)$ we add 2_1 or 2_2 on the right to produce the elements of the class (A). In this way we establish that there are exactly $2C_e(2\nu-2)$ elements in the class (A).

Next, we transform the n-colour even compositions in class (B) by subtracting 2 from h, that is, replacing h_t by $(h-2)_t$ and subtracting 2 from 2ν of $(2\nu)_t$, $1 \le t \le 2\nu - 2$. This transformation also establishes the fact that there are exactly $C_e(2\nu-2)$ elements in class (B).

Finally, we transform the elements in class (C) as follows: Subtract 2_2 from h_t on the right, $h > 2$, $h-1 \le t \le h$, that is, replace h_t by $(h-2)_{(t-2)}$. In this way we will get n-colour even compositions of $2\nu - 2$ whose parts are 2_1,2_2 or $h_t, h-1 \le t \le h$ on the right, except n-colour even compositions in one part only. We also replace $(2\nu)_{(2\nu-1)}$ by $(2\nu-2)_{(2\nu-3)}$ and $(2\nu)_{(2\nu)}$ by $(2\nu-2)_{(2\nu-2)}$, and to get the n-colour even compositions into one part $(2\nu-2)_{(2\nu-3)}$, $(2\nu-2)_{(2\nu-2)}$. To get the remaining n-colour even compositions from $C_e(2\nu-4)$ we add 2 to the right parts, that is, replace h_t by $(h+2)_t$ to get the n-colour even compositions into one part: $(2\nu-2)_t$, $1 \le t \le (2\nu-4)$, and n-colour even compositions of $2\nu - 2$ having h_t on the right, $h > 2$, $1 \le t \le h-2$. We see that the number of n-colour even compositions in class (C) is also equal to $C_e(2\nu-2)$. Hence, $C_e(2\nu)+C_e(2\nu-4) = 4C_e(2\nu-2)$. viz.,$C_e(2\nu) = 4C_e(2\nu-2) - C_e(2\nu-4)$.

Thus, we complete the proof.

And Guo also defined n-colour even self-inverse compositions in [7].

Definition 2.1 ([7]) *An n-colour even composition whose parts read from left to right are identical with when read from right to left is called an n-colour even self -inverse composition.*

Thus, for example, there are 6 n-colour even self-inverse compositions of 4. viz., 4_1, 4_2, 4_3, 4_4, 2_12_1, 2_22_2.

Guo gave the following theorem.

Theorem 2.3 ([7]) *Let $B(e,\nu)$ denote the number of n-colour even self-inverse compositions of ν, let $B(e,q)$ denote the generating function for $B(e,4\nu+2)$. Then*

$$B(e,4\nu+2)=(4\nu+2)+\sum_{t=2}^{4\nu-2}\sum_{m=1}^{\frac{4\nu+2-t}{4}} t2^m\binom{\frac{4\nu+2-t}{4}+m-1}{2m-1}, \quad (10)$$

$$B(e,q)=\sum_{\nu=0}^{\infty}B(e,4\nu+2)q^\nu=\frac{2+2q}{1-4q+q^2}. \quad (11)$$

where $\nu=0,1,2,\cdots;\quad t=4k+2, k=0,1,2,...,\nu-1$.

We shall prove a relation between $B(e,4\nu+2)$ and $C_e(\nu)$.

Theorem 2.4 *Let $B(e,\nu)$ denote the number of n-colour even self-inverse compositions of ν, let $C_e(\nu)$ denote the number of n-colour even compositions of ν. Then*

$$B(e,4\nu+2)=C_e(2\nu)+C_e(2\nu+2).$$

Where, $\nu\geq 1$.

From the generating function of $B(e,4\nu+2)$ and $C_e(\nu)$ we can get the relation easily. Now we give the combinatorial proof. The following proof is also similar to Narang and Agarwal's proof of Theorem3.1 in paper[6].

Proof. Obviously, an even number which is $4\nu+2$ ($\nu=1,2,\cdots$) can have even self-inverse n-colour compositions only when the number of parts is odd. The central part is even say h and it also satisfies $h\equiv 2(mod 4)$. So we split the self-inverse even n-colour compositions of $4\nu+2$ into four classes.

(A) $h=2$, i.e. the central part is $2_t, t=1$.
(B) $h=2$, i.e. the central part is $2_t, t=2$.
(C) $h>2$, i.e. the central part is h_t.
(D) the self-inverse n-colour even compositions in one part only. i.e. $(4\nu+2)_t, 1\leq t\leq 4\nu+2$.

In class(A), For every self-inverse even n-colour composition of $4\nu+2$ has the central part 2_1 and two identical even n-colour compositions of 2ν on each side of the central part 2_1. Then we get an n-colour composition of 2ν by deleting 2_1 and all parts on the right side. This process is one to one. Thus there are $C_e(2\nu)$ n-colour even compositions.

In class(B), an even self-inverse n-colour composition of $4\nu+2$ has the central part 2_2 and two identical even n-colour compositions of 2ν on each side of the central part 2_2. We get an n-colour even composition of $2\nu+2$ by deleting all parts in the right side of the central part 2_2. So we get n-colour even compositions of $2\nu+2$ with right extreme is 2_2.

In class (C), we also split the self-inverse even n-colour compositions into two classes:

(a) the central part is $h_t, h > 2, 1 \le t \le \frac{h+2}{2}$;

(b) the central part is $h_t, h > 2, \frac{h+2}{2} < t \le h$.

Given any self-inverse n-colour even composition in class (a) we replace h_t by $(\frac{h+2}{2})_t$, then we get an even n-colour composition of $2\nu+2$ by deleting all parts on the right of the central part $(\frac{h+2}{2})_t$. Thus we get n-colour even compositions of $2\nu+2$ which have not part 2_1 or 2_2 on the right extreme. This process is one to one.

Next we transform any self-inverse even composition in class (b) as follows: First, we replace h_t by $(\frac{h+2}{2})_{h-t+1}$, then we obtain an n-colour even composition of $2\nu+2$ by deleting all parts on the left of the central part $(\frac{h+2}{2})_{h-t+1}$. Second, we split part $(\frac{h+2}{2})_{h-t+1}$ into two parts: $(\frac{h+2}{2}-2)_{h-t+1}$, 2_1, then we rest $(\frac{h+2}{2}-2)_{h-t+1}$ and 2_1 to make them on each side of the composition which have not part $(\frac{h+2}{2}-2)_{h-t+1}$, and lay 2_1 on the right. Hence, we get an n-colour even composition of $2\nu+2$ with the number of parts is more than 2 and 2_1 on the right. This process is one to one. For example, $2_1 6_5 2_1 \longrightarrow 2_1 4_2 2_1 \longrightarrow 4_2 2_1 \longrightarrow 2_2 2_1 2_1$.

Finally, we transform the even compositions in class(D) as follows: Replace $(4\nu+2)_t$ by $(2\nu+2)_t$ when $1 \le t \le 2\nu+2$. This produces $2\nu+2$ n-colour even compositions of $2\nu+2$ in one part only. Then we replace $(4\nu+2)_t$ by $(2\nu+2)_{4\nu+3-t}$ when $2\nu+2 < t \le 4\nu+2$. After that, we split $(2\nu+2)_{4\nu+3-t}$ into two ordered parts :$(2\nu)_{4\nu+3-t}$, 2_1. In this way we have n-colour even compositions of $2\nu+2$ which have two parts and with part 2_1 on the right. This process is one to one.

Therefore we have all n-colour even compositions of $2\nu+2$ from class (B),(C)and (D).

Hence, we have

$$B(e, 4\nu+2) = C_e(2\nu) + C_e(2\nu+2).$$

We complete the proof.

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