SOME IDENTITIES ON THE EULER NUMBERS ARISING FROM EULER BASIS POLYNOMIALS

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ABSTRACT. In this paper we show that the set $\{E_0(x), E_1(x), \cdots, E_n(x)\}$ of Euler polynomials is a basis for the space of polynomials of degree less than or equal to n. From the properties of Euler basis polynomials, we give some interesting identities on the product of two Bernoulli and Euler polynomials.

1. Introduction

The so-called Euler polynomials $E_n(x)$ may be defined by means of

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1-22]). In the special case, $x=0$, $E_n(0)=E_n$ are called the n-th Euler numbers.

As is well known definition, the n-th Bernoulli polynomials are also defined by the generating function as follows:

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [1,2,3,10,11]). In the special case, $x=0$, $B_n(0)=B_n$ are called the n-th Bernoulli numbers.

From the definition of Bernoulli and Euler polynomials, we note that

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_l, \quad E_n(x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}E_l.$$

For $n \in \mathbb{Z}_+$, we have

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \quad \frac{dE_n(x)}{dx} = nE_{n-1}(x).$$

By the definition of Bernoulli and Euler polynomials, we get

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \qquad E_0 = 1, \quad E_n(1) + E_n = 2\delta_{0,n},$$

where $\delta_{k,n}$ is Kronecker symbol.

In [5], L. Carlitz introduced the following formula for the product of two Bernoulli polynomials:

$$B_m(x)B_n(x) = \sum_r \left\{ \binom{m}{2r} n + \binom{m}{2r} m \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{n}} \right\},$$

where $m, n \in \mathbb{Z}_+$ with $m + n \geq 2$.

In [6], L. Carlitz gave the formula for the product of two Eulerian polynomials:

$$\begin{aligned} H_m(x|\alpha)H_n(x|\alpha^{-1}) &= -(1-\alpha)\sum_{r=0}^{m-1}\binom{m}{r+1}H_{r+1}(\alpha)\frac{B_{m+n-r}(x)}{m+n-r} \\ &\quad - (1-\alpha^{-1})\sum_{s=0}^{n-1}\binom{n}{s+1}H_{s+1}(\alpha^{-1})\frac{B_{m+n-s}(x)}{m+n-s} \\ &\quad + (-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-\alpha)H_{m+n+1}(\alpha), \end{aligned}$$

where $H_m(\alpha)$ are the m-th Eulerian numbers.

In this paper, we show that the set $\{E_0(x), E_1(x), \cdots, E_n(x)\}$ of Euler polynomials is a basis for the space of polynomials of degree less than or equal to n. By using the properties of Euler basis polynomials, we give some interesting formulae for the product of two Bernoulli and Euler polynomials.

2. Some formulae for the product of two Bernoulli and Euler polynomials

Let $\mathbf{P}_n = \{ \sum_i a_i x^i \mid a_i \in \mathbb{Q} \}$ be the space of polynomials of degree less than or equal to n. It is easy to show that

(1)

$$e^{xt} = \frac{1}{2}\frac{2(e^t+1)}{e^t+1}e^{xt} = \frac{1}{2}\frac{2e^{(x+1)t}}{e^t+1} + \frac{1}{2}\frac{2e^{xt}}{e^t+1} = \frac{1}{2}\sum_{n=0}^{\infty}\Big(E_n(x+1)+E_n(x)\Big)\frac{t^n}{n!}.$$

Note that

$$(E+1)^n + E_n = 2\delta_{0,n}, \quad E_0 = 1.$$

From the definition of Euler polynomials, we have

$$\begin{aligned} E_n(x+1) &= \sum_{l=0}^{n}\binom{n}{l}E_l(1)x^{n-l} = x^n E_0(1) - \sum_{l=1}^{n}\binom{n}{l}E_l x^{n-l} \\ &= 2x^n - \sum_{l=0}^{n}\binom{n}{l}E_l x^{n-l} = 2x^n - E_n(x). \end{aligned}$$

Hence

$$E_n(x+1) + E_n(x) = 2x^n,$$

$$E_n(x+1)=\sum_{l=0}^{n}\binom{n}{l}E_l(x)=E_n(x)+\sum_{l=0}^{n-1}E_l(x)\binom{n}{l},$$

and

$$E_n(x+1)+E_n(x)=2E_n(x)+\sum_{l=0}^{n-1}E_l(x)\binom{n}{l}.$$

From (1), we note that

(2) $$x^n=\frac{1}{2}\Big(E_n(x+1)+E_n(x)\Big)=E_n(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x).$$

By (2), we see that $\{E_0(x),E_1(x),\cdots,E_n(x)\}$ span $\mathbf{P}_n$. For $p(x)\in\mathbf{P}_n$, let $p(x)=\sum_{k=0}^{n}b_kE_k(x)$ and $g(x)=p(x+1)+p(x)$. Then we have

(3) $$g(x)=\sum_{k=0}^{n}b_k\Big(E_k(x+1)+E_k(x)\Big)=\sum_{k=0}^{n}2b_kx^k.$$

Thus, from (3), we have

(4) $$g^{(r)}(x)=\sum_{k=0}^{n}2b_kk(k-1)\cdots(k-r+1)x^{k-r},$$

where $g^{(r)}(x)=\frac{d^rg(x)}{dx^r}$ and $r=0,1,2,\cdots,n$.

Let us take $x=0$ in (4). Then we have

(5) $$g^{(r)}(0)=2b_rr!.$$

From (5), we have

(6) $$b_r=\frac{g^{(r)}(0)}{2r!}=\frac{1}{2r!}\Big(p^{(r)}(1)+p^{(r)}(0)\Big).$$

Let us assume that $0=p(x)=\sum_{k=0}^{n}b_kE_k(x)$. Then, by (6), we get

(7) $$b_r=\frac{1}{2r!}\Big(p^{(r)}(1)+p^{(r)}(0)\Big)=0,\quad\text{for}\quad r=0,1,2,\cdots k.$$

From (7), we have $\{E_0(x),E_1(x),\cdots,E_n(x)\}$ is a linearly independent set. Therefore, we obtain the following theorem.

Theorem 1 . *The set $\{E_0(x),E_1(x),\cdots,E_n(x)\}$ of Euler polynomials is a basis for $\mathbf{P}_n$.*

Let us take polynomial $p(x)\in\mathbf{P}_n$, let $p(x)=\sum_{k=0}^{n}b_kE_k(x)$ and $g(x)=p(x+1)+p(x)$. Then we have. as a linear combination of Euler basis polynomials with

(8) $$p(x)=C_0E_0(x)+C_1E_1(x)+\cdots+C_nE_n(x).$$

The equation (8) can be rewritten as a dot product of two variables:

$$(9) \qquad p(x) = \begin{pmatrix} E_0(x) & E_1(x) & \dots & E_n(x) \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}.$$

By (9), we get

$$(10) \quad p(x) = \begin{pmatrix} 1 & x & \dots & x^n \end{pmatrix} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & \dots & a_{1,n+1} \\ 0 & 1 & a_{23} & \dots & \dots & a_{2,n+1} \\ 0 & 0 & 1 & \dots & \dots & a_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n,n+1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix},$$

where the $a_{i,j}$ are the coefficient of the power basis that are used to determine the respective Euler polynomials.

From the definition of Euler polynomials, we note that

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \cdots.$$

In the quadratic case ($n = 2$), the matrix representation is

$$p(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}.$$

In the cubic case ($n = 3$), the matrix representation is

$$p(x) = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

In many applications of Euler polynomials, a matrix formulation for the Euler polynomials seems to be interesting.

Let us take $p(x) = B_n(x) \in \mathbf{P}_n$. By Theorem 1. we see that $p(x)$ is given by $p(x) = \sum_{k=0}^{n} b_k E_k(x)$. From (6), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right) \\ &= \frac{1}{2k!} n(n-1)\cdots(n-k+1)\left(B_{n-k}(1) + B_{n-k}\right) \\ (11) \qquad &= \frac{\binom{n}{k}}{2}\left(B_{n-k}(1) + \delta_{1,n-k} + B_{n-k}\right) \\ &= \begin{cases} \binom{n}{k} B_{n-k} & \text{if } k \neq n-1, \\ 0 & \text{if } k = n-1. \end{cases} \end{aligned}$$

Therefore, by (11), we obtain the following proposition.

Proposition 1 . *For $n \in \mathbb{Z}_+$, we have*

$$B_n(x) = \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} E_k(x) + E_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n-2} \binom{n}{k} B_{n-k} E_k(x).$$

Let us consider polynomials $p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x)$ in $\mathbf{P}_n$. Then we have

$$p^{(r)}(x) = \frac{(n+1)!}{(n+1-r)!} \sum_{k=r}^{n} B_{k-r}(x) B_{n-k}(x), \tag{12}$$

where $r = 0, 1, 2, \cdots, n$.

By Theorem 1, we see that $p(x)$ is given by $p(x) = \sum_{k=0}^{n} b_k E_k(x) \in \mathbf{P}_n$. From (6) and (12), we note that

(13)

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right) \\
&= \frac{1}{2k!} \frac{(n+1)!}{(n-k+1)!} \sum_{l=k}^{n} \left\{ B_{l-k}(1) B_{n-l}(1) + B_{l-k} B_{n-l} \right\} \\
&= \frac{(n+1)\binom{n}{k}}{2(n-k+1)} \left\{ 2 \sum_{l=k}^{n} B_{l-k} B_{n-l} + 2 B_{n-1-k} + \delta_{k,n-2} \right\} \\
&= \begin{cases} \frac{(n+1)\binom{n}{k}}{n-k+1} \left\{ \sum_{l=k}^{n} B_{l-k} B_{n-l} + B_{n-1-k} \right\} & \text{if } k \neq n-2, \\ \frac{(n+1)\binom{n}{n-2}}{3} \left\{ 2 \sum_{l=n-2}^{n} B_{l-n+2} B_{n-l} + B_1 \right\} + \frac{(n^2-1)n}{12} & \text{if } k = n-2. \end{cases}
\end{aligned}$$

Therefore, by (13), we obtain the following theorem.

Theorem 2 . *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n} B_k(x) B_{n-k}(x) \\
&= (n+1) \sum_{k=0}^{n} \frac{\binom{n}{k}}{n-k+1} \left\{ \sum_{l=k}^{n} B_{l-k} B_{n-l} + B_{n-1-k} \right\} E_k(x) + \frac{(n^2-1)n}{12} E_{n-2}(x).
\end{aligned}$$

Let us take $p(x) = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} B_k(x) B_{n-k}(x) \in \mathbf{P}_n$. Then we see that

$$p^{(r)}(x) = 2^r \sum_{k=r}^{n} \frac{B_{k-r}(x) B_{n-k}(x)}{(k-r)!(n-k)!}, \quad (r = 0, 1, 2, \cdots, n). \tag{14}$$

From Theorem 1, we see that $p(x)$ is given by $p(x) = \sum_{k=0}^{n} b_k E_k(x) \in \mathbf{P}_n$. By (6) and (14), we get

(15)

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right) \\
&= \frac{2^k}{2k!}\sum_{l=k}^{n} \frac{B_{l-k}(1)B_{n-l}(1) + B_{l-k}B_{n-l}}{(l-k)!(n-l)!} \\
&= \frac{2^{k-1}}{k!}\sum_{l=k}^{n} \frac{(B_{l-k}+\delta_{1,l-k})(B_{n-l}+\delta_{1,n-l}) + B_{l-k}B_{n-l}}{(l-k)!(n-l)!} \\
&= \frac{2^{k-1}}{k!}\left\{2\sum_{l=k}^{n} \frac{B_{l-k}B_{n-l}}{(l-k)!(n-l)!} + \frac{2B_{n-1-k}}{(n-1-k)!} + \delta_{k,n-2}\right\} \\
&= \begin{cases} \frac{2^k}{k!}\left\{\sum_{l=k}^{n} \frac{B_{l-k}B_{n-l}}{(l-k)!(n-l)!} + \frac{B_{n-1-k}}{(n-1-k)!}\right\} & \text{if } k \neq n-2, \\ \frac{2^{n-2}}{(n-2)!}\left\{\sum_{l=n-2}^{n} \frac{B_{l-n+2}B_{n-l}}{(l-n+2)!(n-l)!} + B_1\right\} + \frac{2^{n-3}}{(n-2)!} & \text{if } k = n-2. \end{cases}
\end{aligned}$$

Therefore, by (15), we obtain the following theorem.

Theorem 3 . *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(x) \\
&= \sum_{k=0}^{n} 2^k \binom{n}{k}\left\{\sum_{l=k}^{n}\binom{n-k}{l-k}B_{l-k}B_{n-l} + n\binom{n-1}{k}B_{n-1-k}\right\}E_k(x) \\
&\quad + n(n-1)2^{n-3}E_{n-2}(x).
\end{aligned}$$

Let us consider polynomials $p(x) = \sum_{k=1}^{n-1} \frac{B_k(x)B_{n-k}(x)}{k(n-k)}$ in $\mathbf{P}_n$. Then, for $k = 0, 1, 2, \cdots, n-1$, we have

$$(16) \quad p^{(k)}(x) = 2C_k B_{n-k}(x) + (n-1)\cdots(n-k)\sum_{l=k+1}^{n-1} \frac{B_{l-k}(x)B_{n-l}(x)}{(l-k)(n-l)},$$

where

$$(17) \qquad C_k = \frac{\sum_{j=1}^{k}(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k)}{(n-k)}.$$

By Theorem 1, we see that $p(x)$ can be written as $p(x)=\sum_{k=0}^{n} b_k E_k(x)$ in $\mathbf{P}_n$. By (6) and (16), we get

$$\begin{aligned} b_k &= \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right) \\ &= \frac{1}{2k!}\left(2C_kB_{n-k}(1)+2C_kB_{n-k}\right) \\ &+ \frac{(n-1)\cdots(n-k)}{2k!}\sum_{l=k+1}^{n-1}\left\{\frac{B_{l-k}(1)B_{n-l}(1)+B_{l-k}B_{n-l}}{(l-k)!(n-l)!}\right\}. \end{aligned} \tag{18}$$

Note that $b_{n-1}=0$. We may assume $0\le k\le n-2$. Then we have

$$\begin{aligned} b_k &= \frac{1}{k!}\left(2C_kB_{n-k}+C_k\delta_{1,n-k}\right) \\ &+ \frac{(n-1)!}{2k!(n-k-1)!}\sum_{l=k+1}^{n-1}\left\{\frac{(B_{l-k}+\delta_{1,l-k})(B_{n-l}+\delta_{1,n-l})+B_{l-k}B_{n-l}}{(l-k)(n-l)}\right\} \\ &= \frac{1}{k!}2C_kB_{n-k}+\binom{n-1}{k}\sum_{l=k+1}^{n-1}\frac{B_{l-k}B_{n-l}}{(l-k)(n-l)}+\binom{n-1}{k}\frac{B_{n-1-k}}{n-1-k}+\frac{\binom{n-1}{k}}{2}\delta_{k,n-2}. \end{aligned}$$

Note that

$$p^{(n)}(x)=\left(p^{(n-1)}(x)\right)'=\left(2C_{n-1}B_1(x)\right)'=2C_{n-1}. \tag{19}$$

From (17), we have

$$\begin{aligned} \frac{C_k}{k!} &= \frac{1}{k!}\sum_{j=1}^{k}\frac{(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k)}{(n-k)!} \\ &= \frac{1}{k!}\sum_{j=1}^{k}\frac{(n-1)!}{(n-k)!(n-j)} = \frac{\binom{n}{k}}{n}\sum_{j=1}^{k}\frac{1}{n-j} \\ &= \frac{\binom{n}{k}}{n}\left\{\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n-k-1}\frac{1}{j}\right\}=\frac{\binom{n}{k}}{n}\left(H_{n-1}-H_{n-k-1}\right), \end{aligned} \tag{20}$$

and

$$C_{n-1}=(n-1)!H_{n-1}, \quad b_n=\frac{p^{(n)}(1)+p^{(n)}(0)}{2n!}=\frac{2C_{n-1}}{n!}, \tag{21}$$

where $H_n=\sum_{i=1}^{n}\frac{1}{i}$.

Therefore, by (18), (20), and (21), we obtain the following theorem.

Theorem 4 . *For $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x)$$
$$= \sum_{k=0}^{n-2} \left\{ \frac{2\binom{n}{k}}{n} (H_{n-1} - H_{n-k-1}) B_{n-k} + \binom{n-1}{k} \sum_{l=k+1}^{n-1} \frac{B_{l-k} B_{n-l}}{(l-k)(n-l)} \right.$$
$$\left. + \frac{\binom{n-1}{k} B_{n-1-k}}{n-1-k} \right\} E_k(x) + \frac{n-1}{2} E_{n-2}(x) + \frac{2}{n} H_{n-1} E_n(x).$$

Let us take $p(x) = \sum_{k=0}^{n} E_k E_{n-k}(x) \in \mathbf{P}_n$. Then we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!} \sum_{l=k}^{n} E_{l-k}(x) E_{n-k}(x). \tag{22}$$

By Theorem 1, $p(x)$ is given by $p(x) = \sum_{k=0}^{n} b_k E_k(x)$. From (6) and (22), we have

(23)

$$b_k = \frac{1}{2k!} \left(p^{(k)}(1) + p^{(k)}(0) \right)$$
$$= \frac{(n+1)!}{2k!(n-k+1)!} \sum_{l=k}^{n} \left\{ E_{l-k}(1) E_{n-l}(1) + E_{l-k} E_{n-l} \right\}$$
$$= \frac{n+1}{2(n-k+1)} \binom{n}{k} \sum_{l=k}^{n} \left\{ (-E_{l-k} + 2\delta_{0,l-k})(-E_{n-l} + 2\delta_{0,n-l}) + E_{l-k} E_{n-l} \right\}$$
$$= \frac{n+1}{n-k+1} \binom{n}{k} \left(\sum_{l=k}^{n} E_{l-k} E_{n-l} - 2E_{n-k} + 2\delta_{n,k} \right).$$

Therefore, by (23), we obtain the following theorem.

Theorem 5 . *For $n \in \mathbb{Z}_+$, we have*

$$\frac{1}{n+1} \sum_{k=0}^{n} E_k(x) E_{n-k}(x)$$
$$= \sum_{k=0}^{n} \frac{\binom{n}{k}}{n-k+1} \left(\sum_{l=k}^{n} E_{l-k} E_{n-l} - 2E_{n-k} \right) E_k(x) + 2E_n(x).$$

Let us take $p(x) = 2^k \sum_{k=0}^{n} \frac{E_k(x) E_{n-k}(x)}{k!(n-k)!} \in \mathbf{P}_n$. Then we have

$$p^{(k)}(x) = 2^k \sum_{l=k}^{n} \frac{E_{l-k}(x) E_{n-l}(x)}{(l-k)!(n-k)!}. \tag{24}$$

By Theorem 1, we see that $p(x)$ can be written as $p(x)=\sum_{k=0}^{n} b_k E_k(x)$ in $\mathbf{P}_n$. From (6) and (24), we note that

$$
\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right)\\
&= \frac{2^{k-1}}{k!}\sum_{l=k}^{n}\frac{1}{(l-k)!(n-l)!}\left\{E_{l-k}(1)E_{n-l}(1)+E_{l-k}E_{n-l}\right\}\\
&= \frac{2^{k-1}}{k!}\sum_{l=k}^{n}\left\{\frac{(-E_{l-k}+2\delta_{0,l-k})(-E_{n-l}+2\delta_{0,n-l})+E_{l-k}E_{n-l}}{(l-k)!(n-l)!}\right\}\\
&= \frac{2^k}{k!}\left\{\sum_{l=k}^{n}\frac{E_{l-k}E_{n-l}}{(l-k)!(n-l)!}-\frac{2E_{n-k}}{(l-k)!}+2\delta_{k,n}\right\}.
\end{aligned}
\tag{25}
$$

Therefore, by (25), we obtain the following theorem.

Theorem 6 . *For $n\in\mathbb{Z}_+$, we have*

$$
\sum_{k=0}^{n}\frac{E_k(x)E_{n-k}(x)}{k!(n-k)!}=\sum_{k=0}^{n}\frac{2^k}{k!}\left(\sum_{l=k}^{n}\frac{E_{l-k}E_{n-l}}{(l-k)!(n-l)!}-\frac{2E_{n-k}}{(n-k)!}\right)E_k(x)+2E_n(x).
$$

Let us take $p(x)=\sum_{l=k}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x)\in\mathbf{P}_n$. Then we have

$$
p^{(k)}(x)=2C_kE_{n-k}(x)+(n-1)\cdots(n-k)\sum_{l=k+1}^{n-1}\frac{E_{l-k}(x)E_{n-l}(x)}{(l-k)(n-l)},\tag{26}
$$

where

$$
C_k=\frac{\sum_{j=1}^{k}(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k)}{n-k}\quad (k=1,2,\cdots,n-1),\tag{27}
$$

and $\quad C_0=0$.

Note that $p^{(n)}(x)=\left(p^{(n-1)}(x)\right)'=\left(2C_{n-1}E_1(x)\right)'=2C_{n-1}$. By (7), we get

$$
\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right)\\
&= \frac{1}{k!}C_k(E_{n-k}(1)+E_{n-k})+\frac{(n-1)\cdots(n-k)}{2k!}\sum_{l=k+1}^{n-1}\frac{E_{l-k}(1)E_{n-l}(1)+E_{l-k}E_{n-l}}{(l-k)(n-l)}\\
&= \binom{n-1}{k}\sum_{l=k+1}^{n-1}\frac{E_{l-k}E_{n-l}}{(l-k)(n-k)},
\end{aligned}
\tag{28}
$$

where $k=1,2,\cdots,n-1$.

Note that

$$
b_n=\frac{1}{2n!}\left(p^{(n)}(1)+p^{(n)}(0)\right)=\frac{1}{2n!}4(n-1)!H_{n-1}=\frac{2}{n}H_{n-1}.\tag{29}
$$

Therefore, by (28), and (29), we obtain the following theorem.

Theorem 7 . *For $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x)$$
$$=\sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1}\frac{\binom{n-1}{k}}{(l-k)(n-l)}E_{l-k}E_{n-l}E_k(x)+\frac{2}{n}H_{n-1}E_n(x).$$

Let us consider polynomial $p(x)$ with $p(x)=\sum_{k=0}^{n}B_k(x)E_{n-k}(x)\in\mathbf{P}_n$. Then we have

$$p^{(k)}(x)=\frac{(n+1)!}{(n-k+1)!}\sum_{l=k}^{n}B_{l-k}(x)E_{n-l}(x), \tag{30}$$

where $k=0,1,2,\cdots,n$. From Theorem 1, we note that $p(x)$ is given by $p(x)=\sum_{k=0}^{n}b_kE_k(x)$ in $\mathbf{P}_n$. By (6) and (30), we get

$$\begin{aligned}
b_k&=\frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right)\\
&=\frac{(n+1)!}{2k!(n-k+1)!}\sum_{l=k}^{n}\left(B_{l-k}(1)E_{n-l}(1)+B_{l-k}E_{n-l}\right)\\
&=\frac{\binom{n+1}{k}}{2}\sum_{l=k}^{n}\left\{(B_{l-k}+\delta_{1,l-k})(-E_{n-l}+2\delta_{0,n-l})+B_{l-k}E_{n-l}\right\}\\
&=\frac{\binom{n+1}{k}}{2}\left(2B_{n-k}-E_{n-k-1}+2\delta_{k,n-1}\right)\\
&=\begin{cases}\frac{\binom{n+1}{k}}{2}\left(-E_{n-k-1}+2B_{n-k}\right) & \text{if } k\neq n-1,\\ 0 & \text{if } k=n-1,\\ n+1 & \text{if } k=n.\end{cases}
\end{aligned}\tag{31}$$

Therefore, by (31), we obtain the following theorem.

Theorem 8 . *For $n \in \mathbb{Z}_+$, we have* l

$$\sum_{k=0}^{n}B_k(x)E_{n-k}(x)=\frac{1}{2}\sum_{k=0}^{n-2}\binom{n+1}{k}\left(-E_{n-k-1}+2B_{n-k}\right)E_k(x)+(n+1)E_n(x).$$

Let us take $p(x)=2^k\sum_{l=k}^{n}\frac{1}{k!(n-k)!}B_k(x)E_{n-k}(x)\in\mathbf{P}_n$. Then, for $k=0,1,2,\cdots,n$, we have

$$p^{(k)}(x)=2^k\sum_{l=k}^{n}\frac{1}{(l-k)!(n-l)!}B_{l-k}(x)E_{n-l}(x). \tag{32}$$

By Theorem 1, let us assume that $p(x)=\sum_{k=0}^{n} b_k E_k(x)$ in $\mathbf{P}_n$. From (6) and (32), we have

$$
\begin{aligned}
b_k &= \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right)\\
&= \frac{2^{k-1}}{k!}\sum_{l=k}^{n}\frac{1}{(l-k)!(n-l)!}\left\{B_{l-k}(1)E_{n-l}(1)+B_{l-k}E_{n-l}\right\}\\
&= \frac{2^{k-1}}{k!}\sum_{l=k}^{n}\left\{\frac{(B_{l-k}+\delta_{1,l-k})(-E_{n-l}+2\delta_{0,n-l})+B_{l-k}E_{n-l}}{(l-k)!(n-l)!}\right\}\\
&= \frac{2^{k-1}}{k!}\left\{\frac{2B_{n-k}}{(n-k)!}-\frac{E_{n-k-1}}{(n-k-1)!}+2\delta_{k,n-1}\right\}\\
&= \begin{cases} \frac{2^{k-1}}{k!}\left(\frac{2B_{n-k}}{(n-k)!}-\frac{E_{n-k-1}}{(n-k-1)!}\right) & \text{if } k\neq n-1,\\ 0 & \text{if } k=n-1.\end{cases}
\end{aligned}
\tag{33}
$$

Therefore, by (33), we obtain the following theorem.

Theorem 9 . *For $n\in\mathbb{Z}_+$, we have*

$$
\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(x)=\sum_{k=0}^{n-2}2^{k-1}\binom{n}{k}\left\{-(n-k)E_{n-k-1}+2B_{n-k}\right\}+2^nE_n(x).
$$

Let us consider polynomials $p(x)$ with $p(x)=\sum_{l=k}^{n-1}\frac{1}{k(n-k)}B_k(x)E_{n-k}(x)\in\mathbf{P}_n$. Then we have

(34)

$$
p^{(k)}(x)=C_k(B_{n-k}(x)+E_{n-k}(x))+(n-1)\cdots(n-k)\sum_{l=k+1}^{n-1}\frac{B_{l-k}(x)E_{n-l}(x)}{(l-k)(n-l)},
$$

where

(35)

$$
C_k=\frac{\sum_{j=1}^{k}(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k)}{n-k}\quad (k=1,2,\cdots,n-1).
$$

Note that $p^{(n)}(x) = 2C_{n-1} = 2(n-1)!H_{n-1}$. By Theorem 1, let us assume that $p(x) = \sum_{k=0}^{n} b_k E_k(x)$. From (6) and (34), we have

(36)

$$\begin{aligned} b_k &= \frac{1}{2k!}\Big(p^{(k)}(1) + p^{(k)}(0)\Big) \\ &= \frac{1}{2k!}C_k\Big\{B_{n-k}(1) + \big(B_{n-k} + E_{n-k}(1)E_{n-k}\big) \\ &\qquad + (n-1)\cdots(n-k)\sum_{l=k+1}^{n-1}\frac{B_{l-k}(1)E_{n-l}(1) + B_{l-k}E_{n-l}}{(l-k)(n-l)}\Big\} \\ &= \frac{1}{2k!}\Big\{C_k\big(2B_{n-k} + \delta_{1,n-k}\big) - (n-1)\cdots(n-k)\sum_{l=k+1}^{n-1}\frac{\delta_{1,l-k}E_{n-l}}{(l-k)(n-l)}\Big\}, \end{aligned}$$

where $k = 1, 2, \cdots, n-1$.

$$= \begin{cases} \frac{1}{2k!}\Big\{2C_kB_{n-k} - (n-1)\cdots(n-k)\frac{E_{n-k-1}}{n-k-1}\Big\} & \text{if } k \neq n-1, \\ 0 & \text{if } k = n-1. \end{cases}$$

By (35),we get

$$\frac{C_k}{k!} = \frac{1}{n}\binom{n}{k}\big(H_{n-1} - H_{n-k-1}\big). \tag{37}$$

It is easy to show that

$$b_n = \frac{1}{2n!}\Big(p^{(n)}(1) + p^{(n)}(0)\Big) = \frac{1}{2n!}4(n-1)!H_{n-1} = \frac{2H_{n-1}}{n}. \tag{38}$$

Therefore, by (36), (37), and (38), we obtain the following theorem.

Theorem 10 . *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} &\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)E_{n-k}(x) \\ &= \sum_{k=0}^{n-2}\Big\{\frac{1}{n}\binom{n}{k}\big(H_{n-1} - H_{n-k-1}\big)B_{n-k} - \frac{1}{2}\binom{n-1}{k}\frac{E_{n-k-1}}{n-k-1}\Big\}E_k(x) \\ &\quad + \frac{2}{n}H_{n-1}E_n(x). \end{aligned}$$

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