The Crossing Number of $K_{2,4,n}$

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Abstract

In this paper we show that the crossing number of the complete tripartite graph $K_{2,4,n}$ is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2n$.

1 Introduction

Computing the crossing number of a given graph is, in general, an elusive problem. Exact values are known only for very restricted classes of graphs. In fact, computing the crossing number of a graph is NP-complete [3]. A good updated survey on crossing numbers is [8].

A longstanding problem of crossing numbers is the Zarankiewicz conjecture which asserts that the crossing number of the complete bipartite graph $K_{m,n}$ is given by

$$Z(m,n)=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor. \tag{1}$$

It is only known to be true for $m \leq 6$ and for $m = 7$ and $n \leq 10$ (see [7] and [9]). Recently, E. deKlerk et al. give a new lower bound for the crossing number of $K_{m,n}$ (see [2]).

It is natural to generalize the Zarankiewicz conjecture and to ask: What is the crossing number for the complete tripartite graph $K_{m,n,l}$? In [1] Asano showed that the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$ are $Z(4,n)+\lfloor\frac{n}{2}\rfloor$ and $Z(5,n)+n$, respectively. In [5] we have applied similar techniques as in [1] to find the crossing numbers of $K_{1,1,1,1,n}$, $K_{1,2,2,n}$, $K_{1,1,1,2,n}$, and $K_{1,4,n}$. In [6] we have proved that the crossing number of $K_{2,4,n}$ is $Z(6,n)+2n$ if it is true for $n \leq 55$, assumed that the Zarankiewicz conjecture is true for $m = 7$. In this paper we prove

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Theorem 1.1. *The crossing number of* $K_{2,4,n}$ *is* $c(K_{2,4,n}) = Z(6,n) + 2n$.

Here are some definitions. Let G be a simple graph with the vertex set V and the edge set E. A *drawing* of a graph G is the image of an injective map from G into the plane such that each vertex is represented by a distinct point and each edge is represented by a simple curve without any vertex drawn in its interior. A drawing is *good* if any two edges share at most one common point including endpoints, and any non-vertex intersection between two edges is a transverse crossing.

A common point of two edges other than an endpoint is a *crossing*. For a good drawing D of $K_{2,4,n}$, we denote by $c(D)$ the total number of crossings. Then the *crossing number* of $K_{2,4,n}$, denoted by $c(K_{2,4,n})$ is the minimum of $c(D)$ among all good drawings D of $K_{2,4,n}$. A drawing D is *optimal* if $c(D) = c(K_{2,4,n})$.

Remark. We often make no distinction between a graph-theoretical object (such as a vertex or an edge) and its drawing. Throughout this work we have taken special care to ensure that no confusion arouses from this practice.

Let (X, Y, Z) be the partition of $K_{2,4,n}$ where $X = \{x_1, x_2\}$, $Y = \{y_1, ..., y_4\}$, and $Z = \{z_1, ..., z_n\}$. Let A and B be subsets of the edge set E. In a drawing D the number of crossings of edges in A with edges in B is denoted by $c(A, B)$. Especially, $c(A, A)$ will be denoted by $c(A)$. We note the following formulas which can be shown easily

$$c(A \cup B) = c(A) + c(B) + c(A, B) \quad (2)$$

$$c(A, B \cup C) = c(A, B) + c(A, C), \quad (3)$$

where A, B, and C are mutually disjoint subsets of E. The set of edges which are incident to a vertex v is denoted by $E(v)$.

2 Crossing number of $K_{2,4,n}$

The idea of the proof of Theorem 1.1 is as follows. If $c(K_{2,4,n}) < Z(6,n)+2n$ then the $K_{2,4}$ in the optimal drawing of $K_{2,4,n}$ must be drawn in some special form, namely, the $K_{2,4}$ must be drawn such that it contains a region with at least 5 vertices on its boundary. By analyzing each one of these drawings of $K_{2,4}$ carefully we can conclude that it is impossible to extend these drawings to a drawing of $K_{2,4,n}$ with crossing number less than $Z(6,n) + 2n$. First we have the following

Lemma 2.1. *There are exactly 8 drawings of* $K_{2,4}$ *such that a region exists with at least 5 vertices on its boundary (see Figure 1).*

Proof. From [4] we know that there are 6 non-isomorphic drawings of $K_{2,3}$ shown in Figure 2.

To obtain a drawing of $K_{2,4}$ from these drawings of $K_{2,3}$ such that there is a region with at least 5 vertices of $K_{2,4}$ on its boundary, the only candidates are D_1, D_2, D_4, and D_5. To obtain a drawing of $K_{2,4}$ from D_1, D_2, D_4, and D_5 we need to draw a new vertex in a region of $K_{2,3}$ and draw new edges connecting the new vertex and the two vertices denoted by • in each D_i of Figure 2.

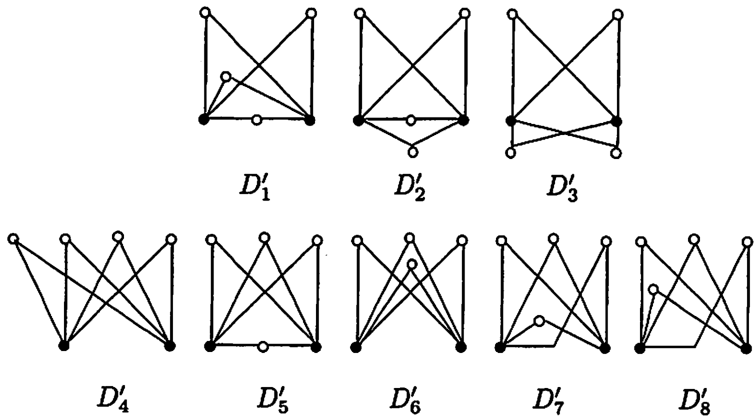


Figure 1.

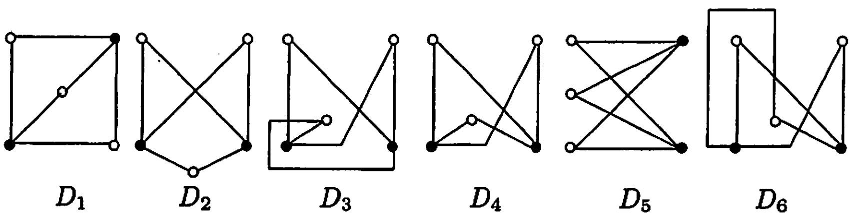


Figure 2.

For D_1 we can assume that the new vertex is located in the unbounded region (Figure 3(a)). Then the possible good drawings are shown in Fig. 3(b) and (c) being isomorphic to D'_1 and D'_2, respectively.

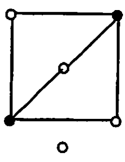


Figure 3(a).

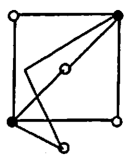


Figure 3(b).

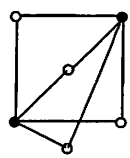


Figure 3(c).

For D_2 the new vertex can be located as in Figure 4(a), (b), or (c), due to symmetry. All possible good drawings for Figure 4(a) are as in Figure 4(d) to (i) being isomorphic to $D_5', D_1', D_3', D_7', D_5'$, and D_2', respectively. From Figure 4(b) and (c) one can obtain only D_1' and D_2', respectively.

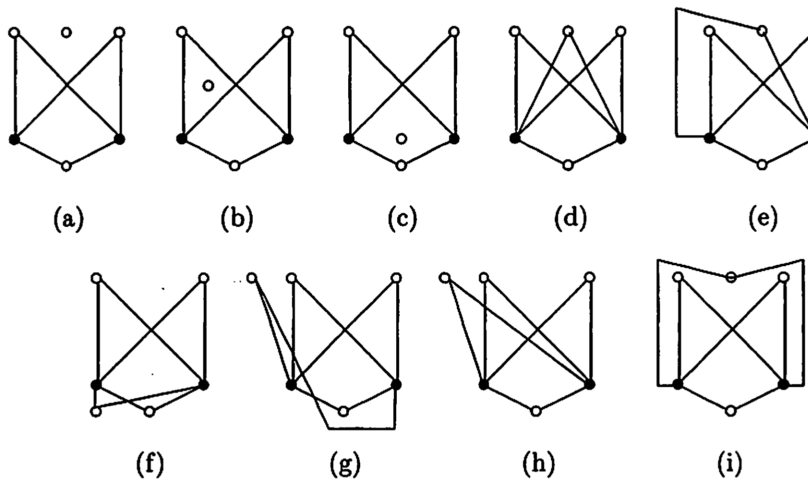


Figure 4.

For D_4 the new vertex must be drawn in the unbounded region as in Figure 5(a). The possible good drawings are as in Figure 5(b) to (f) being isomorphic to D_1', D_7', D_8', D_7', and D_6', respectively.

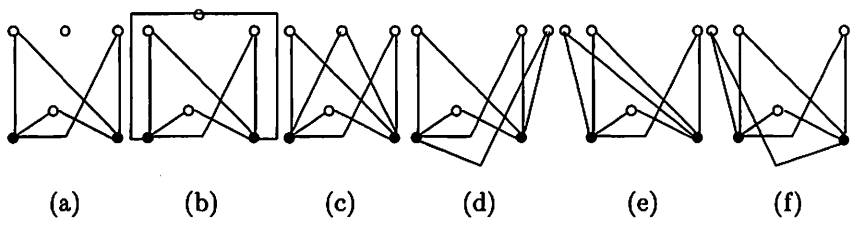


Figure 5.

For D_5 the new vertex can be located as in Figure 6(a) to (d) by symmetry. The possible good drawings from Figure 6(a) are Figure 6(e) to (h), which are isomorphic to D_4', D_8', D_5', and D_6', respectively. The possible drawings from Figure 6(b) to (d) are D_6', D_8', and D_7', respectively. This shows that the drawings in Figure 1 are all the possible drawings. □

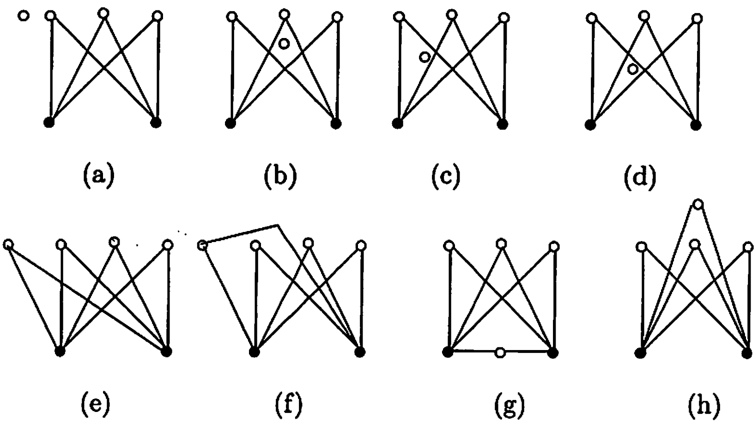


Figure 6.

Proof of Theorem 1.1. First consider a drawing of $K_{2,4,n}$ as in Figure 7 which is the Turán drawing of $K_{6,n}$ with suitable edges being added. One can check that the crossing number of this drawing of $K_{2,4,n}$ is $Z(6,n)+2n$ implying $c(K_{2,4,n}) \leq Z(6,n)+2n$.

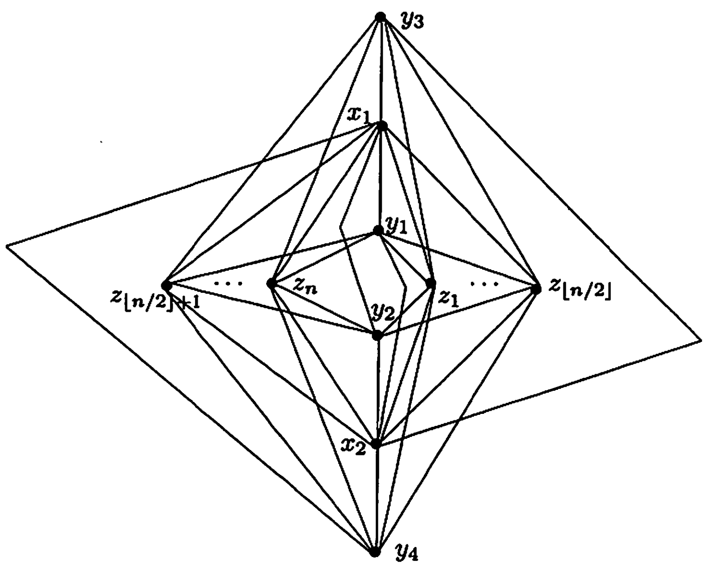


Figure 7.

Therefore it suffices to show that $c(K_{2,4,n}) \geq Z(6,n) + 2n$. We will prove it by induction on n. For $n = 1$, since $K_{2,4,1}$ contains $K_{3,4}$, we have $c(K_{2,4,1}) \geq c(K_{3,4}) = 2$ from [7]. For $n = 2$, since $K_{2,4,2}$ contains $K_{4,4}$, we have $c(K_{2,4,2}) \geq c(K_{4,4}) = 4$ from [7]. Now suppose

$$\begin{array}{rcl} c(K_{2,4,n-2}) & \geq & Z(6,n-2) + 2(n-2); \\ c(K_{2,4,n}) & < & Z(6,n) + 2n, \end{array} \tag{4}$$

where $n \geq 3$. Thus there exists a good drawing D of $K_{2,4,n}$ such that

$$c(D) \leq Z(6,n) + 2n - 1. \tag{5}$$

Let W be the subgraph of $K_{2,4,n}$ induced by $X \cup Y$. From (2) and (3) it follows

$$c(D) = c(W) + c(\bigcup_{i=1}^{n} E(z_i)) + \sum_{i=1}^{n} c(W, E(z_i)). \tag{6}$$

Since $\bigcup_{i=1}^{n} E(z_i)$ is isomorphic to $K_{6,n}$, from [7] we have $c(\bigcup_{i=1}^{n} E(z_i)) \geq Z(6,n)$. Hence, by (5) and (6) we get

$$c(W) + \sum_{i=1}^{n} c(W, E(z_i)) \leq 2n - 1. \tag{7}$$

Therefore $c(W, E(z_i)) \leq 1$ for some $1 \leq i \leq n$. Without loss of generality we assume that $c(W, E(z_1)) \leq 1$. There are two cases to be considered: Case (i): $c(W, E(z_1)) = 0$ and Case (ii): $c(W, E(z_1)) = 1$. Before considering Cases (i) and (ii) we prove

Lemma 2.2. *Let $F = W \cup E(z_1)$. If $c(F) \geq 2$ and $c(F, E(z_i)) \geq 5$ for $2 \leq i \leq n$, then $c(D) \geq Z(6,n) + 2n$.*

Proof of Lemma 2.2. By (2) and (3), we have

$$c(D) = c(F) + c(\bigcup_{i=2}^{n} E(z_i)) + \sum_{i=2}^{n} c(F, E(z_i)). \tag{8}$$

Note that $\bigcup_{i=2}^{n} E(z_i)$ is isomorphic to $K_{6,n-1}$. Thus $c(\bigcup_{i=2}^{n} E(z_i)) \geq Z(6, n-1)$ from [7]. Hence, by (8) and the assumptions we get $c(D) \geq 2 + Z(6,n-1) + 5(n-1) \geq Z(6,n) + 2n$. □

2.1 Case (i). $c(W, E(z_1)) = 0$.

From $c(W, E(z_1)) = 0$ we can conclude that the drawing W in the drawing D has a region with all vertices of $X \cup Y$ on its boundary. From Lemma 2.1 the drawing of W must be as in D'_3, D'_4, or D'_5.

If $W = D'_3$, then F must be drawn as in Figure 8.

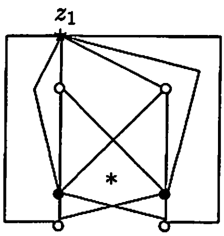


Figure 8.

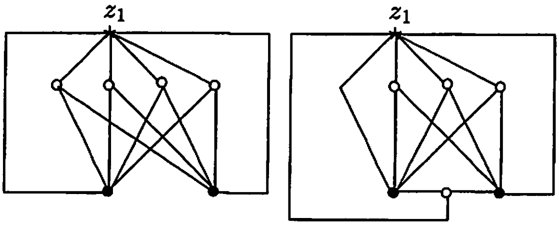


Figure 9.

If z_i for $2 \leq i \leq n$ lies in any region being not marked with $*$, we can check that

$$c(F, E(z_i)) \geq 5. \tag{9}$$

If z_i for $2 \leq i \leq n$ lies lies in any region marked with $*$, we have

$$c(F, E(z_i)) \geq c(W, E(z_i)) \geq 4 \tag{10}$$

since there are four vertices of $X \cup Y$ which are not on the boundary of the region marked with $*$ and the boundary of this region is formed by the edges of W.

We claim that it is impossible for z_i, where $2 \leq i \leq n$, lying in the region marked with $*$, that $c(F, E(z_i)) = 4$. To prove this we may suppose that z_2 lies in the region marked with $*$ and $c(F, E(z_2)) = 4$. From (10),

$$c(W, E(z_2)) = 4 \quad \text{and} \quad c(E(z_1), E(z_2)) = 0. \tag{11}$$

For $3 \leq k \leq n$, $E(z_1) \cup E(z_2) \cup E(z_k)$ is isomorphic to $K_{3,6}$. Hence, from [7] and by (2), (3) we have $c(E(z_1) \cup E(z_2), E(z_k)) + c(E(z_1), E(z_2)) \geq 6$. By (11) it follows

$$c(E(z_1) \cup E(z_2), E(z_k)) \geq 6 \quad \text{for} \quad 3 \leq k \leq n. \tag{12}$$

Let $E' = E - (E(z_1) \cup E(z_2))$. Then by (2), (3) we have

$$\begin{aligned} c(D) &= c(E') + c(E(z_1) \cup E(z_2)) + c(W, E(z_1)) \\ &\quad + c(W, E(z_2)) + \sum_{k=3}^{n} c(E(z_1) \cup E(z_2), E(z_k)). \end{aligned} \tag{13}$$

Note that E' is isomorphic to $K_{2,4,n-2}$. Therefore, from (4), (11), (12), (13), and $c(W, E(z_1)) = 1$ we get $c(D) \geq Z(6, n-2) + 2(n-2) + 1 + 4 + 6(n-2) \geq Z(6, n) + 2n$ which contradicts (5). This proves the claim.

By the claims (10) and (9) we know that $c(F, E(z_i)) \geq 5$ for $2 \leq i \leq n$. From Figure 8, $c(F) = 2$. Hence, by Lemma 2.2 we have $c(D) \geq Z(6, n) + 2n$.

If $W = D'_4$ or D'_5, then F must be drawn as in Figure 9. In both cases, we have $c(F, E(z_i)) \geq 5$ for $2 \leq i \leq n$. Figure 9 implies $c(F) \geq 2$. Hence, by Lemma 2.2 we have $c(D) \geq Z(6, n) + 2n$.

2.2 Case (ii). $c(W, E(z_1)) = 1$.

There exists a region in W such that its boundary contains at least 5 vertices in $X \cup Y$. By Lemma 2.1 the subgraph W must be drawn as one of the D'_i where $1 \leq i \leq 8$ in Figure 1. If $W \neq D'_2$, then by $c(W, E(z_1)) = 1$ and the graph F must be drawn as in Figures 10 to 16 if $W = D'_1, D'_3, D'_4, D'_5, D'_6, D'_7$, and D'_8, respectively. In each of these drawings, we can conclude that $c(F, E(z_i)) \geq 5$ for $2 \leq i \leq n$. Note also that $c(F) \geq 2$ for each of these drawings. Hence, by Lemma 2.2 we have $c(D) \geq Z(6, n) + 2n$.

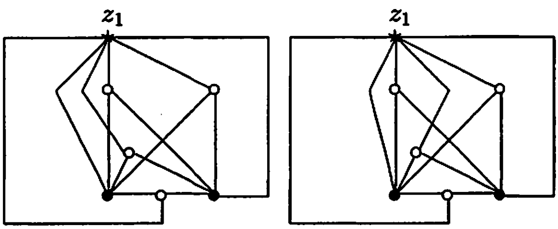


Figure 10.

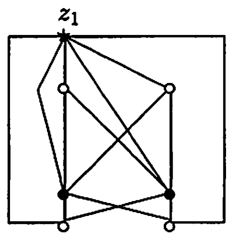


Figure 11.

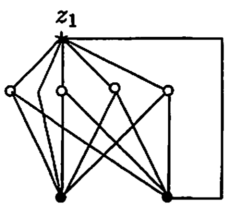


Figure 12.

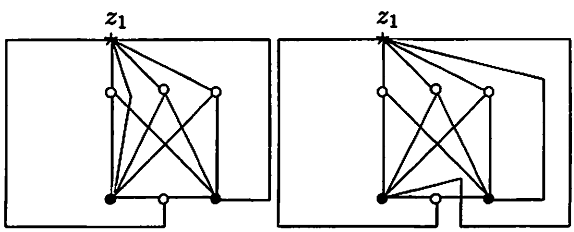


Figure 13.

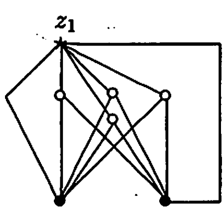


Figure 14.

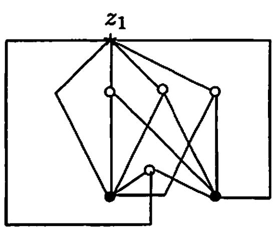


Figure 15.

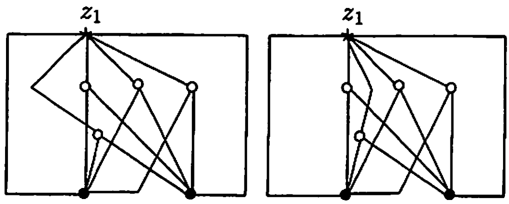


Figure 16.

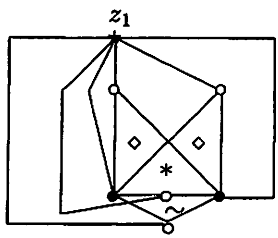


Figure 17.

Finally, we consider $W = D_2'$. By $c(W, E(z_i)) = 1$, the drawing of F must be as in Figure 17. From now on we make the following assumption on the subindex i, $2 \leq i \leq n$. If z_i lies in the region marked with $*$ then

$$c(F, E(z_i)) \geq c(W, E(z_i)) \geq 3. \tag{14}$$

Similar to the proof of the claim right after (10), one can prove that

$$c(F, E(z_i)) \geq 4 \tag{15}$$

if z_i lies in the region marked with $*$. If z_i lies in the region marked with $\sim$ we have

$$c(F, E(z_i)) \geq 4. \tag{16}$$

Note also that

$$c(E_{XY}, E(z_i)) \geq 3 \text{ if } c(F, E(z_i)) \leq 4 \tag{17}$$

if z_i lies in the region marked with $\sim$. To see this, suppose z_i lies in the region marked with $\sim$ and $c(W, E(z_i)) \leq 2$. If z_i lies in the region marked with $\sim$ then z_i lies in a region of the drawing of W which contains exactly four vertices of $X \cup Y$ (see Figure 18) and this implies $c(W, E(z_i)) \geq 2$. Therefore, $c(W, E(z_i)) = 2$. In order to satisfy $c(W, E(z_i)) = 2$ the drawing of $W \cup E(z_i)$ can only be as in Figure 19(a), 19(b), or 19(c). However, if $W \cup E(z_i)$ is drawn as in Figure 19 one can easily see that $c(F, E(z_i)) \geq 5$. This proves (17).

Note also that if z_i lies in the region marked with $\diamond$, then

$$c(F, E(z_i)) \geq c(W, E(z_i)) \geq 4. \tag{18}$$

If z_i lies in the region which is not marked with $\sim$, $*$, and $\diamond$ then we have

$$c(F, E(z_i)) \geq 6. \tag{19}$$

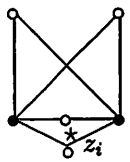
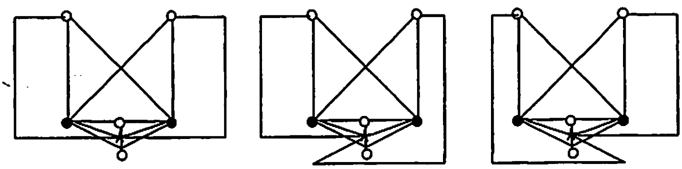


Figure 18.



(a) (b) (c)

Figure 19.

Let l_1 be the number of vertices z_i such that z_i lies in the region marked with $*$. Let l_2 be the number of vertices z_i such that z_i lies in the region marked with $\sim$ and $c(F, E(z_i)) \le 4$. Let l_3 be the number of vertices z_i such that z_i lies in the regions marked with $\diamond$. Therefore, from (14), (17), and (18) the number of vertices z_i such that $c(W, E(z_i)) \ge 3$ is at least $\sum_{j=1}^{3} l_j$. Hence, $\sum_{i=2}^{n} c(W, E(z_i)) \ge 3\sum_{j=1}^{3} l_j + (n - 1 - \sum_{i=1}^{3} l_j)$ since $c(W, E(z_j)) \ge 1$ for $1 \le j \le n$ (see Figure 17). Hence, from (7), $c(W, E(z_1)) = 1$, and $c(W) = 1$ we have

$$\sum_{j=1}^{3} l_j \le \lfloor \frac{n-2}{2} \rfloor. \tag{20}$$

On the other hand, from (15), (16), (18), and (19) we have

$$\sum_{i=2}^{n} c(F, E(z_i)) \ge 4l_1 + 4l_2 + 4l_3 + 6(n - 1 - \sum_{j=1}^{3} l_j). \tag{21}$$

Thus, by (20) and (21) we have

$$\sum_{i=2}^{n} c(F, E(z_i)) \ge 6(n-1) - 2\lfloor \frac{n-2}{2} \rfloor. \tag{22}$$

From (8), (22), and the fact that $c(F) = 2$ (see Figure 17), we have $c(D) \ge 2 + Z(6, n-1) + 6(n-1) - 2\lfloor \frac{n-2}{2} \rfloor \ge Z(6, n) + 2n$. $\square$

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