The Degree-Sum of Adjacent Vertices, Girth and Upper Embeddability *

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Abstract: Combined with the edge-connectivity, this paper investigates the relationship between the degree-sum of adjacent vertices, girth and upper embeddability of graphs, and obtains the main result: Let G be a k-edge-connected simple graph with girth g. If there exist an integer m $(1 \le m \le g)$, such that for any m consecutively adjacent vertices $x_i(i = 1, 2, \cdots, m)$ in any non-chord cycle C of G, it has

$$\sum_{i=1}^{m} d_G(x_i) > \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m$$

where $k = 1, 2, 3,\ n = |V(G)|$, then G is upper embeddable and the upper bound is best possible.

Keywords: Graph; Betti Deficiency number; Upper Embeddability; Girth.
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1 Introduction

For graphical terminology not explained in this paper, readers are referred to [1]. All graphs considered here are finite and undirected and, unless explicitly stated otherwise, they are also connected. A *graph* is denoted by $G = (V(G), E(G))$, with $V(G), E(G)$ be its vertex set and edge set, respectively. In general, we allow graphs to have loops and multiple edges. Graphs which lack both multiple edges and loops will be called *simple*. The minimum length of a cycle in a graph G is the *grith* $g(G)$ of G, or simply by g if the graph is clear from the context. And if G does not contain a

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cycle, we set $g(G) = \infty$. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. For any cycle (not necessarily the shortest one) which does not have any chord, we call it a *non-chord cycle*. For any nonnegative integer x, $\lfloor x \rfloor$ denotes the largest integer no more than x. And the cardinality of a set X will be denoted by $|X|$.

Recall that the *maximum genus* of a graph G, denoted by $\gamma_M(G)$, is the maximum integer k with the property that there exists a cellular embedding of G on the orientable surface S_k of genus k. Since any cellular embedding must have at least one face, then the *Euler's formula* implies that $\gamma_M(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as the *Betti number* (also called the *cycle rank*) of G. A graph G is said to be *upper embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$. It is clear that if G is a tree, then G is upper embeddable, so we will also assume that the graphs considered in this paper have at least one cycle. Since the original investigation of the maximum genus of graphs by Nordhaus, Stewart and White [2], the maximum genus of graphs has received a considerable attention. Many authors were dealing particularly with upper embeddability of graphs [2, 3, 4, 5]. A known result due to Xuong [5] showed that every 4-edge-connected graph is upper embeddable, however, a little weaker condition of k-edge-connectivity ($1 \leq k \leq 3$) cannot guarantee the result [6]. So, the problem of what conditions may be added to get a k-edge-connected ($1 \leq k \leq 3$) upper embedded graph arises naturally. In [7], Huang and Liu studied the relationship between the upper embeddability of graphs and the degree-sum of nonadjacent vertices, and obtained the following results:

Theorem A.[7] *Let G be a 2-edge connected(or 3-edge connected) simple graph. If for any $uv \notin E(G)$ it has*

$$d_G(u) + d_G(v) \geq \frac{2(|V(G)| - 2)}{3} \left(\text{ or } \frac{|V(G)| + 1}{3} \right)$$

then G is upper embeddable. Furthermore, the lower bound is best possible.

Naturally, people may ask whether there exists any relationship between the upper embeddability of graphs and the degree-sum of adjacent vertices. In this paper, we give an affirmative answer to this question, and obtain the following main theorem:

Theorem. *Let G be a k-edge-connected simple graph with girth g. If there exist an integer m ($1 \leq m \leq g$), such that for any m consecutively adjacent vertices x_i ($i = 1, 2, \cdots, m$) in any non-chord cycle C of G, it has*

$$\sum_{i=1}^{m} d_G(x_i) > \frac{mn}{(k-1)^2 + 2} + \frac{km}{g} + (2 - g)m$$

where $k = 1, 2, 3$, $n = |V(G)|$, then G is upper embeddable and the upper bound is best possible.

This paper is organized as follows. Some basic lemmas are given in Section 2. Section 3 is devoted to the proof of Theorem and Corollary. And in the last section, we will remark that the conditions and results of Corollary are better than those of Theorem A when $g \geq 4$, furthermore, some upper embeddable graphs which can't be determined by Theorem A, but can be determined by Corollary, are presented.

2 Basic Lemmas

Let G be a graph and $A \subseteq E(G)$, $G \setminus A$ is the graph obtained from G by removing all edges in A. Let T be a spanning tree of G, denote by $\xi(G,T)$ the number of components of $G \setminus E(T)$ with odd number of edges. The *Betti deficiency* $\xi(G)$ *of the graph* G is defined to be the minimum of $\xi(G,T)$ over all spanning trees T.

In the study of maximum genus, one of the most remarkable facts is that this topological invariant can be characterized in a purely combinatorial form. The following lemma due to Xuong [3] is a basic result in studying the maximum genus of a graph G, which gives a formula on $\gamma_M(G)$ by means of $\xi(G)$ and $\beta(G)$, and also presents a necessary and sufficient condition for an upper embeddable graph.

Lemma 1.[3] *Let G be a graph. Then*

(1) $\gamma_M(G) = \frac{\beta(G)-\xi(G)}{2}$, *and*

(2) *G is upper embeddable if and only if* $\xi(G) \leq 1$.

Lemma 1 tells us that the maximum genus $\gamma_M(G)$ of a graph G is mainly determined by $\xi(G)$.

Let G be a connected graph and $A \subseteq E(G)$, let $c(G \setminus A)$ and $b(G \setminus A)$ denote the number of components of $G \setminus A$ and the number of components of $G \setminus A$ with odd Betti number, respectively. Nebesky [4] gave another combinatorial expression of $\xi(G)$:

Lemma 2.[4] *Let G be a connected graph. Then*

$$\xi(G) = \max_{A \subseteq E(G)} \{c(G \setminus A) + b(G \setminus A) - |A| - 1\}.$$

Let G be a graph and $A \subseteq E(G)$, $F_{i_1}, F_{i_2}, \cdots, F_{i_s}$ be some connected components of $G \setminus A$. Denote by $E(F_{i_1}, F_{i_2}, \cdots, F_{i_s})$ the set of those edges of G whose two end vertices are respectively in two pairwise subgraphs F_{i_j} and F_{i_t}, for $1 \leq j, t \leq s$ and $j \neq t$, and let $E(F, G)$ denote the edge set consisting of all such edges $e \in E(G)$ such that one end vertex of e is in F but the other not in F.

The following result due to Huang [8] provides a structural characterization of a not upper embeddable graph, namely, the graph G with $\xi(G) \geq 2$.

Lemma 3.[8] *Let G be a graph. If G is not upper embeddable, i.e. $\xi(G) \geq 2$, then there exists an edge subset A of G satisfying the following properties:*

(1) $c(G \setminus A) = b(G \setminus A) \geq 2$;

(2) *For any connected component F of $G \setminus A$, F is a vertex-induced subgraph of G;*

(3) $|E(F_{i_1}, F_{i_2}, \cdots, F_{i_s})| \leq 2s-3$ *for any s $(s \geq 2)$ distinct components $F_{i_1}, F_{i_2}, \cdots, F_{i_s}$ of $G \setminus A$;*

(4) $\xi(G) = 2c(G \setminus A) - |A| - 1$.

Under the conditions and conclusions of Lemma 3, let $F_1, F_2, \cdots, F_l$ be all connected components of $G \setminus A$, then we have the following Lemma 4.

Lemma 4. (1) *For any connected component F of $G \setminus A$, if G is k-edge-connected, then $|E(F, G)| \geq k$;*

(2) $|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, G)|$;

(3) *If G is k-edge-connected , then*

$$l = c(G \setminus A) \geq \begin{cases} 2, & for\ \ k = 1, \\ 3, & for\ \ k = 2, \\ 6, & for\ \ k = 3. \end{cases}$$

Proof. It is clear that properties (1) and (2) are true, therefore, we only need to prove property (3).

For $k = 1$, it is obvious that property 3 holds by Lemma 3(1).

For $k = 2$ or 3, by the properties (1) and (2), we know that $|A| = \frac{1}{2}\sum_{i=1}^{l} |E(F_i, G)| \geq \frac{k}{2}l$, So

$$\xi(G) = 2c(G \setminus A) - |A| - 1 \leq 2l - \frac{k}{2}l - 1.$$

Since $\xi(G) \geq 2$, then $2l - \frac{k}{2}l - 1 \geq 2$, which implies

$$l \geq \frac{6}{4-k} = \begin{cases} 3, & for\ k = 2, \\ 6, & for\ k = 3. \end{cases}$$

□

3 The Proof of Theorem and Corollary

Now, we move to the proof of Theorem.

Proof. Suppose that G is not upper embeddable, then there exists an edge set $A \subseteq E(G)$ such that the properties (1)-(4) of Lemma 3 are all satisfied. Let $F_1, F_2, \cdots, F_l$ be all connected components of $G \setminus A$, where $l = c(G \setminus A) \geq 2$. For any vertex $x \in V(F_i)$, $i = 1, 2, \cdots, l$, let $d_A(x) = d_G(x) - d_{F_i}(x)$. Lemma 3(1) tells us that $\beta(F_i) \equiv 1$ (modulo 2) for $1 \leq i \leq l$, and so F_i is not a tree, i.e, F_i has at least one non-chord cycle. First of all, we will prove the following claim.

Claim. For $1 \leq i \leq l$, let $C^i = x_1^i x_2^i \cdots x_s^i x_1^i$ $(s \geq g)$ be any non-chord cycle of F_i, then for any m $(1 \leq m \leq g)$ consecutively adjacent vertices $x_{t_i}^i, x_{t_i+1}^i, \cdots, x_{t_i+m-1}^i \in V(C^i)$, $(1 \leq t_i \leq |V(C^i)|$ and the subindices of x are modulo $|V(C^i)|)$, it has

$$\sum_{j=t_i}^{t_i+m-1} d_A(x_j^i) \leq \frac{m|E(F_i, G)|}{g}.$$

Subproof. Suppose that Claim is false, then there must exist m, such that for any m consecutively vertices on cycle C^i, we have

$$d_A(x_1^i) + d_A(x_2^i) + d_A(x_3^i) + \cdots + d_A(x_m^i) > \frac{m|E(F_i, G)|}{g}$$

$$d_A(x_2^i) + d_A(x_3^i) + d_A(x_4^i) + \cdots + d_A(x_{m+1}^i) > \frac{m|E(F_i, G)|}{g}$$

$$\vdots$$

$$d_A(x_s^i) + d_A(x_1^i) + d_A(x_2^i) + \cdots + d_A(x_{m-1}^i) > \frac{m|E(F_i, G)|}{g}$$

From the summation of the above inequalities, we can easily get

$$m\sum_{j=1}^{s} d_A(x_j^i) > \frac{m|E(F_i, G)|}{g} s$$

Since $s \geq g$, so

$$\sum_{j=1}^{s} d_A(x_j^i) > \frac{|E(F_i, G)|}{g} s \geq \frac{|E(F_i, G)|}{g} g = |E(F_i, G)|$$

But by the definition of $d_A(x_j^i)$, we know

$$\sum_{j=1}^{s} d_A(x_j^i) \leq |E(F_i, G)|$$

This contradiction means that Claim is true.

Now we continue to the proof of Theorem. Since G is simple, and so is F_i, then for any $x \in V(F_i)$, we have

$$d_{F_i}(x) \leq |V(F_i)| - (g-2),$$

By Claim, for any non-chord cycle C^i of F_i, there exist m consecutively adjacent vertices $x^i_{t_i}, x^i_{t_i+1}, \cdots, x^i_{t_i+m-1} \in V(C^i)$, $(1 \leq t_i \leq |V(C^i)|$ and the subindices of x are modulo $|V(C^i)|)$, such that

$$\sum_{j=t_i}^{t_i+m-1} d_A(x^i_j) \leq \frac{m|E(F_i, G)|}{g}.$$

Therefore

$$\begin{aligned}\sum_{j=t_i}^{t_i+m-1} d_G(x^i_j) &= \sum_{j=t_i}^{t_i+m-1} d_{F_i}(x^i_j) + \sum_{j=t_i}^{t_i+m-1} d_A(x^i_j) \\ &\leq m|V(F_i)| - m(g-2) + \frac{m|E(F_i,G)|}{g}\end{aligned}$$

By Lemma 3(3), we have

$$\begin{aligned}\sum_{i=1}^{l}\sum_{j=t_i}^{t_i+m-1} d_G(x^i_j) &\leq m\sum_{i=1}^{l}|V(F_i)| - ml(g-2) + \sum_{i=1}^{l}\frac{m|E(F_i,G)|}{g} \\ &\leq mn - ml(g-2) + \frac{2m(2l-3)}{g}\end{aligned} \tag{1}$$

However, the conditions of Theorem tell us that

$$\sum_{j=t_i}^{t_i+m-1} d_G(x^i_j) > \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m$$

thus

$$\sum_{i=1}^{l}\sum_{j=t_i}^{t_i+m-1} d_G(x^i_j) > \frac{mnl}{(k-1)^2+2} + \frac{kml}{g} + (2-g)ml \tag{2}$$

Combining inequality (1) and inequality (2), we can easily get

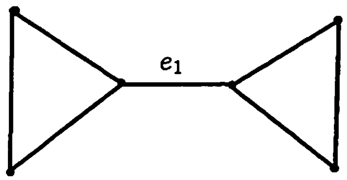
$$\frac{mnl}{(k-1)^2+2} + \frac{kml}{g} + (2-g)ml \leq mn - ml(g-2) + \frac{2m(2l-3)}{g} \tag{3}$$

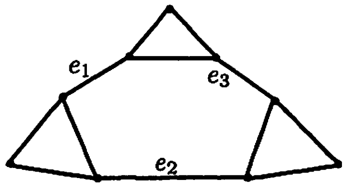
By inequality (3), we have

$$l < \frac{mng - 6m}{\frac{mng}{(k-1)^2+2} + (k-4)m} = \begin{cases} 2, & for\ k=1, \\ 3, & for\ k=2, \\ 6, & for\ k=3. \end{cases}$$

This is a contradiction to Lemma 4(3), so G is upper embeddable.

Next we will show that the bound in Theorem is best possible, that is the strict inequality " $>$ " in Theorem can't be replaced by " $\geq$ ".





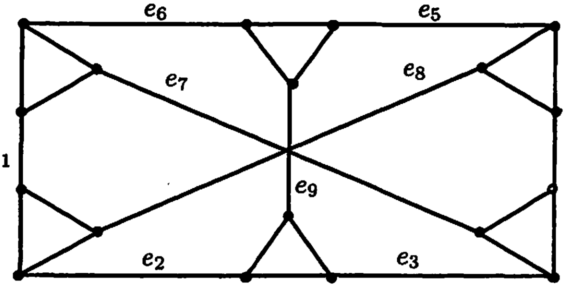


Fig. 1

It can be seen that the graphs $G_k (k = 1, 2, 3)$ shown in Fig.1 are k-edge-connected with $g = 3$. Set $m = 3$, for any 3 consecutively adjacent vertices a_{kj} in any non-chord cycle of G_k, $k, j = 1, 2, 3$, it is easy to check that

$$\sum_{j=1}^{3} d_{G_k}(a_{kj}) \geq 6 + k = \frac{3|V(G_k)|}{(k-1)^2 + 2} + \frac{3k}{3} + 3(2 - 3)$$

We set $A_1 = \{e_1\}$ for G_1; $A_2 = \{e_1, e_2\}$ for G_2; and $A_3 = \{e_1, e_2, \cdots, e_9\}$ for G_3; then

$$c(G_k \setminus A_k) + b(G_k \setminus A_k) - |A_k| - 1 = 2.$$

By Lemma 2, $\xi(G_k) \geq 2$, i.e. G_k is not upper embeddable.

Until now, we have completed the proof of Theorem. □

Corollary. *Let G be a k-edge-connected simple graph with girth g, and $C = x_1 x_2 \ldots x_s x_1$ $(s \geq g)$ be any non-chord cycle of G. If there exist an integer m $(1 \leq m \leq \lfloor \frac{g}{2} \rfloor)$, such that for any m consecutively nonadjacent* *vertices $x_i, x_{i+2}, \ldots, x_{i+2m-2}$ ($1 \le i \le s$, and subindices are modulo s) in C, it has*

$$\sum_{j=0}^{m-1} d_G(x_{i+2j}) > \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m$$

where $k = 1, 2, 3$, $n = |V(G)|$, then G is upper embeddable.

Proof. First, we define a mapping σ on vector $(a_1\, a_2 \ldots a_{s-1}\, a_s)^T$ such that:

$$\sigma \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{s-1} \\ a_s \end{pmatrix} = \begin{pmatrix} a_s \\ a_1 \\ \vdots \\ a_{s-2} \\ a_{s-1} \end{pmatrix}$$

Let $X = (\underbrace{1\ 0\ 1\ 0\ \cdots\ 1}_{2m-1}\ 0\ \cdots\ 0\ )^T_{1\times s}$, and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three matrixes as follows:

$$\mathcal{A} = (\ d_G(x_1)\ \ d_G(x_2)\ \ d_G(x_3)\ \ \cdots\ \ d_G(x_s)\ ),$$

$$\mathcal{B} = (\ X\ \ \sigma(X)\ \ \sigma^2(X)\ \ \cdots\ \ \sigma^{s-1}(X)\ ),$$

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{s\times s}$$

Set $\mathcal{P} = \mathcal{AB} = (p_1\ \ p_2\ \ \cdots\ \ p_s)$. Obviously, for $1 \le i \le s$, $p_i = \mathcal{A}\sigma^{i-1}(X) = \sum_{j=0}^{m-1} d_G(x_{i+2j})$, $\sigma^0(X) = X$. From the conditions of Corollary, we have

$$p_i > \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m \tag{4}$$

Let $\mathcal{Q} = \mathcal{PC} = (q_1\ \ q_2\ \ \cdots\ \ q_s)$, it is quite clear that $q_i = p_i + p_{i+1} = \sum_{j=0}^{2m-1} d_G(x_{i+j})$, $i = 1, 2, \cdots, s$, $i+1$ (modulo s). By inequality (4), we have

$$p_i + p_{i+1} > \frac{2mn}{(k-1)^2+2} + \frac{2km}{g} + (2-g)2m$$

Thus,

$$\sum_{j=0}^{2m-1} d_G(x_{i+j}) > \frac{2mn}{(k-1)^2+2} + \frac{2km}{g} + (2-g)2m$$

So G is upper embeddable by Theorem. □

4 Remarks

In this section, we will prove that the conditions and results of Corollary are better than those of Theorem A when $g \geq 4$ and present some upper embeddable graphs which cannot be determined by Theorem A, but can be determined by Corollary.

Remark 1. First, it is easy to see that the conditions of Corollary are more weaker than those of Theorem A. Second, the lower bounds of Corollary are better than those of Theorem A when $g \geq 4$. Since by taking $m = 2$, $k = 2$, we have

$$\begin{aligned} \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m &= \frac{2n}{(2-1)^2+2} + \frac{4}{g} + (2-g)2 \\ &\leq \frac{2n}{3} - 3 \\ &= \frac{2(n-2)}{3} - \frac{5}{3} \\ &< \frac{2(n-2)}{3} - 1 \end{aligned}$$

And by taking $m = 2, k = 3$, we have

$$\begin{aligned} \frac{mn}{(k-1)^2+2} + \frac{km}{g} + (2-g)m &= \frac{2n}{(3-1)^2+2} + \frac{6}{g} + (2-g)2 \\ &\leq \frac{n}{3} - \frac{5}{2} \\ &= \frac{n+1}{3} - \frac{17}{6} \\ &< \frac{n+1}{3} - 2 \end{aligned}$$

Remark 2. There indeed exist upper embeddable graphs with $g \geq 4$ which can be determined by Corollary, but can't be determined by Theorem A. Let Similar Petersen Graph be the 2-edge connected graph shown in the left hand side of Fig.2, and $C(18, 9)$ be the 3-edge connected graph obtained by adding edges $v_i v_{i+9}$ ($i = 1, 2, \cdots, 9$) to the cycle C_{18}, which is shown in the right hand side of Fig.2. It is a routine task to check that the two above graphs are upper embeddable according to Corollary just by taking $m = 2$, but their upper embeddability can't be determined by Theorem A.

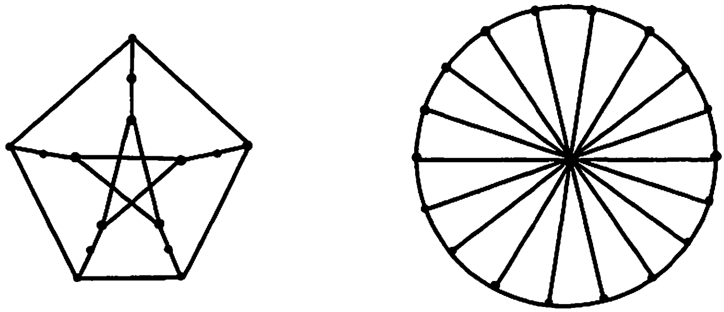


Fig. 2

Remark 3. When $g = 3$, there also exist upper embeddable graphs which can be determined by Corollary, but can't be determined by Theorem A. Let G_4 and G_5 be the two graphs shown in Fig.3. It is seen that G_4 and G_5 are 2-edge connected and 3-edge connected, respectively. It is easy to check that the two graphs are upper embeddable according to Corollary by taking $m = 1$, but their upper embeddability can't be determined by Theorem A.

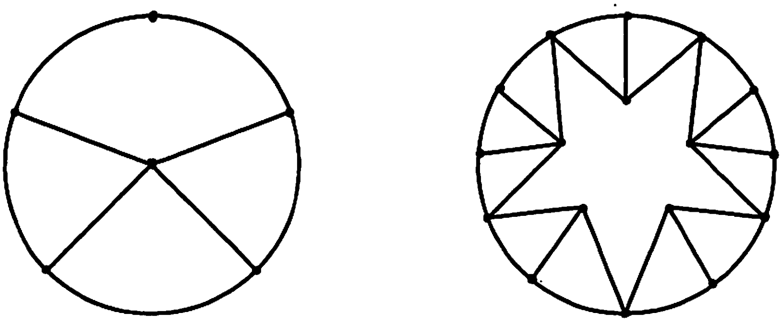


Fig. 3

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