The Relative n-th Commutativity Degree of 2-Generator p-Groups of Nilpotency Class Two

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Abstract

Let H be a subgroup of a finite group G. The relative n-th commutativity degree, denoted as $P_n(H,G)$ is the probability of commuting the n-th power of a random element of H with an element of G. Obviously, if $H = G$ then the relative n-th commutativity degree coincides with the n-th commutativity degree, $P_n(G)$. The purpose of this article is to compute the explicit formula for $P_n(G)$, where G is a 2-generator p-group of nilpotency class two. Furthermore, we observe that if we have two pairs of relative isoclinic groups, then

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they have equal relative n-th commutativity degree.

1 Introduction

Let G be a finite group and n a positive integer. In [8], Mohd Ali and Sarmin introduced the n-th commutativity degree as the probability that the n-th power of a random element of G commute with another element, defined as

$$P_n(G) = \frac{|\{(x,y) \in G^2 : [x^n, y] = 1\}|}{|G|^2}.$$

They focused on $P_n(G)$, where G is a 2-generator 2-group of nilpotency class 2 (see [8]). If $n = 1$ then $P_1(G)$ is the commutativity degree, which was investigated by Erdös and Turan in [2]. Later Erfanian et al. generalized their results to the relative case. They defined the relative n-th commutativity degree of a subgroup H of G as the ratio

$$P_n(H,G) = \frac{|\{(h,g) \in H \times G : [h^n, g] = 1\}|}{|H||G|}.$$

They generalized several facts, which are valid for the commutativity degree in [4]. In [7], Lescot proved that two isoclinic groups have equal commutativity degrees. The concept of isoclinism which determines an equivalence relation on the class of all groups has been introduced by Hall [5] in order to classify p-groups. Isoclinism have been largely studied in the literature, one can refer to [1, 6, 7] for more details. A weaker form of isoclinism is called relative isoclinism (see [3]) which we shall use. First of all, we need the following lemma.

Lemma 1.1. *Let G be a group with an arbitrary subgroup H. Then the map*

$$\gamma(H,G) : H/(Z(G) \cap H) \times G/Z(G) \rightarrow [H,G]$$
$$(\overline{h}, \widetilde{g}) \longmapsto [h,g]$$

is well defined, where $\bar{}$ and $\tilde{}$ denote the natural epimorphisms $H \rightarrow H/(Z(G) \cap H)$ and $G \rightarrow G/Z(G)$, respectively.

Definition 1. Let H_i be a subgroup of G_i for $i = 1, 2$. Then the pair (α, β) is called a relative isoclinism from (H_1, G_1) to (H_2, G_2) whenever

(i) α is an isomorphism from $G_1/Z(G_1)$ to $G_2/Z(G_2)$ such that the restriction of α on $H_1/(Z(G_1) \cap H_1)$ induces an isomorphism from $H_1/(Z(G_1) \cap H_1)$ to $H_2/(Z(G_2) \cap H_2)$.

(ii) β is an isomorphism from $[H_1, G_1]$ to $[H_2, G_2]$, which maps $[h_1, g_1]$ to $[h_2, g_2]$, in which $h_2 \in \alpha(h_1 Z(G_1) \cap H_1)$ and $g_2 \in \alpha(g_1 Z(G_1))$.

If there is such a pair (α, β) with the above properties, then we say that (H_1, G_1) and (H_2, G_2) are relative isoclinic and it is denoted by $(H_1, G_1) \sim (H_2, G_2)$.

It is clear that if $H_1 = G_1$ and $H_2 = G_2$, then the pair (α, β) is an isoclinism between G_1 and G_2. In the next section, we will generalize Lescot's result [7] for two relative isoclinic pairs. In fact, we show that if $(H_1, G_1) \sim (H_2, G_2)$, then $P_n(H_1, G_1) = P_n(H_2, G_2)$. Moreover, we will compute $P_n(H, G)$ and $P_n(G)$ when G is a 2-generator p-group of nilpotency class two.

2 Main Results

In this section, we obtain the exact formula for n-th commutativity degree of a finite 2-generator p-group of nilpotency class two. It is clear that if G is a 2-generator p-group of nilpotency class two, then ${}^{G}/_{Z(G)} \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ and G' is a cyclic group of exponent p^k for some k. We begin with some lemmas.

Lemma 2.1. *Let G be a 2-generator p-group of nilpotency class two. Then*

$$P_n(G) = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|x^{p^m} Z(G)|},$$

where $n = p^m l$ and $\gcd(p, l) = 1$.

Proof. It is easy to get the equation $P_n(G) = P_{p^m}(G)$ and then, the definition of $P_n(G)$ yields:

$$P_n(G) = \frac{1}{|G|} \sum_{x \in G} \frac{|C_G(x^{p^m})|}{|G|} = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|(x^{p^m})^G|}.$$

Let $\phi_{x^{p^m}} : G \longrightarrow [x^{p^m}, G]$ be the epimorphism defined by $\phi_{x^{p^m}}(g) = [x^{p^m}, g]$, where $[x^{p^m}, G] = \langle [x^{p^m}, g] : g \in G \rangle$. Then $ker(\phi_{x^{p^m}}) = C_G(x^{p^m})$ and hence $G/C_G(x^{p^m}) \cong \mathbb{Z}_{|x^{p^m} Z(G)|}$. Therefore, the proof is complete. □

For every p-group X we may define $\Omega_i(X)$ as the group generated by all elements $x \in X$ such that $x^{p^i} = 1$, for every integer $i \geq 0$. We are now ready to state our main result as follows:

Theorem 2.2. *Let G be a finite 2-generator p-group of nilpotency class two and H be a subgroup of G such that $H/(Z(G)\cap H)\cong \mathbb{Z}_{p^s}\times\mathbb{Z}_{p^t}$.*

(i) *If $s\leq m$, then $P_n(H,G)=1$.*

(ii) *If $t\leq m<s$, then $P_n(H,G)=\frac{(s-m)(p-1)+p}{p^{s-m+1}}$.*

(iii) *If $0\leq m<t$, then $P_n(H,G)=\frac{(s-t)(p-1)p^{t-m}+p^{t-m+1}+p^{t-m}-1}{p^{s+t-2m+1}}$,*

where $n=p^m l$, $\gcd(p,l)=1$, p is a prime, s,t are positive integers and $s\geq t$.

Proof. Since two relative isoclinic pairs of groups have equal probabilities, we conclude that $P_n(H,G)=P_n(HZ(G),G)$ (see [6]). Therefore, we may assume that $Z(G)\leq H$. Since $\gcd(p,l)=1$, then we have $P_n(H,G)=P_{p^m}(H,G)$. Now, let $n_i=|\{h\in H:|hZ(G)|=p^i\}|$ and $\Omega_i(H/Z(G))=\Omega_i^*(H)/Z(G)$. Clearly, for $i\geq 1$ we get $n_i=|\Omega_i^*(H)\setminus\Omega_{i-1}^*(H)|$ and there are three cases:

(1) If $i=0$, then $n_0=|Z(G)|$,

(2) if $i\leq t$, then $n_i=|Z(G)|p^{2(i-1)}(p^2-1)$, and

(3) if $t<i\leq s$, then $n_i=|Z(G)|p^{i+t-1}(p-1)$.

Applying the above discussions and Lemma 2.1, we have

$$P_{p^m}(H,G)=\frac{1}{|H|}\sum_{h\in H}\frac{1}{|h^{p^m}Z(G)|}=\frac{1}{|H|}\left(\sum_{i=0}^{m}n_i+\sum_{i=1}^{s-m}\frac{n_{i+m}}{p^i}\right). \quad (1)$$

We now consider three cases:

Case 1: If $m\geq s$, then $H^{p^m}\subseteq Z(G)$ so that $P_{p^m}(H,G)=1$.

Case 2: If $t\leq m<s$, then by using (1) we get:

$$\begin{aligned}P_{p^m}(H,G)&=\frac{1}{|H|}\left(\sum_{i=0}^{m}n_i+\sum_{i=1}^{s-m}\frac{n_{i+m}}{p^i}\right)\\&=\frac{1}{|H|}\left(|H|-\sum_{i=1}^{s-m}n_{i+m}+\sum_{i=1}^{s-m}\frac{n_{i+m}}{p^i}\right)\\&=\frac{(s-m)(p-1)+p}{p^{s-m+1}}.\end{aligned}$$

Case 3: If $0 \leq m < t$, then (1) yields:

$$
\begin{aligned}
P_{p^m}(H,G) &= \frac{1}{|H|}\left(\sum_{i=0}^{m} n_i + \sum_{i=1}^{s-m} \frac{n_{i+m}}{p^i}\right) \\
&= \frac{1}{|H|}\left(|H| - \sum_{i=1}^{s-m} n_{i+m} + \sum_{i=1}^{t-m} \frac{n_{i+m}}{p^i} + \sum_{i=t-m+1}^{s-m} \frac{n_{i+m}}{p^i}\right) \\
&= \frac{(s-t)(p-1)p^{t-m} + p^{t-m+1} + p^{t-m} - 1}{p^{s+t-2m+1}}.
\end{aligned}
$$

The proof is thus complete. □

The following corollary is a direct consequence of Theorem 2.2.

Corollary 2.3. *Let G be a 2-generator p-group of nilpotency class two such that $|G'| = p^k$. Then*

$$
P_n(G) = \begin{cases} 1, & m \geq k, \\ \frac{p^{k+1}+p^k-1}{p^{2k+1}}, & \textit{otherwise}, \end{cases}
$$

where $n = p^m l$, $\gcd(p,l) = 1$ and p is a prime.

We conclude this section by showing that the relative isoclinic pair of groups have the same relative n-th commutativity degrees. The following theorem classifies pairs of groups (H,G) by means of relative isoclinism, where H is a subgroup of G.

Theorem 2.4. *Let H_i be a subgroup of G_i, for $i = 1,2$. If (H_1, G_1) and (H_2, G_2) are relative isoclinic, then $P_n(H_1, G_1) = P_n(H_2, G_2)$.*

Proof. Suppose that (α, β) is an isoclinism between (H_1, G_1) and (H_2, G_2). By the definition 1, we get:

$$
\begin{aligned}
\frac{|H_1||G_1|P_n(H_1,G_1)}{|Z(G_1)\cap H_1||Z(G_1)|} &= \frac{|\{(h_1,g_1)\in H_1\times G_1 : [h_1^n, g_1] = 1\}|}{|Z(G_1)\cap H_1||Z(G_1)|} \\
&= |\{(\overline{h_1}, \widetilde{g_1}) \in \overline{H_1}\times \widetilde{G_1} : \beta(\gamma(H_1,G_1)(\overline{h_1^n}, \widetilde{g_1})) = 1\}| \\
&= |\{(\overline{h_1}, \widetilde{g_1}) \in \overline{H_1}\times \widetilde{G_1} : \gamma(H_2,G_2)\alpha^2(\overline{h_1^n}, \widetilde{g_1}) = 1\}| \\
&= |\{(\overline{\overline{h_2}}, \widehat{g_2}) \in \overline{\overline{H_2}}\times \widehat{G_2} : \gamma(H_2,G_2)(\overline{\overline{h_2^n}}, \widehat{g_2}) = 1\}| \\
&= \frac{|\{(h_2,g_2)\in H_2\times G_2 : [h_2^n, g_2] = 1\}|}{|Z(G_2)\cap H_2||Z(G_2)|} \\
&= \frac{|H_2||G_2|P_n(H_2,G_2)}{|Z(G_2)\cap H_2||Z(G_2)|},
\end{aligned}
$$

as required. □

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