Expanders and the Affine Building of Sp_n

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Abstract

For $n \geq 2$ and a local field K, let Δ_n denote the affine building naturally associated to the symplectic group $\mathrm{Sp}_n(K)$. We compute the spectral radius of the subgraph Y_n of Δ_n induced by the special vertices in Δ_n, from which it follows that Y_n is an analogue of a family of expanders and is non-amenable.

Introduction

In [2], Bien considers the problem of constructing a model for an efficient communication network. Such a model can be represented by a magnifier—a graph with a small number of edges such that every subset of vertices has many distinct neighbors (see [2, p. 6]). For a finite graph X, the "quality" of X as a network can be quantified by its isoperimetric constant $h(X)$ (see [7, p. 1]). Davidoff, Sarnak, and Valette explore the problem of explicitly constructing a family of expanders; i.e., a family $\{X_m\}$ of finite, connected, r-regular ($r \geq 2$) graphs with $\mathrm{Card}(X_m) \to \infty$ as $m \to \infty$ such that there is an $\varepsilon > 0$ satisfying $h(X_m) \geq \varepsilon$ for all m [7, Definition 0.3.3]. Families of expanders have become building blocks in many engineering applications, including network designs, complexity theory, coding theory, and cryptography (see [7, p. 3] and the references cited there). Note that by [7, p. 4], an infinite family of r-regular Ramanujan graphs is not only a family of expanders but is also optimal from the spectral viewpoint.

Let K be a local field, and let Ξ_n denote the affine building naturally associated to $\mathrm{SL}_n(K)$. In [14, Example 3], Saloff-Coste and Woess explicitly calculate the spectral radius of the simple random walk on the one-complex X_n of Ξ_n. This gives the spectral radius $\rho(X_n)$ of X_n [6, Theorem 1]. Then $h(X_n) > 0$ (by [3, Theorem 3.3]) and the number of vertices in X_n contained in concentric balls grows exponentially with the radius of the balls (by [3, Theorem 2.2]); alternatively, X_n is *expanding*. Note that $h(X_n) > 0$ implies X_n non-amenable (see the paragraph following Theorem 2.7).

Let Δ_n denote the affine building naturally associated to the symplectic group $\mathrm{Sp}_n(K) \leq \mathrm{SL}_{2n}(K)$. In this paper, we consider the subgraph Y_n of Δ_n induced by the special vertices in Δ_n (all the vertices in Ξ_n are special). Using the techniques of [14, Example 3], we compute $\rho(Y_n)$. Then $h(Y_n) > 0$ (by [3, Theorem 3.3]) and the number of vertices in Y_n contained in concentric balls grows exponentially with the radius of the balls (by [3, Theorem 2.2]); hence, Y_n is also expanding and non-amenable.

After completing this work, we learned that it is also possible to derive the formula in Proposition 2.6 using the techniques and results of James Parkinson. Parkinson's approach is quite general, as it takes a building-theoretic perspective rather than the group-theoretic one we use. As in [14, Example 3], Parkinson's approach is through a simple random walk: by [12, Theorem 6.3] and general facts about C^*-algebras, the spectral radius of an isotropic random walk (of which a simple random walk is an example) on an arbitrary thick, regular, affine building of irreducible type is $\widehat{A}(1)$, where A is the transition operator of the random walk and $\widehat{A}$ its Gelfand transform. To express $\widehat{A}(1)$ for the graph Y_n in terms of the order q of the residue field of K (as in Proposition 2.6 below), one identifies and uses the underlying root system of the building Δ_n, together with results about the Macdonald spherical functions defined in [12, p. 580]. In contrast, our approach is through the natural association of Δ_n with $\mathrm{Sp}_n(K)$—in particular, the transitive action of $\mathrm{GSp}_n(K)$ (the analogue of $\mathrm{GL}_{2n}(K)$ for $\mathrm{Sp}_n(K)$) on the special vertices in Δ_n (see Proposition 1.4). As a result, we deduce properties about $\mathrm{Sp}_n(K)$ and $\mathrm{GSp}_n(K)$—for example, we produce a solvable subgroup of $\mathrm{GSp}_n(K)$ that acts transitively on the vertices in Y_n. In addition, we characterize the set of vertices in Y_n adjacent to a given one in terms of orbits (Proposition 2.4).

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1 The Affine Building Δ_n of $\mathrm{Sp}_n(K)$

Fix a local field K with discrete valuation "ord," valuation ring $\mathcal{O}$, uniformizer π, and residue field $k \cong \mathbb{F}_q$. The affine building Δ_n naturally associated to $\mathrm{Sp}_n(K)$ can be modeled as an n-dimensional simplicial complex as follows (see [9, pp. 336 - 337]). Fix a $2n$-dimensional K-vector space V endowed with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$, and recall that a subspace U of V is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$. A *lattice* in V is a free, rank $2n$, $\mathcal{O}$-submodule of V, and two lattices L and L' in V are *homothetic* if $L' = \alpha L$ for some $\alpha \in K^\times$; write $[L]$ for the homothety class of the lattice L. A lattice L is *primitive* if $\langle L, L \rangle \subseteq \mathcal{O}$ and $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $L/\pi L$. Then a *vertex* in Δ_n is a homothety class of lattices in V with a representative L such that there is a primitive lattice L_0 with $\langle L, L \rangle \subseteq \pi\mathcal{O}$

and $\pi L_0 \subseteq L \subseteq L_0$; equivalently, $L/\pi L_0$ is a totally isotropic k-subspace of $L_0/\pi L_0$. Two vertices $t, t' \in \Delta_n$ are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$, $\langle L', L'\rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq L \subseteq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subseteq L \subseteq L_0$. Thus, a maximal simplex or *chamber* in Δ_n has $n+1$ vertices $t_0, \ldots, t_n$ with representatives $L_i \in t_i$ such that L_0 is primitive, $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \le i \le n$, and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$.

Recall that a basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V is *symplectic* if $\langle u_i, w_j\rangle = \delta_{ij}$ (Kronecker delta) and $\langle u_i, u_j\rangle = 0 = \langle w_i, w_j\rangle$ for all i, j. If a 2-dimensional, totally isotropic subspace U of V is a hyperbolic plane, then a frame is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines (1-dimensional K-subspaces) in V such that

1. $\lambda_i^1 + \lambda_i^2$ is a hyperbolic plane for all $1 \le i \le n$,
2. $\lambda_i^1 + \lambda_i^2$ is orthogonal to $\lambda_j^1 + \lambda_j^2$ for all $i \ne j$, and
3. $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$.

A vertex $t \in \Delta_n$ lies in the apartment specified by the frame $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ if for any representative $L \in t$, there are lattices M_i^j in λ_i^j for all i, j such that $L = M_1^1 + M_1^2 + \cdots + M_n^1 + M_n^2$. The following lemma is easily established.

Lemma 1.1.

1. *Every symplectic basis for V specifies an apartment of Δ_n.*
2. *If Σ is an apartment of Δ_n, there is a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V such that every vertex in Σ has the form*

$$[\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n]$$

for some $a_i, b_i \in \mathbb{Z}$.

Since π is fixed, if $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis for V, follow [16, p. 3411] and write $(a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ for the lattice $\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$ and $[a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}}$ for its homothety class. Then the lattice $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ is primitive if and only if $a_i + b_i = 0$ for all i by [16, p. 3411], and $[L]$ is a *special* vertex in Δ_n if and only if $a_i + b_i = \mu$ is constant for all i by [16, Corollary 3.4]. Note that by [16, p. 3412], a chamber in Δ_n has exactly two special vertices.

Proposition 1.2. *Every special vertex in Δ_n is contained in exactly*

$$\prod_{m=1}^{n} \frac{q^{2m}-1}{q-1}$$

chambers in Δ_n.

Proof. Let $t \in \Delta_n$ be a special vertex. Then the number of chambers in Δ_n containing t is the number of chambers in the spherical $C_n(k)$ building (cf. [5, p. 138]). By [13, p. 6], a chamber in the spherical $C_n(k)$ building is a maximal flag of non-trivial, totally isotropic subspaces of a $2n$-dimensional k-vector space endowed with a non-degenerate, alternating bilinear form. An obvious modification of the proof of [15, Proposition 2.4] finishes the proof. □

Let $C \in \Delta_n$ be a chamber with vertices $t_0, \ldots, t_n$, and let $L_i \in t_i$ be representatives such that L_0 is primitive, $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $1 \le i \le n$, and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. Let Σ be an apartment of Δ_n containing C and $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 1.1. For all $0 \le j \le n$, let

$$L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}.$$

Lemma 1.3. *The two special vertices in C are $[L_0]$ and $[L_n]$.*

Proof. The fact that $[L_0]$ is special follows from [16, Corollary 3.4] and [16, p. 3411]. To see that $[L_n]$ is special, note that if L_j represents a special vertex in C for $1 \le j \le n$, then $a_i^{(j)} + b_i^{(j)} = \mu$ for all i (by [16, Corollary 3.4]), where $\mu \in \{1, 2\}$ (since $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$). But $\mu = 2$ implies $L_j = \pi L_0$, which is impossible. Thus, $a_i^{(j)} + b_i^{(j)} = 1$ for all i and $L_j/\pi L_0 \cong k^n$; hence, $j = n$. □

Let

$$\begin{aligned}\mathrm{GSp}_n(K) = \{g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(K) : A, B, C, D \in M_n(K),\ A^tC = C^tA, \\ B^tD = D^tB,\ A^tD - C^tB = \nu(g)I_n \text{ for some } \nu(g) \in K^\times\};\end{aligned}$$

alternatively, abuse notation and think of $\mathrm{GSp}_n(K)$ as

$$\begin{aligned}\{g \in \mathrm{GL}_K(V) : \forall\, v_1, v_2 \in V, \exists\, \nu(g) \in K^\times \text{ such that} \\ \langle gv_1, gv_2 \rangle = \nu(g)\langle v_1, v_2 \rangle\}.\end{aligned}$$

Then $\mathrm{Sp}_n(K)$ consists of those $g \in \mathrm{GSp}_n(K)$ such that $\nu(g) = 1$; alternatively, $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$. Let $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ be a symplectic basis for V and $g \in \mathrm{GSp}_n(K)$. If $t \in \Delta_n$ is a vertex with representative $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$, define

$$gt = [\mathcal{O}\pi^{a_1}gu_1 + \cdots + \mathcal{O}\pi^{a_n}gu_n + \mathcal{O}\pi^{b_1}gw_1 + \cdots + \mathcal{O}\pi^{b_n}gw_n].$$

Note that

$$\mathcal{B}_g := \{\nu(g)^{-1}gu_1, \ldots, \nu(g)^{-1}gu_n, gw_1, \ldots, gw_n\}$$

is a symplectic basis for V; hence, $m = \text{ord}(\nu(g))$ implies $gL = (a_1 + m, \dots, a_n + m; b_1, \dots, b_n)_{\mathcal{B}_g}$.

Proposition 1.4. *The group* $\text{GSp}_n(K)$ *acts transitively on the special vertices in* Δ_n.

Proof. Note that if $\text{GSp}_n(K)$ acts on the special vertices in Δ_n, then [16, Proposition 3.3] implies that the action is transitive. We thus show that $\text{GSp}_n(K)$ acts on the special vertices in Δ_n. Let $t \in \Delta_n$ be a special vertex and $L \in t$ a representative such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$. Let Σ be an apartment of Δ_n containing t and $[L_0]$, and let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 1.1. Then [16, p. 3411], the last lemma, and [16, Corollary 3.4] imply

$$L_0 = (c_1, \dots, c_n; -c_1, \dots, -c_n)_{\mathcal{B}}$$

and

$$L = (a_1, \dots, a_n; \mu - a_1, \dots, \mu - a_n)_{\mathcal{B}},$$

where $\mu \in \{1, 2\}$. Let $g \in \text{GSp}_n(K)$ with $\text{ord}(\nu(g)) = m$. Since $gt = [a_1 + m, \dots, a_n + m; \mu - a_1, \dots, \mu - a_n]_{\mathcal{B}_g}$, [16, Corollary 3.4] implies that it suffices to show gt is a vertex in Δ_n. First suppose $m = 2r$ for some $r \in \mathbb{Z}$. Then $\pi^{-r}gL_0$ is primitive, $\langle \pi^{-r}gL, \pi^{-r}gL\rangle \subseteq \pi\mathcal{O}$, and $\pi^{-r}g(\pi L_0) \subseteq \pi^{-r}gL \subseteq \pi^{-r}gL_0$; i.e., gt is a vertex in Δ_n. Now suppose $m = 2r + 1$. If $\mu = 1$, then $\pi^{-r-1}gL$ is primitive and gt is a vertex in Δ_n. Otherwise, $\mu = 2$, and $\langle \pi^{-r-1}gL, \pi^{-r-1}gL\rangle \subseteq \pi\mathcal{O}$. Let $\pi M_0 = (a_1 + r, \dots, a_n + r; \mu - a_1 - r, \dots, \mu - a_n - r)_{\mathcal{B}_g}$. Then M_0 is primitive and $\pi M_0 \subseteq \pi^{-r-1}gL \subseteq M_0$; i.e., gt is a vertex in Δ_n. Thus, $\text{GSp}_n(K)$ acts on the special vertices in Δ_n. □

Call two distinct, incident vertices in Δ_n *adjacent.*

Proposition 1.5. *The group* $\text{GSp}_n(K)$ *takes adjacent special vertices in* Δ_n *to adjacent special vertices in* Δ_n.

Proof. Let $t, t' \in \Delta_n$ be adjacent special vertices, and let $L \in t$ and $L' \in t'$ be representatives such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$, $\langle L', L'\rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq L \subsetneq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subsetneq L \subseteq L_0$. By Lemma 1.3, either $L' = L_0$ or $L = L_0$; i.e., either $\pi L' \subsetneq L \subsetneq L'$ with L' primitive or $\pi L \subsetneq L' \subsetneq L$ with L primitive. By symmetry, assume $\pi L \subsetneq L' \subsetneq L$ with L primitive, and let $g \in \text{GSp}_n(K)$ with $\text{ord}(\nu(g)) = m$. If $m = 2r$ for some $r \in \mathbb{Z}$, then (as in the last proposition) $\pi^{-r}gL$ is primitive, $\langle \pi^{-r}gL', \pi^{-r}gL'\rangle \subseteq \pi\mathcal{O}$, and $\pi^{-r}g(\pi L) \subsetneq \pi^{-r}gL' \subsetneq \pi^{-r}gL$. Similarly, if $m = 2r+1$, then $\pi^{-r-1}gL'$ is primitive, $\langle \pi^{-r}gL, \pi^{-r}gL\rangle \subseteq \pi\mathcal{O}$, and $\pi(\pi^{-r-1}gL') \subsetneq \pi^{-r}gL \subsetneq \pi^{-r-1}gL'$. □

Lemma 1.6. *If $t, t' \in \Delta_n$ are adjacent special vertices and t has a primitive representative L, then there is a representative $L' \in t'$ with $\pi L \subsetneq L' \subsetneq L$ such that the number of chambers in Δ_n containing both t and t' equals the number of maximal flags of non-trivial, proper k-subspaces of $L'/\pi L$.*

Proof. Since a chamber $C \in \Delta_n$ containing both t and t' has $n+1$ vertices $t_0, \ldots, t_n$ that have representatives $L_i \in t_i$ such that L_0 is primitive, $\langle L_i, L_i \rangle \subseteq \pi \mathcal{O}$ for all $1 \leq i \leq n$, and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$, Lemma 1.3 implies L_0 and L_n represent the two special vertices in C; in particular, $t = t_0$ and $t' = t_n$. Let $L = L_0$ and $L' = L_n$. Then $\pi L \subsetneq L' \subsetneq L$ and varying $L_1, \ldots, L_{n-1}$ over all lattices in V contained in L and containing πL such that $\langle L_i, L_i \rangle \subseteq \pi \mathcal{O}$ for all $1 \leq i \leq n-1$ and $\pi L \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L'$ gives all the chambers in Δ_n containing both t and t'. □

Proposition 1.7. *If $t \in \Delta_n$ is a special vertex, then t is adjacent to exactly $\prod_{m=1}^{n} (q^m + 1)$ distinct special vertices in Δ_n.*

Proof. Let $t \in \Delta_n$ be a special vertex. By Proposition 1.2, the number of chambers in Δ_n containing t is $\prod_{m=1}^{n} ((q^{2m} - 1)/(q-1))$. Since this counts a special vertex $t' \in \Delta_n$ adjacent to t more than once if there is more than one chamber in Δ_n containing both t and t', the last lemma and [15, Proposition 2.4] finish the proof. □

Proposition 1.8. *Let L be a lattice in V. Then $\mathrm{GSp}_n(\mathcal{O}) := \mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$ can be identified with $\{g \in \mathrm{GSp}_n(K) : gL = L\}$, where g acts on L as the matrix of a linear transformation with respect to a fixed basis for L.*

Proof. It suffices to prove the proposition for any lattice L in V. Let $\mathcal{B} = \{e_1, \ldots, e_{2n}\}$ be the standard unit basis for V, and let $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$. Let $g \in \mathrm{GSp}_n(K)$ such that $gL = L$. Then $g \in \mathrm{GL}_{2n}(\mathcal{O})$; i.e., $g \in \mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$. On the other hand, any element of $\mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$ fixes L. □

We end this section with some terminology used in Proposition 2.1. By [5, p. 13], two chambers in Δ_n are *adjacent* if they have a common codimension-one face, and a *gallery* in Δ_n connecting the chambers $C, C' \in \Delta_n$ is a sequence $C = C_0, \ldots, C_m = C'$ of chambers in Δ_n such that C_i and C_{i+1} are adjacent for all $0 \leq i \leq m-1$. Recall that any two chambers in a building can be connected by a gallery.

2 The Isoperimetric Constant of Y_n

The graph theory notation and terminology used in this section is primarily from [17, p. 7]; any terminology not defined there is from [4, pp. 1 – 4], except for the definition of walk, which is from [11, p. 2]. In particular, a graph is a finite or countably infinite set X of vertices, together with a symmetric neighborhood or adjacency relation $\sim$. Let X be a connected, r-regular (r finite) graph with infinitely many vertices. As in [3, p. 116], if X' is a subset of X, let $\partial X'$ denote the set of edges in X incident to exactly one vertex in X'. Then the *isoperimetric constant* of X is $h(X) := \inf(\text{Card}(\partial X')/\text{Card}(X'))$, where the infimum is over all finite, non-empty subsets X' of X. Note that $h(X)$ is related to the spectral radius of X: recall that the adjacency operator $A(X)$ of X acts on the Hilbert space of functions $f : X \to \mathbb{C}$ such that $\sum_{x \in X} |f(x)|^2 < \infty$. Then the spectrum of $A(X)$ is $\{\lambda \in \mathbb{C} : A(X) - \lambda I \text{ is not invertible}\}$, and the *spectral radius* of X is

$$\rho(X) := \sup\{|\lambda| : \lambda \text{ is in the spectrum of } A(X)\};$$

equivalently, $\rho(X) = \|A(X)\|$, the norm of $A(X)$, by [10, p. 252] ($A(X)$ is bounded by [10, Theorem 3.2] and can be shown to be self-adjoint). Then by [3, Theorem 3.1], for a connected, r-regular graph X with infinitely many vertices, to show $h(X) > 0$, it suffices to compute the spectral radius $\rho(X)$ of X and show $\rho(X) < r$.

Let Y_n be the subgraph of Δ_n induced by the *special* vertices in Δ_n. Then by Proposition 1.7, Y_n is $(\prod_{m=1}^{n} (q^m + 1))$-regular. Moreover, [16, p. 3414] implies that Y_n has infinitely many vertices.

Proposition 2.1. *The graph Y_n is connected.*

Proof. Let $t \neq t' \in Y_n$. Since there is nothing to prove if t and t' are in a common chamber in Δ_n, assume that no chamber in Δ_n contains both t and t'. Let $C, C' \in \Delta_n$ be chambers such that $t \in C$ and $t' \in C'$, and let $C = C_0, \ldots, C_m = C'$ be a gallery in Δ_n connecting C and C'. For all $0 \leq i \leq m - 1$, let S_i be the set of special vertices contained in C_i and C_{i+1}. Let $t_0 = t$, $t_{m+1} = t'$, and $t_i \in S_{i-1}$ for all $1 \leq i \leq m$. Then t_i and t_{i+1} are incident vertices in Δ_n for all i; hence, for each $0 \leq i \leq m$, either $t_i = t_{i+1}$ or t_i and t_{i+1} are adjacent vertices in Y_n. But we can remove vertices from the sequence $t_0, \ldots, t_{m+1}$ until we are left with a walk in Y_n connecting t and t' (since Y_n has no loops, we need successive vertices to be adjacent). □

If G is a group acting on a set S and $a, b \in S$, write G_a for $\text{Stab}_G(a) = \{g \in G : ga = a\}$ and $G_a b$ for $\{gb : g \in G_a\}$. The main tool that we use is the following reformulation of a special case of [17, Theorem (12.10)].

Theorem 2.2. *Let X be a locally finite, regular graph. If there is a solvable group Q that acts transitively on X as a group of automorphisms and has a left Haar measure $\mu(\cdot)$, then the spectral radius $\rho(X)$ of X is*

$$\rho(X) = \sum_{t' \sim t_0} \sqrt{\frac{\mathrm{Card}(Q_{t'} t_0)}{\mathrm{Card}(Q_{t_0} t')}}, \tag{1}$$

where t_0 is any vertex in X.

Recall that $\mathrm{PGSp}_n(K) = \mathrm{GSp}_n(K)/K^\times$, where we identify $K^\times$ with the scalar matrices of $\mathrm{GSp}_n(K)$. As in [17, Example (12.20)], write g as a matrix in $\mathrm{GSp}_n(K)$ while thinking of it as an element of $\mathrm{PGSp}_n(K)$ consisting of all its non-zero multiples. Let Q be the image in $\mathrm{PGSp}_n(K)$ of Q', where

$$Q' = \{(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}) \in \mathrm{GSp}_n(K) : A \in M_n(K) \text{ is upper triangular}\},$$

a minimal parabolic subgroup of $\mathrm{GSp}_n(K)$, and let V be a $2n$-dimensional K-vector space with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$. We want to apply Theorem 2.2 to the graph Y_n with Q as above and $t_0 = o = [\mathcal{O}^{2n}] = [0, \dots, 0; 0, \dots, 0]_{\mathcal{B}_0}$, where $\mathcal{B}_0 := \{e_1, \dots, e_n, f_1, \dots, f_n\}$ is the standard symplectic basis for V ($f_i = e_{n+i}$ for all i). Note that a modification of the proof of Proposition 1.3.7 of [1] shows that for any $h \in \mathrm{GSp}_n(K)$, there is a $g \in \mathrm{Sp}_n(\mathcal{O})$ such that $gh^{-1} \in Q'$; furthermore, $(hg^{-1})o = ho$ by Proposition 1.8. This, together with Proposition 1.4, implies that Q acts transitively on the vertices in Y_n. The fact that Q acts on Y_n as a group of automorphisms follows from the fact that Q' does (see Proposition 1.5). Since the group of upper triangular matrices in $\mathrm{GL}_{2n}(K)$ is solvable, so is Q. Verifying that Q has a left Haar measure is a straightforward exercise involving topological groups.

Since we take $t_0 = o$ in (1), it suffices to determine $\mathrm{Card}(Q_{t'}o)$ and $\mathrm{Card}(Q_o t')$ for vertices $t' \in Y_n$ with $t' \sim o$. For a symplectic basis $\mathcal{B}$ for V and Σ the apartment of Δ_n specified by $\mathcal{B}$ as in Lemma 1.1, follow [17, Example (12.20)] and write $U(\mathcal{B})$ for the subgraph of Y_n induced by the special vertices in Σ (see Figure 1 for a partial picture of $U(\mathcal{B})$ when $n = 2$). Let $\mathcal{E}(n) = \{0, 1\}^n$. Then the neighbors of o in $U(\mathcal{B}_0)$ are

$$x_{\underline{\varepsilon}} := [\varepsilon_1, \dots, \varepsilon_n; 1 - \varepsilon_1, \dots, 1 - \varepsilon_n]_{\mathcal{B}_0},$$

where $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{E}(n)$, and Proposition 1.8 implies that the stabilizer of o in $\mathrm{PGSp}_n(K)$ can be identified with $\mathrm{PGSp}_n(\mathcal{O})$. It follows that if $g = (g_{ij}) \in Q_o = Q \cap \mathrm{PGSp}_n(\mathcal{O})$ and $|\cdot|_K$ is the absolute value of K, normalized such that $|\pi|_K = 1/q$, then

$$|g_{ii}|_K = 1, \qquad |g_{ij}|_K \le 1 \text{ if } i < j \text{ (with } 1 \le i \le n) \text{ or } 2n \ge i > j \ge n+1, \tag{2}$$

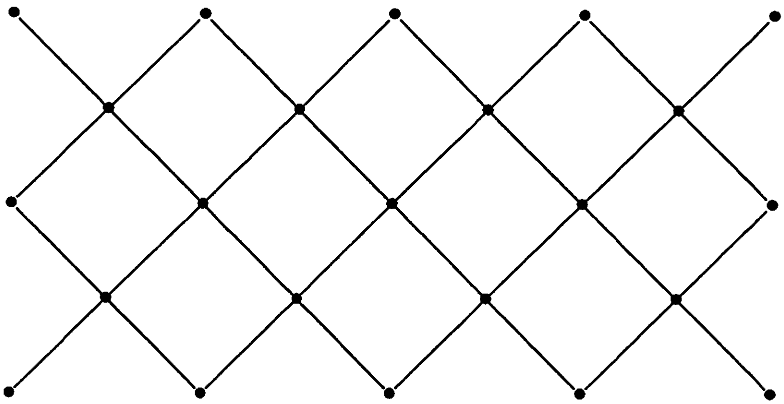

Figure 1: Subgraph of Y_2.

and $g_{ij} = 0$ otherwise. For $\underline{\varepsilon} \in \mathcal{E}(n)$, let

$$g_{\underline{\varepsilon}} := \operatorname{diag}(\pi^{\varepsilon_1}, \dots, \pi^{\varepsilon_n}, \pi^{1-\varepsilon_1}, \dots, \pi^{1-\varepsilon_n}),$$

and note that $g_{\underline{\varepsilon}} o = x_{\underline{\varepsilon}}$ and $Q_{x_{\underline{\varepsilon}}} = g_{\underline{\varepsilon}} Q_o g_{\underline{\varepsilon}}^{-1}$. Then (2) and a similar analysis for an element of $Q_{x_{\underline{\varepsilon}}}$ imply that for $h = (h_{ij}) \in Q_o \cap Q_{x_{\underline{\varepsilon}}}$,

$$|h_{ij}|_K = 1 \text{ if } i = j, \tag{3}$$

$$|h_{ij}|_K \leq \begin{cases} q^{-\max\{0, \varepsilon_i - \varepsilon_j\}} & \text{if } 1 \leq i < j \leq n, \\ q^{-\max\{0, \varepsilon_i - (1-\varepsilon_{j-n})\}} & \text{if } 1 \leq i \leq n < j \leq 2n, \\ q^{-\max\{0, -\varepsilon_{i-n} + \varepsilon_{j-n}\}} & \text{if } n+1 \leq j < i \leq 2n, \end{cases} \tag{4}$$

and $h_{ij} = 0$ otherwise. In addition, by the orbit-stabilizer theorem,

$$\operatorname{Card}(Q_o x_{\underline{\varepsilon}}) = \frac{\mu(Q_o)}{\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})} \qquad \text{and} \qquad \operatorname{Card}(Q_{x_{\underline{\varepsilon}}} o) = \operatorname{Card}(Q_o x_{\underline{1} - \underline{\varepsilon}}),$$

where $\mu(\cdot)$ is a left Haar measure on Q and $\underline{1} = (1, \dots, 1) \in \mathcal{E}(n)$.

Consequently, to determine $\operatorname{Card}(Q_o x_{\underline{\varepsilon}})$ and $\operatorname{Card}(Q_{x_{\underline{\varepsilon}}} o)$, it suffices to find $\mu(Q_o)$ and $\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})$. Since we can completely characterize both Q_o and $Q_o \cap Q_{x_{\underline{\varepsilon}}}$ in terms of $|\cdot|_K$, $\mu(Q_o)$ (resp., $\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})$) is the product of the Haar measures of the unconstrained, non-constant entries of an element $g \in Q_o$ (resp., of $g \in Q_o \cap Q_{x_{\underline{\varepsilon}}}$). Write $\operatorname{vol}(g_{ij})$ for the Haar measure of the (i, j)-entry of $g = (g_{ij})$, with vol normalized such that $\operatorname{vol}(\mathcal{O}) = 1$, $\operatorname{vol}(\pi\mathcal{O}) = 1/q$, and $\operatorname{vol}(\mathcal{O}^\times) = 1$. It follows from (2) that

$$\mu(Q_o) = 1.$$

For $\underline{\varepsilon} \in \mathcal{E}(n)$, let $|\underline{\varepsilon}|_n := \sum_{i=1}^{n} \varepsilon_i$. Then

$$M(\underline{\varepsilon}, n) := \sum_{1 \leq i < j \leq n} \max\{0, \varepsilon_i - \varepsilon_j\}$$

counts the number of 0s that follow each 1 in $\underline{\varepsilon}$. Moreover,

$$\sum_{1 \leq i \leq n < j \leq i+n} \max\{0, \varepsilon_i - (1 - \varepsilon_{j-n})\} = \sum_{1 \leq j \leq i \leq n} \max\{0, (\varepsilon_i + \varepsilon_j) - 1\}$$

adds the number of 1s in $\underline{\varepsilon}$ to the number of 1s that follow each 1 in $\underline{\varepsilon}$; i.e., if $|\underline{\varepsilon}|_n = m$, then the last sum is $m(m+1)/2$. Let $\underline{\varepsilon} \in \mathcal{E}(n)$ with $|\underline{\varepsilon}|_n = m$ and $h = (h_{ij}) \in Q_o \cap Q_{x_{\underline{\varepsilon}}}$. Then (3) and (4) imply

$$\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}}) = q^{-M(\underline{\varepsilon}, n)} q^{-m(m+1)/2};$$

hence,

$$\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) = q^{M(\underline{\varepsilon}, n)} q^{m(m+1)/2} \tag{5}$$

and

$$\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o) = q^{M(\underline{1} - \underline{\varepsilon}, n)} q^{(n-m)(n-m+1)/2}.$$

Since $M(\underline{1} - \underline{\varepsilon}, n)$ counts the number of 0s that precede each 1 in $\underline{\varepsilon}$, $M(\underline{\varepsilon}, n) + M(\underline{1} - \underline{\varepsilon}, n)$ is the product of the number of zeros in $\underline{\varepsilon}$ and $|\underline{\varepsilon}|_n$. Consequently,

$$\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) \mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o) = q^{n(n+1)/2}. \tag{6}$$

Lemma 2.3. *Let $\underline{\varepsilon} \neq \underline{\varepsilon}' \in \mathcal{E}(n)$. Then for all $g \in Q_o$, $x_{\underline{\varepsilon}'} \neq g x_{\underline{\varepsilon}}$.*

Proof. Let $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\underline{\varepsilon}' = (\varepsilon_1', \ldots, \varepsilon_n')$, and let $1 \leq \ell \leq n$ such that $\varepsilon_\ell \neq \varepsilon_\ell'$. If $x_{\underline{\varepsilon}'} = g x_{\underline{\varepsilon}}$ for some $g = (g_{ij}) \in Q_o$, then $g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} g \in Q_{x_{\underline{\varepsilon}}}$, which is impossible since by (2), the (ℓ, ℓ)-entry of $g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} g$ satisfies

$$\left| g_{\ell\ell} \pi^{\varepsilon_\ell - \varepsilon_\ell'} \right|_K \in \{q^{-1}, q\},$$

contradicting (3). □

It follows that for $\underline{\varepsilon}, \underline{\varepsilon}' \in \mathcal{E}(n)$, the sets $Q_o x_{\underline{\varepsilon}}$ and $Q_o x_{\underline{\varepsilon}'}$ are disjoint if $\underline{\varepsilon} \neq \underline{\varepsilon}'$.

Proposition 2.4. *The set of vertices in Y_n adjacent to o is $N(o) = \bigcup_{\underline{\varepsilon} \in \mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}$.*

Proof. First note that by the last lemma and (5),

$$\text{Card}\left(\bigcup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}\right) = \sum_{m=0}^{n} q^{m(m+1)/2}\left(\sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} q^{M(\underline{\varepsilon},n)}\right).$$

Let

$$W(n,m) = \sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} q^{M(\underline{\varepsilon},n)},$$

and note that by [17, p. 135], $W(n,m) = \binom{n}{m}_q$, the number of m-dimensional subspaces of an n-dimensional $\mathbb{F}_q$-vector space, for all $0 \le m \le n$ (see [17, p. 133] for the formula for $\binom{n}{m}_q$). The proof now follows from Proposition 1.7 since for all $n \ge 1$,

$$\prod_{m=1}^{n} (1+q^m) = \sum_{i=0}^{n} q^{i(i+1)/2}\binom{n}{i}_q. \qquad \square$$

Corollary 2.5. *If $t \in \Delta_n$ is a vertex with a primitive representative, then the set of vertices in Y_n adjacent to t is $\bigcup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_t x_{\underline{\varepsilon}}$.*

Proof. This follows from the last proposition since we only used the fact that o has a primitive representative. $\square$

Proposition 2.6. *The spectral radius of Y_n is $\rho(Y_n) = 2^n q^{n(n+1)/4}$.*

Proof. This follows from (1), Proposition 2.4, (6), and the fact that

$$\frac{\text{Card}(Q_{x_{\underline{\varepsilon}}} o)}{\text{Card}(Q_o x_{\underline{\varepsilon}})} = \frac{\text{Card}(Q_{g x_{\underline{\varepsilon}}} o)}{\text{Card}(Q_o g x_{\underline{\varepsilon}})}$$

for all $g \in Q_o$. $\square$

Theorem 2.7. *The isoperimetric constant of Y_n satisfies $h(Y_n) > 0$.*

Proof. By [3, Theorem 3.1] and the last proposition, it suffices to show that $2^n q^{n(n+1)/4} < \prod_{m=1}^{n} (q^m + 1)$, which is straightforward. $\square$

Let X be a connected graph with infinitely many vertices and of bounded degree. Following [8, p. 2480], for a connected, induced subgraph X' of X with at least one edge, let $\sigma(X')$ denote the set of vertices in X' adjacent to a vertex in X not in X'. Then X is *amenable* if $\inf(\text{Card}(\sigma(X'))/\text{Card}(X'))$ $= 0$, where the infimum is over all finite, connected, induced subgraphs X' of X with at least one edge. Note that if X is finite and has at

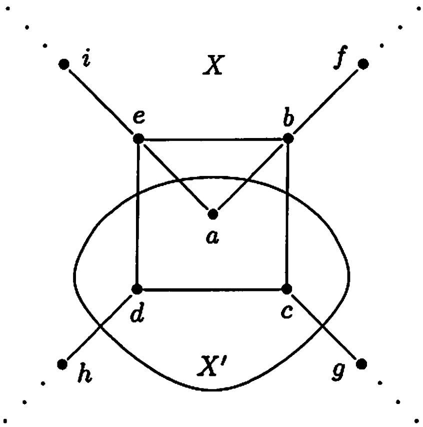

Figure 2: In general, $\mathrm{Card}(\sigma(X')) \leq \mathrm{Card}(\partial X')$.

least one edge, then X is trivally amenable. Furthermore, if X' is a finite, connected, induced subgraph of X with at least one edge, then $\mathrm{Card}(\sigma(X')) \leq \mathrm{Card}(\partial X')$ (see Figure 2). On the other hand, since X has bounded degree, X amenable implies $h(X) = 0$.

Corollary 2.8. *The graph Y_n is non-amenable.*

Proof. As noted above, X amenable implies $h(X) = 0$; hence, $h(Y_n) > 0$ implies Y_n non-amenable. □

For $x \in Y_n$, let $B_i(x) = \{y \in Y_n : \mathrm{d}(x,y) \leq i\}$, where $\mathrm{d}(x,y)$ is the (graph) distance between x and y.

Corollary 2.9. *There is a constant $C > 1$ such that for all $x \in Y_n$ and for all $i \in \mathbb{Z}^{\geq 0}$,* $\mathrm{Card}(B_i(x)) > C^i$.

Proof. This follows from the last theorem and [3, Theorem 2.2]. □

Finally, note that the building Δ_n is a subcomplex of Ξ_{2n} (compare the description of Δ_n given in Section 1 with the description of Ξ_n in [13, p. 115]); hence, Y_n is a subgraph of the one-complex X_{2n} of Ξ_{2n}. Since both Y_n and X_{2n} are expanding, it is natural to ask about their relative expansion properties. It is straightfoward to show that $\rho(Y_n) < \rho(X_{2n})$ for all $n \geq 2$, but since the degree of any vertex in Y_n is also strictly less than the degree of any vertex in X_{2n}, it is unclear what this reveals. On the other hand, the analogue of Corollary 2.9 holds for X_{2n}. Then Theorems

2.2 and 3.1 of [3] provide a constant $C(Y_n)$ (resp., $C(X_{2n})$) as in Corollary 2.9 (resp., as in the analogue of Corollary 2.9 for X_{2n}) in terms of $\rho(Y_n)$ and the degree of any vertex in Y_n (resp., in terms of $\rho(X_{2n})$ and the degree of any vertex in X_{2n}). This thus raises the question of whether there is any relationship between $C(Y_n)$ and $C(X_{2n})$. For example, our data for $2 \leq n \leq 5$ and $q = p^i$, where $1 \leq i \leq 5$ and p is one of the first five primes, indicates that $C(Y_n) < C(X_{2n})$; is this always the case?

References

[1] A. Andrianov. *Quadratic Forms and Hecke Operators*, volume 286 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1987.

[2] F. Bien. Constructions of telephone networks by group representations. *Notices of the American Mathematical Society*, 36(1):5 – 22, 1989.

[3] N. Biggs, B. Mohar, and J. Shawe-Taylor. The spectral radius of infinite graphs. *The Bulletin of the London Mathematical Society*, 20(2):116 – 120, 1988.

[4] B. Bollobás. *Graph Theory: An Introductory Course*, volume 63 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1979.

[5] K. Brown. *Buildings.* Springer-Verlag, New York, 1989.

[6] D. Cartwright, P. Solé, and A. Żuk. Ramanujan geometries of type $\widetilde{A}_n$. *Discrete Mathematics*, 269(1 – 3):35 – 43, 2003.

[7] G. Davidoff, P. Sarnak, and A. Valette. *Elementary Number Theory, Group Theory, and Ramanujan Graphs*, volume 55 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2003.

[8] G. Elek and G. Tardos. On roughly transitive amenable graphs and harmonic Dirichlet functions. *Proceedings of the American Mathematical Society*, 128(8):2479 – 2485, 2000.

[9] P. Garrett. *Buildings and Classical Groups.* Chapman & Hall, London, 1997.

[10] B. Mohar. The spectrum of an infinite graph. *Linear Algebra and its Applications*, 48:245 – 256, 1982.

[11] M. Murty. Ramanujan graphs. *Journal of the Ramanujan Mathematical Society*, 18(1):33 – 52, 2003.

[12] J. Parkinson. Spherical harmonic analysis on affine buildings. *Mathematische Zeitschrift*, 253(3):571 – 606, 2006.

[13] M. Ronan. *Lectures on Buildings*, volume 7 of *Perspectives in Mathematics.* Academic Press, Inc., Boston, MA, 1989.

[14] L. Saloff-Coste and W. Woess. Transition operators, groups, norms, and spectral radii. *Pacific Journal of Mathematics*, 180(2):333 – 367, 1997.

[15] A. Schwartz and T. Shemanske. Maximal orders in central simple algebras and Bruhat-Tits buildings. *Journal of Number Theory*, 56(1):115 – 138, 1996.

[16] T. Shemanske. The arithmetic and combinatorics of buildings for Sp_n. *Transactions of the American Mathematical Society*, 359(7):3409 – 3423, 2007.

[17] W. Woess. *Random Walks on Infinite Graphs and Groups*, volume 138 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2000.

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