ON THE NUMBER OF SUBPERMUTATIONS WITH FIXED ORBIT SIZE

ABDALLAH LARADJI ABDULLAHI UMAR *

Abstract

Consider an n-set, say $X_n = \{1, 2, \ldots, n\}$. An exponential generating function and recurrence relation for the number of subpermutations of X_n, whose orbits are of size at most $k \geq 0$ are obtained. Similar results for the number of nilpotent subpermutations of nilpotency index at most k, and exactly k are also given, along with arithmetic and asypmtotic formulas for these numbers.
[1] [2]

1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$. Then a (partial) transformation $\alpha :$ $\mathrm{Dom}\,\alpha \subseteq X_n \longrightarrow \mathrm{Im}\,\alpha \subseteq X_n$ is said to be *full* or *total* if $\mathrm{Dom}, \alpha = X_n$; otherwise it is called *strictly* partial. A partial

*The second-named author acknowledges with thanks the support of the Sultan Qaboos University research grant for Project No. IGCS/DOMS/09/13

[1] *MSC2010*: 20M18, 20M20, 05A10, 05A15.

[2] *Key Words*: partial one-one transformation, subpermutation, digraph, component, orbit, cycle, path, partial derangement, partial identity, nilpotent.

transformation is *nilpotent* if $\alpha^k = \emptyset$ (the empty or zero map) for some positive integer k. The *nilpotency index* of a nilpotent α is k, if $\alpha^k = \emptyset$ and $\alpha^{k-1} \neq \emptyset$. The set of partial one-to-one transformations of X_n, with composition as binary operation, is known as the *symmetric inverse semigroup* and is denoted by I_n. Partial one-to-one transformations are also called *subpermutations*, see Cameron and Deza [3]. A (sub)permutation without fixed points is called a *(partial) derangement*. Each subpermutation α (of X_n) can be pictured as a digraph on n vertices with ij an edge of α if $i\alpha = j$. Each component of such a digraph is called an *orbit*, and they are of two types: *cycles* (including 1-cycles or fixed points) and *simple paths*. Note that whilst the size and length of a cycle are the same, the length of a path is its size minus one, except for the empty path which has size and length equal to zero. Moreover, the nilpotency index of a subpermutation coincides with the maximum size of its constituent paths, except the empty subpermutation or zero whose nilpotency index is one.

As far back as 1987, Gomes and Howie [7] remarked that very little has been written on I_n. Despite the appearance of the books of Lipscomb [12] and Ganyushkin and Mazorchuk [6] and numerous papers (for example, [1, 4, 5, 8, 9, 10, 11]), the study of I_n is still in its infancy relative to its cousin S_n, the permutation group on an n-set or even T_n, the full transformation semigroup on an n-set.

It is known (see for example, Wilf [14]) that $\sigma(n, k)$, the number of permutations of n objects the size (or length) of all of whose cycles is at most k, has exponential generating function

$$e^{x+x^2/2+\cdots+x^k/k}, \tag{1}$$

and satisfies the recurrence

$$\sigma(n+1, k) = \sum_{j=0}^{k-1} \frac{n!}{(n-j)!}\sigma(n-j, k). \tag{2}$$

Our aim in this note is to obtain similar results for $p\sigma(n,k)$, the number of subpermutations of X_n with orbits of size at most $k(>0)$. As a by-product, we obtain a recurrence relation and the exponential generating function for $\nu(n,k)$, the number of nilpotent subpermutations of n objects with nilpotency index at most k and those with nilpotency index exactly k.

Let $b(n,k)$ be the number of subpermutations on X_n all of whose orbits are of size at most k and without fixed points, so that they may contain j-cycles for $1 < j \leq k$, but not 1-cycles. In other words, $b(n,k)$ is the number of partial derangements of X_n all of whose orbits are of size at most k. Then $b(n,k) = b(n,n)$ if $k > n$, and it is clear that $p\sigma(n,k) = \sum_{i=0}^{n} \binom{n}{i} b(n-i,k)$, since each subpermutation on X_n can be decomposed into a partial identity component and a partial derangement component. The following lemma is needed:

Lemma 1.1 *Let k be a positive integer. Then, a sequence (u_n) with $u_0 = 1$ satisfies the recurrence relation*

$$u_n = c_1 u_{n-1} + c_2(n-1)u_{n-2} + \cdots + c_k(n-1)(n-2)\cdots(n-k+1)u_{n-k}$$

if and only if its exponential generating function is

$$\sum_{n\geq 0} \frac{u_n}{n!} x^n = e^{c_1 x + c_2 x^2/2 + \cdots + c_k x^k/k}.$$

Proof. Let the sequence (u_n) satisfy the above recurrence relation and let $f(x) = \sum_{n\geq 0} \frac{u_n}{n!} x^n$. Then

$$\begin{aligned} xf'(x) &= \sum_{n\geq 1} \frac{u_n x^n}{(n-1)!} = c_1 x \sum_{n\geq 1} \frac{u_{n-1} x^{n-1}}{(n-1)!} + c_2 x^2 \sum_{n\geq 2} \frac{u_{n-2} x^{n-2}}{(n-2)!} + \\ &\cdots + c_k x^k \sum_{n\geq k} \frac{u_{n-k} x^{n-k}}{(n-k)!} \\ &= (c_1 x + c_2 x^2 + \cdots + c_k x^k) f(x), \end{aligned}$$

i. e., $f'(x) = (c_1 + c_2 x + \cdots + c_k x^{k-1}) f(x)$. Integrating gives

$$f(x) = e^{c_1 x + c_2 x^2/2 + \cdots + c_k x^k/k},$$

as required. The converse is easily proved by reversing the above steps. □

2 Subpermutations with Fixed Orbit Size

Proposition 2.1 *Let $p\sigma(n,k)$ be the number of subpermutations of X_n with orbit size at most k. Then the exponential generating function of $p\sigma(n,k)$ is*

$$e^{2x+3x^2/2+\cdots+(k+1)x^k/k}.$$

Proof. For each partial derangement α of X_n with all its orbits of size at most k, we consider two cases.

Case 1. n is in some j-cycle $(2 \le j \le k)$. Thus we have $(n-1)b(n-2,k)+(n-1)(n-2)b(n-3,k)+\cdots$ $+(n-1)(n-2)\cdots(n-k+1)b(n-k,k)$.

Case 2. n is not in any j-cycle. In this case either, n is not in any path or n is in a path size 2 or 3 or $\cdots$ or k, since there is a unique path of size 0, the empty path and no path of size 1. Thus we have $b(n-1,k)+2(n-1)b(n-2,k)+\cdots$ $+k(n-1)(n-2)\cdots(n-k+1)b(n-k,k)$, such maps in this case.

We therefore obtain

$$\begin{aligned} b(n,k) &= b(n-1,k)+3(n-1)b(n-2,k)+\cdots \\ &+ (k+1)(n-1)(n-2)\cdots(n-k+1)b(n-k,k). \end{aligned}$$

By Lemma 1.1, we infer that $g_k(x)$, the exponential generating function of $b(n,k)$ is

$$g_k(x)=e^{x+3x^2/2+\cdots+(k+1)x^k/k}.$$

Now using the fact (mentioned above) that

$$p\sigma(n,k)=\sum_{i=0}^{n}\binom{n}{i}b(n-i,k),$$

we see that the exponential generating function of $p\sigma(n,k)$ is

$$e^{2x+3x^2/2+\cdots+(k+1)x^k/k},$$

as required. □

Using Lemma 1.1 it is not difficult to deduce the following recurrence relation for $p\sigma(n,k)$:

Proposition 2.2 *Let $p\sigma(n,k)$ be the number of subpermutations of X_n with orbit size at most k. Then for $n \geq k \geq 2$, $p\sigma(n,k)$ satisfies the recurrence relation*

$$\begin{aligned} p\sigma(n,k) &= 2p\sigma(n-1,k) + 3(n-1)p\sigma(n-2,k) + \cdots \\ &\quad +(k+1)(n-1)(n-2)\cdots(n-k+1)p\sigma(n-k,k), \end{aligned}$$

where $p\sigma(n,0) = 1$, $p\sigma(n,1) = 2^n$ and $p\sigma(n,n+r) = p\sigma(n,n)$ for all nonnegative r.

Proposition 2.3 *Let $p\sigma(n,k)$ be the number of subpermutations of X_n with orbit size at most k. Then*

$$p\sigma(n,k) = \sum_{n_1+2n_2+\cdots+kn_k=n} \frac{2^{n_1}(\frac{3}{2})^{n_2}\cdots(\frac{k+1}{k})^{n_k} n!}{n_1!n_2!\cdots n_k!},$$

where $n_1, n_2, \cdots, n_k \geq 0$.

Proof. Let the exponential generating function of $p\sigma(n,k)$ be $g(x)$. Then

$$\begin{aligned} g(x) &= \sum_{n\geq 0} \frac{p\sigma(n,k)x^n}{n!} = e^{2x}\cdot e^{3x^2/2}\cdot\ldots\cdot e^{(k+1)x^k/k} \\ &= \sum_{n_1+2n_2+\cdots+kn_k=n} \frac{2^{n_1}(\frac{3}{2})^{n_2}\cdots(\frac{k+1}{k})^{n_k} x^n}{n_1!n_2!\cdots n_k!}. \end{aligned}$$

Hence

$$p\sigma(n,k) = \sum_{n_1+2n_2+\cdots+kn_k=n} \frac{2^{n_1}(\frac{3}{2})^{n_2}\cdots(\frac{k+1}{k})^{n_k} n!}{n_1!n_2!\cdots n_k!}$$

where $n_1, n_2, \ldots, n_k \geq 0$. □

Cameron [2, pp. 69–60] proved that for $n > 1$, $\sigma(n, 2)$ is even and $\sigma(n, 2) > \sqrt{(n!)}$. We have similar results for $p\sigma(n, k)$.

Lemma 2.4 *Let $p\sigma(n, k)$ be the number of subpermutations of X_n with orbit size at most k. Then*

$$p\sigma(n,k) \text{ is } \begin{cases} \text{even if } n \text{ is odd,} \\ \text{odd if } n \text{ is even.} \end{cases}$$

Proof. First observe that from the recurrence (in Proposition 2.2) $p\sigma(n, k)$ is clearly even if n is odd. However, if n is even, then for $k \geq 2$

$$p\sigma(n,k) \equiv p\sigma(n-2,k) \equiv p\sigma(2,k) \equiv p\sigma(2,2) = 7 \ (mod\, 2),$$

that is, $p\sigma(n, k)$ is odd. □

Proposition 2.5 *Let $\nu(n, k)$ be the number of nilpotents of I_n with nilpotency index at most k. Then the exponential generating function of $\nu(n, k)$ is*

$$\sum_{n \geq 0} \nu(n,k) \frac{x^n}{n!} = e^{x+x^2+\ \cdots\ +x^k}.$$

Proof. Decomposing a subpermutation α of X_n all of whose orbits are of size at most k into a (full) permutation component and a nilpotent component, we obtain that

$$p\sigma(n,k) = \sum_{i=0}^{n} \binom{n}{i} \sigma(i,k) \nu(n-i,k).$$

Now from Eqn.(1) and Proposition 2.1, we see that the exponential generating function of $\nu(n, k)$ is

$$e^{2x+3x^2/2+\ \cdots\ +(k+1)x^k/k-(x+x^2/2+\ \cdots\ +x^k/k)} = e^{x+x^2+\ \cdots\ +x^k},$$

as required. □

Corollary 2.6 *[13, A000262]. The exponential generating function for the number of nilpotent subpermutations of X_n is* $e^{x+x^2+x^3+\cdots} = e^{x/(1-x)}$.

Again, using Lemma 1.1 it is not difficult to deduce the following recurrence relation for $\nu(n,k)$:

Proposition 2.7 *Let $\nu(n,k)$ be the number of nilpotents of I_n with nilpotency index at most k. Then for $n \geq k \geq 2$, $\nu(n,k)$ satisfies the recurrence relation*

$$\begin{aligned} \nu(n,k) &= \nu(n-1,k) + 2(n-1)\nu(n-2,k) + \cdots \\ &+ k(n-1)\cdots(n-k+1)\nu(n-k,k), \end{aligned}$$

with $\nu(n,1) = 1$ and $\nu(n,n+r) = \nu(n,n)$ for all nonnegative r.

Using the same argument as in the proof of Proposition 2.3 we obtain the following corresponding result.

Lemma 2.8 *For $n \geq 0$, we have*

$$\begin{aligned} &\nu(n,k) \\ = \; & n! \sum_{n_1+2n_2+\cdots+kn_k=n} \frac{1}{n_1!n_2!\cdots n_k!} \\ = \; & \sum_{n_1+2n_2+\cdots+kn_k=n} \binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k} \frac{n!}{(n_1+n_2+\cdots+n_k)!}, \end{aligned}$$

where $n_1, n_2, \ldots, n_k \geq 0$.

Lemma 2.9 *For $n \geq k \geq 2$ we have $n(n-1) | \nu(n,k) - 1$.*

Proof. From the expression for $\nu(n,k)$ in Lemma 2.8, we see that if $n_1 \neq n$ then $2n_2 + 3n_3 + \cdots + kn_k \geq 1$, that is,

$$2(n_2+n_3+\cdots) + n_3 + 2n_4 + \cdots + (k-2)n_k \geq 1.$$

So, if $k \geq 2$

$$2(n_2 + n_3 + \cdots) + n_3 + 2n_4 + \cdots + (k-2)n_k \geq 2,$$

which implies $n - (n_1 + n_2 + \cdots + n_k) \geq 2$. Hence $\frac{n!}{(n_1+n_2+\cdots+n_k)!}$ is divisible by $n(n-1)$.

If $n_1 = n$ then we get the first term in the sum expression for $\nu(n,k)$ equals to 1. This implies that if $k \geq 2$ then $n(n-1)|\nu(n,k) - 1$. □

Using the above lemma and the fact that if $k = 2$, we have $i(n,2) = \nu(n,2) - 1$, we see that for $k \geq 3$,

$$i(n,k) = \nu(n,k) - \nu(n,k-1) = (\nu(n,k) - 1) - (\nu(n,k-1) - 1),$$

where all terms on the right hand side are divisible by $n(n-1)$. Hence, for all $k \geq 2$, we see that $n(n-1)|i(n,k)$. Thus, we have proved the following:

Proposition 2.10 *Let $i(n,k)$ be the number of nilpotents in I_n of index exactly k. Then for all $k \geq 2$, we have that $n(n-1)|i(n,k)$. In particular, for all $k \geq 2$, $i(n,k)$ is always even.*

Proposition 2.11 *Let $i(n,k)$ be the number of nilpotents in I_n of index exactly k. Then $i(n,1) = 1$, $i(n,n+r) = i(n,n)$ for all nonnegative r, and for $n \geq k \geq 2$, $i(n,k)$ satisfies the recurrence relation $i(n,k) = i(n-1,k) + 2(n-1)i(n-2,k) + \cdots + k(n-1)\cdots(n-k+1)i(n-k,k)$.*

Proof. This follows from the obvious fact that $i(n,k) = \nu(n,k) - \nu(n,k-1)$, for $k \geq 2$ and Proposition 2.7. □

Remark 2.12 *The triangular arrays of numbers $b(n,k)$, $p\sigma(n,k)$, $\nu(n,k)$ and $i(n,k)$ are as at the time of submitting this paper not in Sloane [13]. However, $b(n,n)$ is [13, A144085]; $p\sigma(n,n)$ is [13, A002720]; $\nu(n,n)$ is [13, A000262]; and $i(n,n)$ is [13, A000142].*

$n\backslash k$	0	1	2	3	4	5	6	$\Sigma b(n,k)$
0	1							1
1	1	1						2
2	1	1	4					6
3	1	1	10	18				30
4	1	1	46	78	108			234
5	1	1	94	486	636	780		1998
6	1	1	784	3096	4896	5760	6600	21138

Table 2.1 Some computed values for $b(n,k)$

$n\backslash k$	0	1	2	3	4	5	6	$\Sigma p\sigma(n,k)$
0	1							1
1	1	2						3
2	1	4	7					12
3	1	8	26	34				69
4	1	16	115	179	209			520
5	1	32	542	1102	1402	1546		4625
6	1	64	2809	7609	10759	12487	13327	47056

Table 2.2 Some computed values for $p\sigma(n,k)$

$n\backslash k$	1	2	3	4	5	6	$\Sigma\nu(n,k)$
1	1						1
2	1	3					4
3	1	7	13				21
4	1	25	49	73			148
5	1	81	261	381	501		1225
6	1	331	1531	2611	3331	4051	11856

Table 2.3 Some computed values for $\nu(n,k)$

$n\backslash k$	1	2	3	4	5	6	$\Sigma i(n,k)$
1	1						1
2	1	2					3
3	1	6	6				13
4	1	24	24	24			73
5	1	80	180	120	120		501
6	1	330	1200	1080	720	720	4051

Table 2.4 Some computed values for $i(n,k)$

3 An Inequality and Asymptotic Result

In this section we prove results for $p\sigma(n,2)$ analogous to those for $\sigma(n,2)$ from [2] and [14].

Proposition 3.1 *For $n \geq 4$, $p\sigma(n,2) > 6(\frac{3}{2})^{n-1}\sqrt{n!}$*

Proof. The proof is by induction. First note that $6(3/2)^3(\sqrt{4!}) = \frac{3^4\sqrt{6}}{2} < 115 = p\sigma(4,2)$ and $6(3/2)^4\sqrt{5!} = \frac{3^5\sqrt{30}}{4} < 542 = p\sigma(5,2)$. Now consider

$$\begin{aligned}
p\sigma(n,2) &= 2p\sigma(n-1,2) + 3(n-1)p\sigma(n-2,2) \\
&> 2\cdot 6(\frac{3}{2})^{n-2}\sqrt{(n-1)!} + 3(n-1)\cdot 6(\frac{3}{2})^{n-3}\sqrt{(n-2)!} \\
&= 6(\frac{3}{2})^{n-3}[3\sqrt{(n-1)!} + 3\sqrt{n-1}\sqrt{(n-1)!}] \\
&= 3\cdot 6(\frac{3}{2})^{n-3}\sqrt{(n-1)!}[1+\sqrt{n-1}] \\
&> 3\cdot 6(\frac{3}{2})^{n-3}\sqrt{n!} \\
&> 6(\frac{3}{2})^{n-1}\sqrt{n!},
\end{aligned}$$

as required. Note that we have used the following inequality: $\sqrt{n} < 1 + \sqrt{n-1}$. □

Proposition 3.2 *For $n \geq 2$, we have*

$$p\sigma(n,2) \equiv \begin{cases} 1 \ (mod\ 6) & \text{if } n \text{ is even,} \\ 2 \ (mod\ 6) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From the recurrence

$$p\sigma(n+1,2) - p\sigma(n-1,2) = 3(n+1)p\sigma(n-1,2) + 6(n-1)p\sigma(n-2,2),$$

we see that $p\sigma(n+1,2) \equiv (3n+4)p\sigma(n-1,2)\ (mod\ 6)$. Now, if n is odd then $p\sigma(n+1,2) \equiv p\sigma(n-1,2)\ (mod\ 6)$, and if n is even then $p\sigma(n+1,2) \equiv 4p\sigma(n-1,2)\ (mod\ 6)$. But $p\sigma(1,2) = 2$, and so $p\sigma(n,2) \equiv 2\ (mod\ 6)$, for all n odd. Also $p\sigma(2,2) = 7$, and so $p\sigma(n,2) \equiv 1\ (mod\ 6)$, for all n even. □

It is known that $\sigma(n,2)$, the number of involutions in S_n, satisfies

$$\sigma(n,2) \sim \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4},$$

as $n \to \infty$, [14, (5.4.14)]. This asymptotic formula can be obtained using Hayman's method, see for example, Wilf [14]. The same method can be applied to obtain a similar result for $p\sigma(n,2)$. However, we need the following special case of a more general result due to Hayman, see Wilf [14].

Theorem 3.3 *Let $f(z) = e^{p(z)}$, where $p(z)$ is a nonconstant polynomial with nonnegative real coefficients. Then the coefficients a_n of the Taylor series of $f(z)$ satisfy*

$$a(n) \sim (2\pi(rp'(r) + r^2p''(r)))^{-1/2} e^{p(r)} r^{-n},$$

as $n \to \infty$, where r is the positive real root of $rp'(r) = n$.

Combined with Stirling's approximation formula:

$$n! \sim (\frac{n}{e})^n \sqrt{2\pi n},$$

as $n \to \infty$, the above theorem gives the result:

$$p\sigma(n,2) \sim \frac{e^{r-n/2}}{\sqrt{2}}(\frac{n}{r})^n \sqrt{\frac{1}{1-\frac{r}{n}}},$$

where $r = \frac{\sqrt{1+3n}-1}{3}$ is the positive root of $rp'(r) = n$ with $p(r) = 2r + 3r^2/2$. Now since $\frac{r}{n} \to 0$ as $n \to \infty$, we obtain after some routine manipulation the following result.

Proposition 3.4 *Let $p\sigma(n,2)$ be the number of subpermutations of X_n with orbit size at most 2. Then as $n \to \infty$, we have*

$$p\sigma(n,2) \sim \frac{1}{\sqrt{2}} e^{r-n/2}(\sqrt{1+3n}+1)^n,$$

where $r = \frac{\sqrt{1+3n}-1}{3}$.

An element α in I_n is called a *quasi-idempotent* if $\alpha^4 = \alpha^2$, that is, α^2 is an idempotent. Clearly all idempotents and involutions are quasi-idempotents. We have

Proposition 3.5 *The number of quasi-idempotents in I_n is $p\sigma(n,2)$.*

Proof. We show that α in I_n is a quasi-idempotent if and only if all the orbits of α are of size at most 2. If α has a 3-cycle say, $(a_1a_2a_3)$ then clearly α^2 contains the 3-cycle $(a_1a_3a_2)$ and so α^2 is not an idempotent, that is, α is not a quasi-idempotent. Similarly, if α has a simple path of length 3 say, $(a_1a_2a_3]$ then clearly α^2 contains the path $(a_1a_3]$ and so α^2 is not an idempotent, that is, α is again, not a quasi-idempotent. The converse is clear. □

Acknowledgements. The authors wish to thank King Fahd University of Petroleum and Minerals and Sultan Qaboos University. We wish to express our sincere appreciation to Prof. G. Korvin for bringing reference [14] to our attention.

References

[1] D. Borwein, S. Rankin and L. Renner, Enumeration of injective partial transformations. *Discrete Math.* **73** (1989), 291–296.

[2] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, 1994.

[3] P. J. Cameron and M. Deza, On permutation geometries. *J. London Math. Soc* **20** (1979), 373–386.

[4] O. Ganyushkin and V. Mazorchuk, Combinatorics of nilpotents in symmetric inverse semigroups. *Ann. Comb.* **8** **(2)** (2004), 161–175.

[5] O. Ganyushkin and V. Mazorchuk, Combinatorics and distributions of partial injections. *Australas. J. Combin.* **34** (2006), 161–186.

[6] O. Ganyushkin and V. Mazorchuk, *Classical Finite Transformation Semigroups: An Introduction*, Springer, London, 2009.

[7] G. M. S. Gomes and J. M. Howie, Nilpotents in finite symmetric inverse semigroups. *Proc. Edinburgh Math. Soc.* **30** (1987), 383–395.

[8] S. Janson and V. Mazorchuk, Some remarks on the combinatorics of IS_n. *Semigroup Forum* **70** (2005), no. 3, 391–405.

[9] Y. Kochubinska, Combinatorics of partial wreath power of finite inverse symmetric semigroup IS_d. *Algebra Discrete Math.* (2007), no. 1, 49–60.

[10] A. Laradji and A. Umar, On the number of nilpotents in the partial symmetric semigroup. *Comm. Algebra* **32** (2004), 3017–3023.

[11] A. Laradji and A. Umar, A. Combinatorial results for the symmetric inverse semigroup. *Semigroup Forum* **75** (2007), no. 1, 221–236.

[12] S. Lipscomb, *Symmetric Inverse Semigroups*, Mathematical Surveys of The American mathematical Society, no. 46, Providence, R. I., 1996.

[13] N. J. A. Sloane (Ed.), *The On-Line Encyclopedia of Integer Sequences*, 2011. Available at http://oeis.org/.

[14] H. Wilf, *Generatingfunctionolgy*, third edition. A. K. Peters, Ltd., Wellesley, MA, 2006.

A. Laradji
Department of Mathematics & Statistics
King Fahd University of Petroleum & Minerals
Dhahran 31261 – SAUDI ARABIA
E-mail:`alaradji@kfupm.edu.sa`

and

A. Umar
Department of Mathematics and Statistics
Sultan Qaboos University
Al-Khod, PC 123 – OMAN
E-mail:`aumarh@squ.edu.om`