BIVARIATE GAUSSIAN FIBONACCI AND LUCAS POLYNOMIALS

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ABSTRACT. In this study we define and study the Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials. We give generating function, Binet formula, explicit formula and partial derivation of these polynomials. By defining these bivariate polynomials for special cases $F_n(x,1)$ is the Gaussian Fibonacci polynomials, $L_n(x,1)$ is the Gaussian Lucas polynomials, $F_n(1,1)$ is the Gaussian Fibonacci numbers and $L_n(1,1)$ is the Gaussian Lucas numbers defined in [19].

1. INTRODUCTION

Fibonacci Polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $f_n(x)$ studied by Catalan are defined by the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

where $f_0(x) = 0$, $f_1(x) = 1$, and $n > 2$. Notice that $f_n(1) = F_n$, the nth Fibonacci number.

Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$l_n(x) = xl_{n-1}(x) + l_{n-2}(x)$$

where $l_0(x) = 2$, $l_1(x) = x$ and $n \geq 2$.

Bivariate Fibonacci polynomials are defined by

$$f_n(x,y) = xf_{n-1}(x,y) + yf_{n-2}(x,y)$$

where $f_0(x,y) = 0$, $f_1(x,y) = 1$, and $n \geq 2$. By these polynomials $f_n(x,1)$ are the Fibonacci polynomials and $f_n(1,1) = F_n$, the nth Fibonacci number.

Bivariate Lucas polynomials are defined by

$$l_n(x,y) = xl_{n-1}(x,y) + yl_{n-2}(x,y)$$

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where $l_0(x, y) = 2$, $l_1(x, y) = x$, and $n \geq 2$. See that $l_n(x, 1)$ are the Lucas polynomials and $l_n(1, 1) = L_n$, the nth Lucas number.

The Gaussian Fibonacci sequence in [19] is $GF_0 = i$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. One can see that

$$GF_n = F_n + iF_{n-1}$$

where F_n is the usual nth Fibonacci number.

The Gaussian Lucas sequence in [19] is defined similar to Gaussian Fibonacci sequence as $GL_0 = 2-i$, $GL_1 = 1+2i$, and $GL_n = GL_{n-1}+GL_{n-2}$ for $n > 1$. Also it can be seen that

$$GL_n = L_n + iL_{n-1}$$

where L_n is the usual nth Lucas number.

The complex Fibonacci numbers and Gaussian Fibonacci numbers are studied by some other authors [11, 12, 16]. The complex Fibonacci polynomials were defined and studied in [18] by Horadam. Harman [12] give a new approach toward the extension of Fibonacci numbers into the complex plane. Before this study there were two different methods for defining such numbers studied by Horadam [17] and Berzsenyi [1]. Harman [12] generalized both of the methods. In [2, 3, 6, 7] theories of the generalized Fibonacci and Lucas polynomials are developed. Yu and Liang [26] derive some identities involving the partial derivative sequences of the bivariate Fibonacci polynomials $F_n(x, y)$ and the bivariate Lucas polynomials $L_n(x, y)$. Djordjevic [4, 5] considered the generating functions, explicit formulas and partial derivative sequences of the generalized Fibonacci and Lucas polynomials. Good [8] points out that the square root of the Golden Ratio is the real part of a simple periodic continued fraction but using (complex) Gaussian integers $a + ib$ instead of the natural integers. Tuglu et al. [24] study the bivariate Fibonacci and Lucas p–polynomials ($p \geq 0$ is integer) from which, specifying x, y and p, bivariate Fibonacci and Lucas polynomials, bivariate Pell and Pell-Lucas polynomials, Jacobsthal and Jacobsthal–Lucas polynomials, Fibonacci and Lucas $p-$polynomials, Fibonacci and Lucas p–numbers, Pell and Pell–Lucas p–numbers and Chebyshev polynomials of the first and second kind, are obtained. Webb and Parberry [27] study the divisibility properties of the Fibonacci polynomial sequence. Hoggatt and Long [13] obtain the results including results of Webb and Parberry and generalize the results to bivariate Fibonacci polynomials. For more information one can see [9, 10, 14, 15, 20, 21, 22, 23, 25]

In this article we define and study the Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials $GF_n(x, y)$ and $GL_n\ (x, y)$. We give generating function, Binet formula, explicit formula and partial derivation of these polynomials. Special cases of these bivariate polynomials are Gaussian Fibonacci polynomials $F_n(x, 1)$, Gaussian Lucas polynomials

$L_n(x,1)$, Gaussian Fibonacci numbers $F_n(1,1)$ and Gaussian Lucas numbers $L_n(1,1)$ defined in [19].

2. Bivariate Gaussian Fibonacci and Lucas Polynomials

Definition 1. *The Bivariate Gaussian Fibonacci polynomials $GF_n(x,y)$ are defined by the following recurrence relation*

$$GF_{n+1}(x,y) = xGF_n(x,y) + yGF_{n-1}(x,y), \quad n \geq 1 \tag{2.1}$$

with initial conditions $GF_0(x,y) = i$ and $GF_1(x,y) = 1$.

It can be easily seen that

$$GF_n(x,y) = f_n(x,y) + iyf_{n-1}(x,y)$$

where $f_n(x,y)$ is the nth bivariate Fibonacci Polynomial.

Definition 2. *The bivariate Gaussian Lucas polynomials $\{GL_n(x,y)\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GL_{n+1}(x,y) = xGL_n(x,y) + yGL_{n-1}(x,y), \quad n \geq 1 \tag{2.2}$$

with initial conditions $GL_0(x,y) = 2 - ix$ and $GL_1(x,y) = x + 2iy$.

Also

$$GL_n(x,y) = l_n(x,y) + iyl_{n-1}(x,y)$$

where $l_n(x,y)$ is the nth bivariate Lucas Polynomial.

For later use the first few terms of the sequence are as shown in the following tables

n	$GF_n(x,y)$
0	i
1	1
2	$x + iy$
3	$x^2 + y + ixy$
4	$x^3 + 2xy + iy(x^2 + y)$
5	$x^4 + 3x^2y + y^2 + iy(x^3 + 2xy)$
$\vdots$	$\vdots$

and

n	$GL_n(x,y)$
0	$2 - ix$
1	$x + i2y$
2	$x^2 + 2y + iy(x)$
3	$x^3 + 3xy + iy\left(x^2 + 2y\right)$
4	$x^4 + 4x^2y + 2y^2 + iy\left(x^3 + 3xy\right)$
5	$x^5 + 5x^3y + 5xy^2 + iy\left(x^4 + 4x^2y + 2y^2\right)$
$\vdots$	$\vdots$

2.1. Some Properties of Bivariate Gaussian Fibonacci and Lucas Polynomials.

Theorem 1. *The generating function for bivariate Gaussian Fibonacci Polynomials is*

$$g(t)=\sum_{n=0}^{\infty} GF_n(x,y)t^n=\frac{t+i(1-xt)}{1-xt-yt^2}$$

and for bivariate Gaussian Lucas Polynomials is

$$h(t)=\sum_{n=0}^{\infty} GL_n(x,y)t^n=\frac{2-xt-i(x-2yt-x^2t)}{1-xt-yt^2}.$$

Proof. Let $g(t)$ be the generating function of the Bivariate Gaussian Fibonacci polynomial sequence $GF_n\,(x,y)$ then

$$\begin{aligned} g(t)-xtg(t)-yt^2g(t) &= GF_0(x,y)+tGF_1(x,y)-xtGF_0(x,y) \\ &\quad +\sum_{n=2}^{\infty} t^n\left[GF_n(x,y)-xGF_{n-1}(x,y)\right. \\ &\quad \left.-yGF_{n-2}(x,y)\right] \\ &= t+i(1-xt) \end{aligned}$$

By taking $g(t)$ parenthesis we get

$$g(t)=\frac{t+i(1-xt)}{1-xt-yt^2}.$$

The proof is completed. □

Binet's formulas are well known and studied in the theory of Fibonacci numbers. Now we can get the Binet formula of bivariate Gaussian Fibonacci and Lucas polynomials. Let $\alpha(x,y)$ and $\beta(x,y)$ be the roots of the characteristic equation

$$t^2-xt-y=0$$

of the recurrence relation (2.1). Then

$$\alpha(x,y)=\frac{x+\sqrt{x^2+4y}}{2},\quad \beta(x,y)=\frac{x-\sqrt{x^2+4y}}{2}.$$

Note that $\alpha(x,y)+\beta(x,y)=x$ and $\alpha(x,y)\beta(x,y)=-y$. Now we can give the Binet formula for the bivariate Gaussian Fibonacci and Lucas polynomials.

Theorem 2. *For* $n\geq 0$

$$GF_n(x,y)=\frac{\alpha^n(x,y)-\beta^n(x,y)}{\alpha(x,y)-\beta(x,y)}+iy\frac{\alpha^{n-1}(x,y)-\beta^{n-1}(x,y)}{\alpha(x,y)-\beta(x,y)}$$

and

$$GL_n(x,y) = \alpha^n(x,y) + \beta^n(x,y) + iy\left(\alpha^{n-1}(x,y) + \beta^{n-1}(x,y)\right).$$

Proof. Theorem can be proved by mathematical induction on n. □

Theorem 3. *The explicit formula of bivariate Gaussian Fibonacci Polynomials is*

$$\begin{aligned} GF_{n+1}(x,y) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^{n-2k} y^k \\ &\quad + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1} y^{k+1}. \end{aligned}$$

Theorem 4. *The explicit formula of bivariate Gaussian Lucas Polynomials is*

$$\begin{aligned} GL_n(x,y) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k \\ &\quad + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-k-1} \binom{n-k-1}{k} x^{n-2k-1} y^{k+1}. \end{aligned}$$

Theorem 5. *Let $D_n(x,y)$ denote the $n \times n$ tridiagonal matrix as*

$$D_n(x,y) = \begin{bmatrix} 1 & i & 0 & \cdots & 0 \\ -y & x & y & \ddots & \vdots \\ 0 & -1 & x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y \\ 0 & \cdots & 0 & -1 & x \end{bmatrix}, n \geq 1$$

and let $D_0(x,y) = i$. Then $\det D_n(x,y) = GF_n(x,y)$, $n \geq 0$.

Proof. By induction on n we can prove the theorem. For $n = 1$ and $n = 2$,

$$\begin{aligned} \det D_1(x,y) &= 1 = GF_1(x,y) \\ \det D_2(x,y) &= x + iy = GF_2(x,y). \end{aligned}$$

Assume that

$$\det D_{n-1}(x,y) = GF_{n-1}(x,y)$$

and

$$\det D_{n-2}(x,y) = GF_{n-2}(x,y).$$

Then

$$\begin{aligned}\det D_n(x,y) &= x\det D_{n-1}(x,y) + y\det D_{n-2}(x,y)\\ &= xGF_{n-1}(x,y) + yGF_{n-2}(x,y)\\ &= GF_n(x,y).\end{aligned}$$

□

Theorem 6. *Let $H_n(x,y)$ denote the $n \times n$ tridiagonal matrix defined as*

$$H_n(x,y) = \begin{bmatrix} 2-ix & 2i & 0 & \cdots & 0 \\ 1-y & i & y & \ddots & \vdots \\ 0 & -1 & x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y \\ 0 & \cdots & 0 & -1 & x \end{bmatrix}, \quad n \geq 1.$$

Then $\det H_n(x,y) = GL_{n-1}(x,y)$, $n \geq 0$.

Now we introduce the matrix $Q(x,y)$ that plays the role of the Q-matrix. Let $Q(x,y)$ and P denote the 2×2 matrices defined as

$$Q(x,y) = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix}.$$

Then we can give the following theorem:

Theorem 7. *Let $n \geq 1$. Then*

$$Q(x,y)^n P = \begin{bmatrix} GF_{n+1}(x,y) & f_{n+1}(x,y) \\ GF_n(x,y) & f_n(x,y) \end{bmatrix}$$

where $f_n(x,y)$ is the nth bivariate Fibonacci Polynomial.

Proof. We can prove the theorem by induction on n. For $n = 1$

$$\begin{aligned}\begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix} &= \begin{bmatrix} x+iy & x \\ 1 & 1 \end{bmatrix}\\ &= \begin{bmatrix} GF_2(x,y) & f_2(x,y) \\ GF_1(x,y) & f_1(x,y) \end{bmatrix}.\end{aligned}$$

Assume that the theorem holds for $n = k$, that is

$$Q(x,y)^k P = \begin{bmatrix} GF_{k+1}(x,y) & f_{k+1}(x,y) \\ GF_k(x,y) & f_k(x,y) \end{bmatrix}.$$

Then for $n = k + 1$ we have

$$\begin{aligned} Q(x,y)^{k+1} P &= \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GF_{k+1}(x,y) & f_{k+1}(x,y) \\ GF_k(x,y) & f_k(x,y) \end{bmatrix} \\ &= \begin{bmatrix} GF_{k+2}(x,y) & f_{k+2}(x,y) \\ GF_{k+1}(x,y) & f_{k+1}(x,y) \end{bmatrix}. \end{aligned}$$

□

We can extend the definition of bivariate Gaussian Fibonacci polynomials and bivariate Gaussian Lucas polynomials to negative subscripts.

Definition 3. *For* $n \geq 1$

$$\begin{aligned} GF_{-n}(x,y) &= f_{-n}(x,y) + iyf_{-n-1}(x,y) \\ &= (-1)^{n-1}(f_n(x,y) - iyf_{n+1}(x,y) \end{aligned}$$

and

$$\begin{aligned} GL_{-n}(x,y) &= l_{-n}(x,y) + iyl_{-n-1}(x,y) \\ &= (-1)^{n-1}(l_n(x,y) - iyl_{n+1}(x,y) \end{aligned}$$

where $f(x,y)$ *and* $l(x,y)$ *are the bivariate Fibonacci and bivariate Lucas polynomials defined above.*

Theorem 8. *(Cassini Identity) For* $n \geq 1$

$$GF_{n-1}(x,y)GF_{n+1}(x,y) - GF_n^2(x,y) = (1 + y - ix)(-1)^n y^{n-1}.$$

Proof. We can prove the theorem by induction on n.
For $n = 1$

$$\begin{aligned} GF_0 GF_2 - GF_1^2 &= i(x + iy) - 1 \\ &= -1 - y + ix \\ &= (1 + y - ix)(-1)^1 y^0 \end{aligned}$$

and thus the theorem holds. Suppose that the theorem is true for an arbitrary positive integer k, that is

$$GF_{k-1}(x,y)GF_{k+1}(x,y) - GF_k^2(x,y) = (1 + y - ix)(-1)^k y^{k-1}.$$

Then for $k+1$

$$\begin{aligned}
& GF_k(x,y)GF_{k+2}(x,y) - GF_{k+1}^2(x,y) \\
= & \left(\frac{GF_{k+1}(x,y) - yGF_{k-1}(x,y)}{x}\right) \\
& \times (xGF_{k+1}(x,y) + yGF_k(x,y)) \\
& -GF_{k+1}^2(x,y) \\
= & GF_{k+1}^2(x,y) + \frac{y}{x}GF_{k+1}(x,y)GF_k(x,y) \\
& -yGF_{k-1}(x,y)GF_{k+1}(x,y) \\
& -\frac{y^2}{x}GF_{k-1}(x,y)GF_k(x,y) - GF_{k+1}^2(x,y) \\
= & \frac{y}{x}GF_{k+1}(x,y)GF_k(x,y) \\
& -\frac{y^2}{x}GF_{k-1}(x,y)GF_k(x,y) \\
& -y(1+y-ix)(-1)^k y^{k-1} - yGF_k^2(x,y) \\
= & \frac{y}{x}GF_{k+1}(x,y)GF_k(x,y) \\
& +(1+y-ix)(-1)^{k+1}y^k - yGF_k^2(x,y) \\
& -\frac{y^2}{x}GF_{k-1}(x,y)GF_k(x,y) \\
= & \frac{y}{x}GF_{k+1}(x,y)GF_k(x,y) \\
& +(1+y-ix)(-1)^{k+1}y^k \\
& -\frac{y}{x}GF_k(x,y)GF_{k+1}(x,y) \\
= & (1+y-ix)(-1)^{k+1}y^k.
\end{aligned}$$

This completes the proof. □

Theorem 9. *For $n \geq 1$*

$$GL_{n-1}(x,y)GL_{n+1}(x,y) - GL_n^2(x,y) = \left(x^2+4y\right)(1+y-ix)(-1)^{n+1}y^{n-1}.$$

Theorem 10. *For $n \geq 1$*

$$GL_n^2(x,y) - \left(x^2+4y\right)GF_n^2(x,y) = 4y^n(1+y-ix)(-1)^n.$$

Proof. These theorems can be proved by induction on n. □

Taking $x = y = 1$ in above theorems we get:

Corollary 1. [19] *If $x = y = 1$ then*

$$GF_{n+1}GF_{n-1} - GF_n^2 = (2-i)(-1)^n.$$

Corollary 2. [19] *If $x = y = 1$ then*

$$GL_{n+1}GL_{n-1} - GL_n^2 = 5\,(2-i)\,(-1)^{n+1}.$$

Corollary 3. [19] *If $x = y = 1$ then*

$$GL_{n+1}^2 - 5GF_n^2 = 4\,(2-i)\,(-1)^n.$$

Theorem 11. *For $n \geq 1$*

$$GL_n(x,y) = GF_{n+1}(x,y) + yGF_{n-1}(x,y).$$

Proof. We proceed by induction on n. Theorem is true for $n = 1$. Suppose that theorem is true for $n - 1$. Then

$$\begin{aligned}
GL_n(x,y) &= xGL_{n-1}(x,y) + yGL_{n-2}(x,y) \\
&= x\,(GF_n(x,y) + yGF_{n-2}(x,y)) \\
&\quad +y\,(GF_{n-1}(x,y) + yGF_{n-3}(x,y)) \\
&= xGF_n(x,y) + xyGF_{n-2}(x,y) \\
&\quad +yGF_{n-1}(x,y) + y^2GF_{n-3}(x,y) \\
&= xGF_n(x,y) + yGF_{n-1}(x,y) \\
&\quad +xyGF_{n-2}(x,y) + y^2GF_{n-3}(x,y) \\
&= xGF_n(x,y) + yGF_{n-1}(x,y) \\
&\quad +y\,(xGF_{n-2}(x,y) + yGF_{n-3}(x,y)) \\
&= GF_{n+1}(x,y) + yGF_{n-1}(x,y).
\end{aligned}$$

□

Theorem 12. *For $n \geq 1$*

$$GF_n(x,y) = \frac{GL_{n+1}(x,y) + yGL_{n-1}(x,y)}{x^2+4y}.$$

Theorem 13. *The sum of the bivariate Gaussian Fibonacci and bivariate Gaussian Lucas polynomials are given as:*

(i) $\displaystyle\sum_{k=1}^{n} x^{n-k}GF_k(x,y) = \frac{1}{y}\left[GF_{n+2}(x,y) - x^n(x+iy)\right]$

(ii) $\displaystyle\sum_{k=1}^{n} x^{n-k}GL_k(x,y) = \frac{1}{y}\left[GL_{n+2}(x,y) - x^n(x^2+2y+ixy)\right]$

Proof. (i) For $n \geq 2$ we have

$$GF_{n-1}(x,y) = \frac{1}{y}GF_{n+1}(x,y) - \frac{x}{y}GF_n(x,y).$$

Then from this equation

$$\begin{aligned}
x^{n-1}GF_1(x,y) &= \frac{x^{n-1}}{y}GF_3(x,y) - \frac{x^n}{y}GF_2(x,y) \\
x^{n-2}GF_2(x,y) &= \frac{x^{n-2}}{y}GF_4(x,y) - \frac{x^{n-1}}{y}GF_3(x,y) \\
&\vdots \\
xGF_{n-1}(x,y) &= \frac{x}{y}GF_{n+1}(x,y) - \frac{x^2}{y}GF_n(x,y) \\
GF_n(x,y) &= \frac{1}{y}GF_{n+2}(x,y) - \frac{x}{y}GF_{n+1}(x,y).
\end{aligned}$$

By adding both sides of the equations we get

$$\sum_{k=1}^{n} x^{n-k}GF_k(x,y) = \frac{1}{y}\left[GF_{n+2}(x,y) - x^n(x+iy)\right].$$

This completes the proof. □

Theorem 14. *For* $n \geq 0$

$$GF_n(x,y)GL_n(x,y) = (x+2iy)f_{2n-1}(x,y) + y(1-y)f_{2n-2}(x,y).$$

Proof. From the Binet formulas of $GF_n(x,y)$ and $GL_n(x,y)$ we have

$$\begin{aligned}
GF_n(x,y)GL_n(x,y) &= \left(\frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)}\right. \\
&\quad \left. + iy\frac{\alpha^{n-1}(x,y) - \beta^{n-1}(x,y)}{\alpha(x,y) - \beta(x,y)}\right) \\
&\quad \times \left(\alpha^n(x,y) + \beta^n(x,y)\right. \\
&\quad \left. + iy\left(\alpha^{n-1}(x,y) + \beta^{n-1}(x,y)\right)\right) \\
&= \frac{\alpha^{2n}(x,y) - \beta^{2n}(x,y)}{\alpha(x,y) - \beta(x,y)} \\
&\quad + 2iy\frac{\alpha^{2n-1}(x,y) - \beta^{2n-1}(x,y)}{\alpha(x,y) - \beta(x,y)} \\
&\quad - y^2\frac{\alpha^{2n-2}(x,y) - \beta^{2n-2}(x,y)}{\alpha(x,y) - \beta(x,y)} \\
&= f_{2n}(x,y) + 2iyf_{2n-1}(x,y) - y^2 f_{2n-2}(x,y) \\
&= xf_{2n-1}(x,y) + yf_{2n-2}(x,y) \\
&\quad + 2iyf_{2n-1}(x,y) - y^2 f_{2n-2}(x,y) \\
&= (x+2iy)f_{2n-1}(x,y) + y(1-y)f_{2n-2}(x,y)
\end{aligned}$$

□

Corollary 4. [19] *Taking $x = y = 1$ in above theorem we get*

$$GF_n GL_n = (1 + 2i)F_{2n-1}.$$

Theorem 15. *For the partial derivatives we have*

$$\frac{\partial GL_n(x,y)}{\partial x} = nGF_n(x,y) - iyf_{n-1}(x,y)$$

and

$$\frac{\partial GL_n(x,y)}{\partial y} = nGF_{n-1}(x,y) + if_n(x,y)$$

where $f_n(x, y)$ is the nth bivariate Fibonacci polynomial.

Proof. From the partial derivations of the explicit formula of bivariate Gaussian Lucas polynomials theorem can be proved. □

Theorem 16. *For $m \geq 1$ and $n \geq 0$*

$$GF_{m+n}(x,y) = f_m(x,y)GF_{n+1}(x,y) + yf_{m-1}(x,y)GF_n(x,y).$$

Corollary 5. *The norm of the bivariate Gaussian Fibonacci polynomials is*

$$f_n^2(x,y) + y^2 f_{n-1}^2(x,y) = (1-y)f_n^2(x,y) + yf_{2n-1}(x,y)$$

where $f_n(x, y)$ is the nth bivariate Fibonacci polynomial.

3. Conclusion

In this study we generalized the Gaussian Fibonacci and Lucas numbers which were defined in [19] to bivariate Gaussian Fibonacci and bivariate Gaussian Lucas Polynomials. We give many interesting properties of these polynomials. Special cases of these bivariate polynomials are Gaussian Fibonacci polynomials $F_n(x, 1)$, Gaussian Lucas polynomials $L_n(x, 1)$, Gaussian Fibonacci numbers $F_n(1, 1)$ and Gaussian Lucas numbers $L_n(1, 1)$ defined in [19].

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References

[1] Berzsenyi, G. "Gaussian Fibonacci numbers". Fibonacci Quart. 15 (1977), no. 3, 233 236.

[2] Catalini, M., "Generalized Bivariate Fibonacci Polynomials", arXiv:math.Co/0211366 v2, (2004).

[3] Catalini, M., "Some Formulae for Bivariate Fibonacci and Lucas Polynomials",arXiv:math.Co/0406323 v1, (2004).

[4] Djordjevic G.B., "Some properties of a class of polynomials", Matematiqki Vesnik 49 (1997) 265 271.

[5] Djordjevic G.B., "Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers", The Fibonacci Quarterly 42 (2) (2004) 106 113.

[6] Djordjevic G.B., "Some Properties of Partial Derivatives of Generalized Fibonacci and Lucas Polynomials", The Fibonacci Quarterly, 39 (2): 138-14 (2001).
[7] Frei, G., "Binary Lucas and Fibonacci Polynomials I.", Mathematische Nachrichten, 96: 83-112, (1980).
[8] Good, I. J. "Complex Fibonacci and Lucas numbers, continued fractions, and the square root of the golden ratio". Fibonacci Quart. 31 (1993), no. 1, 7 20.
[9] Gould H. W. , "Combinatorial Identities", Morgantown, W. Va., 50-90, (1972).
[10] G. Berzsenyi. "Sums of Products of Generalized Fibonacci Numbers." The Fibonacci Quarterly 13 (1975):343-344.
[11] G, Berzsenyi, "Gaussian Fibonacci Numbers." The Fibonacci Quarterly 15 (1977):233-236.
[12] Harman, C. J. "Complex Fibonacci numbers". Fibonacci Quart. 19 (1981), no. 1, 82 86.
[13] Hoggatt V.E. Jr.; Long C.T., "Divisibility properties of generalized Fibonacci polynomials". Fibonacci Quarterly, 12: 113-120, (1974).
[14] Hoggatt, V. E. Jr., & Marjorie Bicknell. "Roots of Fibonacci Polynomials."The Fibonacci Quarterly 11, no. 3 (1973):271-74.
[15] Horadam, A. F. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." Duke Math. J. 32 (1965):437-46.
[16] Horadam, A. F. "A Generalized Fibonacci Sequence," American Math. Monthly 68 (1961):455˜459,
[17] Horadam, A. F. "Complex Fibonacci Numbers and Fibonacci Quaternions." American Math. Monthly 70 (1963):289-291.
[18] Horadam, A. F.; Horadam, E. M. "Roots of recurrence-generated polynomials". Fibonacci Quart. 20 (1982), no. 3, 219 226.
[19] Jordan, J. H. "Gaussian Fibonacci and Lucas numbers". Fibonacci Quart. 3 1965 315 318.
[20] Koshy T. "Fibonacci and Lucas Numbers with Applications," A Wiley-Interscience Publication, (2001).
[21] Nalli, Aysc; Haukkanen, Pentti "On generalized Fibonacci and Lucas polynomials". Chaos Solitons Fractals 42 (2009), no. 5, 3179 3186.
[22] Pethe, S.; Horadam, A. F. "Generalised Gaussian Fibonacci numbers". Bull. Austral. Math. Soc. 33 (1986), no. 1, 37 48.
[23] Tan, M., Zhang, Y., "A Note on Bivariate and Trivariate Fibonacci Polynomials", Southeast Asian Bulletin of Mathematics, 32: 975-990, (2005).
[24] Tuglu, Naim; Kocer, E. Gokcen; Stakhov, Alexey "Bivariate Fibonacci like p-polynomials". Appl. Math. Comput. 217 (2011), no. 24, 10239 10246.
[25] Vajda, S., "Fibonacci and Lucas Numbers, and the Golden Section Theory and Applications", Ellis Harwood Limitted, (1989).
[26] Yu, Hongquan; Liang, Chuanguang "Identities involving partial derivatives of bivariate Fibonacci and Lucas polynomials". Fibonacci Quart. 35 (1997), no. 1, 19 23.
[27] Webb, W.A.; Parberry, E. A., "Divisibility properties of Fibonacci polynomials", The Fibonacci Quarterly, 7: 457-463, (1969).

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