PARTIAL FRACTION DECOMPOSITION AND DETERMINANT IDENTITIES

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ABSTRACT. By means of partial fraction decomposition method, we evaluate a very general determinant of formal shifted factorial fractions, which covers numerous binomial determinantal identities.

1. INTRODUCTION AND MOTIVATION

Mathematicians and physicists often come across determinants they need to evaluate. For example, it is well-known that Cauchy's celebrated double alternant [9]

$$\det_{0\leq i,j\leq n}\left[\frac{1}{x_i+y_j}\right]=\frac{\prod_{0\leq i<j\leq n}(x_i-x_j)(y_i-y_j)}{\prod_{0\leq i,\,j\leq n}(x_i+y_j)}$$

has fundamental applications to the multiple elliptic hypergeometric series [18] and the theory of symmetric functions [19, Chapter 1]. Several generalizations have appeared in the literatures [5, 8, 12, 15, 17]. In particular, it has recently been generalized by the first author [13] via divided differences.

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As a further extension, this paper will establish a very general determinantal identity (see Theorem 1 in the next section) by means of partial fraction decomposition method. The usefulness of our main theorem will be shown through exemplifying a large number of binomial determinantal formulae, that have been important in enumerative combinatorics (cf. [6, 16, 25]), particularly in plane partitions (see [4, 11, 20, 24] for example).

The paper will be organized as follows. The next section will be devoted to the proof of a generalized Cauchy determinant with the method of partial fraction decomposition as well as to the exposition of few limiting cases. As applications of our main theorem, the third section will collect ten classes of binomial determinantal identities, including several interesting ones contained in Amdeberhan-Zeilberger [3] and the Hankel determinants on Catalan numbers due to Aigner [1] and Radoux [23]. Finally in the fourth section, five classes of determinant identities will be illustrated with the entries of matrices containing more binomial coefficients.

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ will stand for the sets of natural numbers and nonnegative integers, respectively. For two indeterminate x and n, the shifted factorial will be defined by the following Γ-function quotient $(x)_n := \Gamma(x+n)/\Gamma(x)$. It reads, in particular for $n \in \mathbb{N}_0$, explicitly as

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{with} \quad n \in \mathbb{N}.$$

The binomial coefficient will accordingly be defined by

$$\binom{x}{n} = \frac{(1+x-n)_n}{(1)_n} = \frac{\Gamma(x+1)}{\Gamma(n+1)\Gamma(x-n+1)}$$

which reduces to the usual one when n is a nonnegative integer.

2. Partial Fraction Decomposition

For two sequences $\{\alpha_k, \gamma_k\}_{k\geq 0}$, define the generalized shifted factorials by

$$\langle x|\alpha\rangle_0 = 1 \quad \text{and} \quad \langle x|\alpha\rangle_n = \prod_{k=0}^{n-1}(x+\alpha_k) \quad \text{with} \quad n \in \mathbb{N}, \tag{1a}$$

$$\langle y|\gamma\rangle_0 = 1 \quad \text{and} \quad \langle y|\gamma\rangle_n = \prod_{k=0}^{n-1}(y+\gamma_k) \quad \text{with} \quad n \in \mathbb{N}. \tag{1b}$$

When $\alpha_k = \gamma_k = k$ for $k \in \mathbb{N}_0$, they will reduce to the usual shifted factorials. For the upper triangular matrix given by $\alpha = [\alpha_{ij}]_{0\leq i\leq j<\infty}$, denote its j-th column by $\alpha_j = (\alpha_{0j}, \alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{jj})$. Then our main theorem may be stated as follows.

Theorem 1 (Generalized Cauchy determinant). *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinantal identity:*

$$\det_{0\le i,j\le n}\left[\frac{\langle x_i|\alpha_j\rangle_j}{\langle x_i|\gamma\rangle_{j+1}}\right] = \frac{\prod_{0\le i<j\le n}(x_i - x_j)(\alpha_{ij}-\gamma_j)}{\prod_{0\le i,\,j\le n}(x_i+\gamma_j)}.$$

This theorem generalizes Cauchy's double alternant displayed in the introduction. In fact, the very special case of Theorem 1 with $\alpha_{ij} = \gamma_i$ for $i, j \in \mathbb{N}_0$ results in Cauchy's double alternant. Observing that the proof below depends on the Cauchy double alternant, we can therefore say that they are substantially equivalent.

Proof. Expanding the rational function in partial fractions, we have

$$\frac{\langle x_i|\alpha_j\rangle_j}{\langle x_i|\gamma\rangle_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(x_i+\alpha_{\iota j})}{\prod_{k=0}^{j}(x_i+\gamma_k)} = \sum_{k=0}^{j}\frac{w_{kj}}{x_i+\gamma_k}$$

where the connected coefficients are determined by the limit relation

$$w_{kj} = \lim_{x_i\to-\gamma_k}(x_i+\gamma_k)\frac{\langle x_i|\alpha_j\rangle_j}{\langle x_i|\gamma\rangle_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(\alpha_{\iota j}-\gamma_k)}{\prod_{\iota=0:\,\iota\ne k}^{j}(\gamma_\iota-\gamma_k)}.$$

This leads us to the following determinantal factorization

$$\det_{0\le i,j\le n}\left[\frac{\langle x_i|\alpha_j\rangle_j}{\langle x_i|\gamma\rangle_{j+1}}\right] = \det_{0\le i,k\le n}\left[\frac{1}{x_i+\gamma_k}\right]\times\det_{0\le k,j\le n}\left[w_{kj}\right].$$

For the matrix $[w_{kj}]_{0\le k,j\le n}$ is upper triangular, its determinant is equal to the product of its diagonal entries:

$$\det_{0\le k,j\le n}\left[w_{kj}\right] = \prod_{j=0}^{n} w_{jj} = \prod_{0\le i<j\le n}\frac{\alpha_{ij}-\gamma_j}{\gamma_i-\gamma_j}.$$

While the first determinant can be evaluated by Cauchy's double alternant. Their product yields the determinant identity stated in Theorem 1. □

Shifting the γ-parameters by $\gamma_k \to \gamma_{k-1}$, we may state the determinant identity in Theorem 1 in the following more convenient form.

Proposition 2. *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinantal identity:*

$$\det_{0\le i,j\le n}\left[\frac{\langle x_i|\alpha_j\rangle_j}{\langle x_i|\gamma\rangle_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)(\alpha_{ij}-\gamma_{j-1})}{\prod_{k=0}^n\langle x_k|\gamma\rangle_n}.$$

When $\{\alpha_{ij}\}_{i\in\mathbb{N}_0}$ and $\{\gamma_k\}_{k\in\mathbb{N}_0}$ are sequences with the common constant difference between consecutive terms, this proposition reduces to a determinant evaluation appeared in Normand [21, Lemma 3]. Furthermore, we have the following interesting determinantal identities.

Corollary 3 ($\gamma_k\to\infty$ in Proposition 2: see Chu-Diclaudio [14, Thm 3.3]).

$$\det_{0\le i,j\le n}\left[\langle x_i|\alpha\rangle_j\right] = \prod_{0\le i<j\le n}(x_j-x_i).$$

Proof. The limiting process can be carried out as follows. Putting $\gamma_k=\gamma$, then multiplying by $\gamma^{\binom{n+1}{2}}$ across the equation displayed in Proposition 2 and finally letting $\gamma\to\infty$, we confirm the identity stated in the corollary. □

Corollary 4 ($\alpha_{ij}\to\infty$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[\frac{1}{\langle x_i|\gamma\rangle_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)}{\prod_{k=0}^n\langle x_k|\gamma\rangle_n}.$$

This can similarly be done by first putting $\alpha_{ij}=\alpha$, then dividing by $\alpha^{\binom{n+1}{2}}$ across the equation displayed in Proposition 2 and finally letting $\alpha\to\infty$.

Corollary 5 ($\alpha_{ij}=y_j/c$ and $\gamma_k=\gamma$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[\frac{(cx_i+y_j)^j}{(x_i+\gamma)^j}\right] = \prod_{0\le i<j\le n}(x_i-x_j)\frac{\prod_{k=1}^n(y_k-c\gamma)^k}{\prod_{k=0}^n(x_k+\gamma)^n}.$$

Corollary 6 ($\alpha_{ij}=(i+y_j)/c$ and $\gamma_k=k+\gamma$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[\frac{(cx_i+y_j)_j}{(x_i+\gamma)_j}\right] = \prod_{0\le i<j\le n}(x_i-x_j)\frac{\prod_{k=1}^n(c+y_k-ck-c\gamma)_k}{\prod_{k=0}^n(x_k+\gamma)_n}.$$

3. Binomial Determinantal Identities

Applying exclusively Corollary 6, we show now ten classes of binomial determinantal identities. We should point out that our objective is to focus on a unified treatment of binomial determinants, instead of finding new results.

§3.1. Expressing the binomial coefficient in terms of shifted factorials

$$\binom{X_i - j}{A + BX_i} = \binom{X_i}{A + BX_i}\frac{(A + BX_i - X_i)_j}{(-X_i)_j}$$

we find the determinant identity

$$\det_{0\le i,j\le n}\left[\binom{X_i - j}{A + BX_i}\right] = \prod_{k=0}^{n}\frac{\binom{X_k}{A+BX_k}}{(-X_k)_n} \tag{2a}$$

$$\times \prod_{0\le i<j\le n}(X_j - X_i)(1 + A - B + Bj + i - j). \tag{2b}$$

In particular for $X_i = a + bi$, we have the binomial determinantal formula:

$$\det_{0\le i,j\le n}\left[\binom{a + bi - j}{c + di}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(-a - bk)_n} \tag{3a}$$

$$\times \prod_{0\le i<j\le n}(b - d + bc - ad + bi + dj - bj). \tag{3b}$$

§3.2. Rewriting the binomial coefficient in terms of shifted factorials

$$\binom{A + BX_i}{X_i - j} = (-1)^j\binom{A + BX_i}{X_i}\frac{(-X_i)_j}{(1 + A + BX_i - X_i)_j}$$

we get the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\binom{A + BX_i}{X_i - j}\right] = \frac{\prod_{0\le i<j\le n}(X_j - X_i)(A+Bi-i+j)}{\prod_{k=0}^{n}(1+A+BX_k-X_k)_n}\prod_{k=0}^{n}\binom{A + BX_k}{X_k}. \tag{4}$$

For $X_i = c + di$, it reads as the following binomial determinantal identity:

$$\det_{0\le i,j\le n}\left[\binom{a + bi}{c + di - j}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(1+a-c+bk-dk)_n}\prod_{0\le i<j\le n}(ad-bc+bi-di+dj). \tag{5}$$

§3.3. Reformulating the binomial coefficient in terms of shifted factorials

$$\binom{A+BX_i-j}{X_i-j} = \binom{A+BX_i}{X_i}\frac{(-X_i)_j}{(-A-BX_i)_j}$$

we obtain the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\binom{A+BX_i-j}{X_i-j}\right] = \prod_{k=0}^{n}\frac{\binom{A+BX_k}{X_k}}{(-A-BX_k)_n}\prod_{0\le i<j\le n}(X_j-X_i)(1+A+Bi-j). \quad (6)$$

Its special case $X_i = c+di$ leads to the following binomial determinantal identity:

$$\det_{0\le i,j\le n}\left[\binom{a+bi-j}{c+di-j}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(-a-bk)_n}\prod_{0\le i<j\le n}(d+ad-bc+bi-dj). \quad (7)$$

§3.4. Applying the binomial relation

$$\binom{X_i+j}{A+BX_i} = \binom{X_i}{A+BX_i}\frac{(1+X_i)_j}{(1-A-BX_i+X_i)_j}$$

we have the determinant evaluation

$$\det_{0\le i,j\le n}\left[\binom{X_i+j}{A+BX_i}\right] = \prod_{k=0}^{n}\frac{\binom{X_k}{A+BX_k}}{(1-A-BX_k+X_k)_n} \quad (8a)$$

$$\times \prod_{0\le i<j\le n}(X_j-X_i)(-1-A+B+Bi-i+j). \quad (8b)$$

In particular for $X_i = a+bi$, this gives the binomial determinantal identity:

$$\det_{0\le i,j\le n}\left[\binom{a+bi+j}{c+di}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(1+a-c+bk-dk)_n} \quad (9a)$$

$$\times \prod_{0\le i<j\le n}(d-b+ad-bc+di-bi+bj). \quad (9b)$$

§3.5. Observing the binomial relation

$$\binom{A+BX_i}{X_i+j} = (-1)^j\binom{A+BX_i}{X_i}\frac{(-A-BX_i+X_i)_j}{(1+X_i)_j}$$

we recover the determinant identity due to Ostrowski [22] (see [7,10] also)

$$\det_{0\le i,j\le n}\left[\binom{A+BX_i}{X_i+j}\right] = \prod_{k=0}^{n}\frac{\binom{A+BX_k}{X_k}}{(1+X_k)_n}\prod_{0\le i<j\le n}(X_i-X_j)(A-Bj-i+j). \quad (10)$$

Its case corresponding to $X_i = c + di$ reads as

$$\det_{0\le i,j\le n}\left[\binom{a+bi}{c+di+j}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(1+c+dk)_n}\prod_{0\le i<j\le n}(bc-ad+di+bj-dj). \quad (11)$$

§3.6. By invoking the binomial relation

$$\binom{A+BX_i+j}{X_i+j} = \binom{A+BX_i}{X_i}\frac{(1+A+BX_i)_j}{(1+X_i)_j}$$

we establish the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\binom{A+BX_i+j}{X_i+j}\right] = \prod_{k=0}^{n}\frac{\binom{A+BX_k}{X_k}}{(1+X_k)_n}\prod_{0\le i<j\le n}(X_i-X_j)(1+A+i-Bj). \quad (12)$$

For $X_i = c + di$, it results in the following binomial determinantal identity:

$$\det_{0\le i,j\le n}\left[\binom{a+bi+j}{c+di+j}\right] = \prod_{k=0}^{n}\frac{k!\binom{a+bk}{c+dk}}{(1+c+dk)_n}\prod_{0\le i<j\le n}(bc-ad-d-di+bj). \quad (13)$$

§3.7. By means of the binomial relation

$$\binom{2X_i+2j}{X_i+j} = 4^j\binom{2X_i}{X_i}\frac{(\frac{1}{2}+X_i)_j}{(1+X_i)_j}$$

we get the determinant evaluation

$$\det_{0\le i,j\le n}\left[\binom{2X_i+2j}{X_i+j}\right] = \prod_{k=0}^{n}\frac{(2k)!(2X_k)!}{k!X_k!(X_k+n)!}\prod_{0\le i<j\le n}(X_j-X_i). \quad (14)$$

When $X_i = a + bi$, it reduces to the identity due to Amdeberhan and Zeilberger [3, Eq 3]:

$$\det_{0\le i,j\le n}\left[\binom{2a+2bi+2j}{a+bi+j}\right] = b^{\binom{n+1}{2}}\prod_{k=0}^{n}\frac{(2k)!\binom{2a+2bk}{a+bk}}{(1+a+bk)_n}. \quad (15)$$

Similarly, we can show the determinant identity on the generalized Catalan numbers

$$\det_{0\le i,j\le n}\left[\frac{1}{1+\delta+X_i+j}\binom{\delta+2X_i+2j}{X_i+j}\right] \quad (16a)$$

$$= \prod_{k=0}^{n}\frac{(1+2k)!(\delta+2X_k)!}{k!X_k!(1+\delta+X_k+n)!}\prod_{0\le i<j\le n}(X_j-X_i) \quad (16b)$$

where $\delta = 0, 1$. Let C_n stand for the usual Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then the last formula contains, for $\delta = 0$, the following well-known Hankel determinants of Catalan numbers (see Aigner [1] and Radoux [23] for example):

$$\det_{0\le i,j\le n}\Big[\;C_{i+j}\;\Big] = 1, \tag{17}$$

$$\det_{0\le i,j\le n}\Big[C_{i+j+1}\Big] = 1, \tag{18}$$

$$\det_{0\le i,j\le n}\Big[C_{i+j+2}\Big] = n+2. \tag{19}$$

In fact, it is trivial to see that for $\delta = 0$ and $X_k = k + \varepsilon$ with $\varepsilon = 0, 1, 2$, the determinant in (16a) reduces to the Hankel determinants respectively displayed in the last three equations. However, it is not so obvious that the corresponding product in (16b) becomes the right members of the last three equations. Here we limit to prove the first one because the others can be shown analogously. For $\delta = 0$ and $X_k = k$, what we need to confirm is equivalent to the identity

$$\prod_{k=0}^{n}\frac{(2k)!(1+2k)!}{k!(1+n+k)!} = 1. \tag{20}$$

For a real number x, denote by $\lfloor x\rfloor$ the greatest integer $\le x$. According to the parity of k, we can manipulate the denominator product as follows

$$\begin{aligned}
&\prod_{k=0}^{n}\Big\{k!(1+n+k)!\Big\} = \prod_{k=0}^{n}\Big\{k!(1+2n-k)!\Big\}\\
&= \prod_{k=0}^{\lfloor\frac{n}{2}\rfloor}\Big\{(2k)!(1+2n-2k)!\Big\}\times\prod_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\Big\{(1+2k)!(2n-2k)!\Big\}\\
&= \Big\{\prod_{k=0}^{\lfloor\frac{n}{2}\rfloor}(2k)!\prod_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(2n-2k)!\Big\}\times\Big\{\prod_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(1+2k)!\prod_{k=0}^{\lfloor\frac{n}{2}\rfloor}(1+2n-2k)!\Big\}\\
&= \Big\{\prod_{k=0}^{\lfloor\frac{n}{2}\rfloor}(2k)!\prod_{k=\lfloor\frac{n+2}{2}\rfloor}^{n}(2k)!\Big\}\times\Big\{\prod_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(1+2k)!\prod_{k=\lfloor\frac{n+1}{2}\rfloor}^{n}(1+2k)!\Big\}
\end{aligned}$$

where we have appealed to the relation $\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n+1}{2}\rfloor = n$ for a natural number n. The last expression is clearly equal to the numerator product displayed in (20).

§3.8. In view of the binomial relation

$$\binom{X_i+Y_j+j}{A+BX_i+Y_j}\binom{X_i+Y_j}{A+BX_i+Y_j}^{-1} = \frac{(1+X_i+Y_j)_j}{(1-A-BX_i+X_i)_j}$$

we find the following determinantal identity

$$\det_{0\le i,j\le n}\left[\binom{X_i+Y_j+j}{A+BX_i+Y_j}\binom{X_i+Y_j}{A+BX_i+Y_j}^{-1}\right] \tag{21a}$$

$$= \frac{\prod_{0\le i<j\le n}(X_i-X_j)\{1+A-B+(1-B)Y_j+(1-B)i-j\}}{\prod_{k=0}^{n}(1-A-BX_k+X_k)_n}. \tag{21b}$$

§3.9. According to the binomial relation

$$\binom{A+BX_i+Y_j}{j}\binom{X_i+j}{j}^{-1} = (-1)^j \times \frac{(-A-BX_i-Y_j)_j}{(1+X_i)_j}$$

we derive the following binomial determinantal identity

$$\det_{0\le i,j\le n}\left[\binom{A+BX_i+Y_j}{j}\binom{X_i+j}{j}^{-1}\right] \tag{22a}$$

$$= \prod_{k=0}^{n}\frac{1}{(1+X_k)_n}\prod_{0\le i<j\le n}(X_j-X_i)(-A-Y_j+Bj+i). \tag{22b}$$

When $X_i = y+bi$ and $Y_j = cj$, it recovers the binomial determinantal identity due to Amdeberhan and Zeilberger [3, Eq 6]:

$$\det_{0\le i,j\le n}\left[\frac{1}{i!}\binom{x+ai+cj}{j}\binom{y+bi+j}{j}^{-1}\right] \tag{23a}$$

$$= \prod_{k=0}^{n}\frac{(y+bk)!}{(y+bk+n)!}\prod_{0\le i<j\le n}\{(y+j)a-(x+cj-i)b\}. \tag{23b}$$

§3.10. Similarly, the binomial relation

$$\binom{A+BX_i-j}{n-j}\binom{X_i+Y_j}{j} = (-1)^n\frac{(-A-BX_i)_n}{j!(n-j)!}\times\frac{(-X_i-Y_j)_j}{(-A-BX_i)_j}$$

leads us to the following determinantal identities

$$\det_{0\le i,j\le n}\left[i!^2\binom{A+BX_i-j}{n-j}\binom{X_i+Y_j}{j}\right] \tag{24a}$$

$$= \prod_{0\le i<j\le n}(X_j-X_i)(1+A-BY_j+Bi-j), \tag{24b}$$

$$\det_{0\le i,j\le n}\left[i!^2\binom{A+BX_i+Y_j}{n-j}\binom{X_i+j}{j}\right] \tag{25a}$$

$$= \prod_{0\le i<j\le n}(X_j-X_i)\{A-(1+n-j)B+Y_{n-j}-i\}. \tag{25b}$$

We remark that the last two determinant identities are equivalent with the column index being inverted by $j \to n-j$. When $X_i = x+ai$ and $Y_j = cj$, they reduce, respectively, to the following binomial determinantal identities due to Amdeberhan and Zeilberger [3, Eqs 4-5]

$$\det_{0\le i,j\le n}\left[i!\binom{x+ai+cj}{j}\binom{y+bi-j}{n-j}\right] \tag{26a}$$

$$= \prod_{0\le i<j\le n}\{(y-j+1)a-(x+cj-i)b\}, \tag{26b}$$

$$\det_{0\le i,j\le n}\left[i!\binom{x+ai+j}{j}\binom{y+bi+cj}{n-j}\right] \tag{27a}$$

$$= \prod_{0\le i<j\le n}\{(y+cn-i-cj)a-(x+n-j+1)b\}; \tag{27b}$$

where the last equation corrects an error appeared in Amdeberhan and Zeilberger [3, Eq 5]. In particular for $a=1$, $b=c=-1$, $x=0$ and $y=2n$, the last result reads as

$$\det_{0\le i,j\le n}\left[\binom{i+j}{i}\binom{2n-i-j}{n-i}\right] = \frac{(2n+1)!^n}{\prod_{k=1}^{2n}k!} \tag{28}$$

which is conjectured by Kuperberg and Propp in their work on plane partition enumeration and verified by Amdeberhan and Ekhad [2] through Dodgson's rule.

4. Duplicate Determinantal Identities

Performing the parameter replacements in Proposition 2

$$\begin{aligned} x_k &\to (a+x_k)(c-x_k)-ac,\\ \gamma_k &\to \frac{(ad+\gamma_k)(cd+\gamma_k)}{d^2},\\ \alpha_{ij} &\to \frac{(ab+\alpha_{ij})(bc+\alpha_{ij})}{b^2}; \end{aligned}$$

and then applying factorizations

$$\begin{aligned} x_i+\gamma_k &\to \frac{(ad+dx_i+\gamma_k)(cd-dx_i+\gamma_k)}{d^2},\\ x_i+\alpha_{ij} &\to \frac{(ab+bx_i+\alpha_{ij})(bc-bx_i+\alpha_{ij})}{b^2},\\ x_i-x_j &\to (x_i-x_j)(c-a-x_i-x_j),\\ \alpha_{ij}-\gamma_k &\to \frac{(d\alpha_{ij}-b\gamma_k)(d\alpha_{ij}+b\gamma_k+abd+bcd)}{b^2d^2}; \end{aligned}$$

we can reformulate the result as the following determinantal identity.

Proposition 7. *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinantal identity:*

$$\det_{0\le i,j\le n}\left[\frac{\langle ab+bx_i|\alpha_j\rangle_j\langle bc-bx_i|\alpha_j\rangle_j}{\langle ad+dx_i|\gamma\rangle_j\langle cd-dx_i|\gamma\rangle_j}\right]=\frac{\prod_{0\le i<j\le n}(x_i-x_j)(c-a-x_i-x_j)}{\prod_{k=0}^n\langle ad+dx_k|\gamma\rangle_n\langle cd-dx_k|\gamma\rangle_n}$$
$$\times\prod_{0\le i<j\le n}(d\alpha_{ij}-b\gamma_{j-1})(abd+bcd+d\alpha_{ij}+b\gamma_{j-1}).$$

This identity contains the following three interesting special cases. Among them, the first two limiting cases can be justified similarly as those for Corollaries 3 and 4.

Corollary 8 ($b=1$ and $\gamma_k\to\infty$ in Proposition 7).

$$\det_{0\le i,j\le n}\left[\langle a+x_i|\alpha_j\rangle_j\langle c-x_i|\alpha_j\rangle_j\right]=\prod_{0\le i<j\le n}(x_j-x_i)(c-a-x_i-x_j).$$

Corollary 9 ($d=1$ and $\alpha_{ij}\to\infty$ in Proposition 7).

$$\det_{0\le i,j\le n}\left[\frac{1}{\langle a+x_i|\gamma\rangle_j\langle c-x_i|\gamma\rangle_j}\right]=\frac{\prod_{0\le i<j\le n}(x_i-x_j)(c-a-x_i-x_j)}{\prod_{k=0}^n\langle a+x_k|\gamma\rangle_n\langle c-x_k|\gamma\rangle_n}.$$

Corollary 10 ($b = d = 1$, $\alpha_{ij} = y_j$ and $\gamma_j = z$ in Proposition 7).

$$\det_{0\le i,j\le n}\left[\frac{(a+x_i+y_j)^j(c-x_i+y_j)^j}{(a+x_i+z)^j(c-x_i+z)^j}\right] = \prod_{k=0}^{n}\frac{(y_k-z)^k(a+c+y_k+z)^k}{(a+x_k+z)^n(c-x_k+z)^n}$$
$$\times \prod_{0\le i<j\le n}(x_i-x_j)(c-a-x_i-x_j).$$

Alternatively, performing the parameter replacements in Proposition 2

$$\begin{aligned} x_k &\to ax_k + c/x_k,\\ \gamma_k &\to \frac{\gamma_k}{d}+\frac{acd}{\gamma_k},\\ \alpha_{ij} &\to \frac{\alpha_{ij}}{b}+\frac{abc}{\alpha_{ij}}; \end{aligned}$$

and then applying factorizations

$$\begin{aligned} x_i+\gamma_k &\to \frac{(adx_i+\gamma_k)(cd/x_i+\gamma_k)}{d\gamma_k},\\ x_i+\alpha_{ij} &\to \frac{(abx_i+\alpha_{ij})(bc/x_i+\alpha_{ij})}{b\alpha_{ij}},\\ x_i-x_j &\to (x_i-x_j)(a-c/x_ix_j),\\ \alpha_{ij}-\gamma_k &\to \frac{(b\gamma_k-d\alpha_{ij})(abcd-\alpha_{ij}\gamma_k)}{bd\alpha_{ij}\gamma_k}; \end{aligned}$$

we can show the following multiplicative determinantal identity.

Proposition 11. *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinantal identity:*

$$\det_{0\le i,j\le n}\left[\frac{\langle abx_i|\alpha_j\rangle_j\langle bc/x_i|\alpha_j\rangle_j}{\langle adx_i|\gamma\rangle_j\langle cd/x_i|\gamma\rangle_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)(a-c/x_ix_j)}{\prod_{k=0}^{n}\langle adx_k|\gamma\rangle_n\langle cd/x_k|\gamma\rangle_n}$$
$$\times \prod_{0\le i<j\le n}(b\gamma_{j-1}-d\alpha_{ij})(abcd-\alpha_{ij}\gamma_{j-1}).$$

Corollary 12 ($b = 1$ and $\gamma_k \to \infty$ in Proposition 11).

$$\det_{0\le i,j\le n}\Big[\langle ax_i|\alpha_j\rangle_j\langle c/x_i|\alpha_j\rangle_j\Big] = \prod_{0\le i<j\le n}\Big\{\alpha_{ij}(x_j-x_i)(a-c/x_ix_j)\Big\}.$$

Corollary 13 ($d = 1$ and $\alpha_{ij} \to \infty$ in Proposition 11).

$$\det_{0\le i,j\le n}\left[\frac{1}{\langle ax_i|\gamma\rangle_j\langle c/x_i|\gamma\rangle_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)(a-c/x_ix_j)}{\prod_{k=0}^{n}\langle ax_k|\gamma\rangle_n\langle c/x_k|\gamma\rangle_n}\prod_{\ell=1}^{n}\gamma_{\ell-1}^{\ell}.$$

By means of these determinant identities, we can further evaluate the following five binomial determinants.

§4.1. Expressing the binomial coefficients in terms of shifted factorials

$$\frac{\binom{x_i+y_j}{j}\binom{a-x_i+y_j}{j}}{\binom{x_i+b}{j}\binom{a-x_i+b}{j}} = \frac{(-x_i-y_j)_j(-a+x_i-y_j)_j}{(-x_i-b)_j(-a+x_i-b)_j}$$

and then applying Proposition 7, we establish the determinant identity

$$\det_{0\le i,j\le n}\left[\frac{\binom{x_i+y_j}{j}\binom{a-x_i+y_j}{j}}{\binom{x_i+b}{j}\binom{a-x_i+b}{j}}\right] = \prod_{k=0}^{n}\frac{(y_k-b)_k(2+a+b+y_k-2k)_k}{(x_k-a-b)_n(-x_k-b)_n} \tag{29a}$$

$$\times \prod_{0\le i<j\le n}(x_i-x_j)(a-x_i-x_j) \tag{29b}$$

which can be further specialized to the following binomial determinant:

$$\det_{0\le i,j\le n}\left[\frac{\binom{a+bi-j}{j}\binom{c-bi-j}{j}}{i!^2\binom{a+bi}{j}\binom{c-bi}{j}}\right] = (-b)^{\binom{n+1}{2}}\frac{\prod_{0\le i<j\le n}(1+a+c-i-2j)(a-c+bi+bj)}{\prod_{k=0}^{n}(-a-bk)_n(-c+bk)_n}. \tag{30}$$

§4.2. Rewriting the binomial coefficients in terms of shifted factorials

$$\frac{\binom{a+c+x_i}{j}\binom{x_i-b}{n-j}}{\binom{a+b+x_i}{j}\binom{x_i-c}{n-j}} = \frac{(b-x_i)_n(1-c+x_i-n)_j(-x_i-a-c)_j}{(c-x_i)_n(1-b+x_i-n)_j(-x_i-a-b)_j}$$

we get from Proposition 7 the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\frac{\binom{a+c+x_i}{j}\binom{x_i-b}{n-j}}{\binom{a+b+x_i}{j}\binom{x_i-c}{n-j}}\right] = \prod_{k=0}^{n}\frac{(c-b)_k(b-x_k)_n(1+a+b+c+n-2k)_k}{(c-x_k)_n(-a-b-x_k)_n(1-b-n+x_k)_n} \tag{31a}$$

$$\times \prod_{0\le i<j\le n}(x_i-x_j)(n-a-1-x_i-x_j) \tag{31b}$$

which contains, as special case, the following determinantal identity:

$$\det_{0\le i,j\le n}\left[\frac{\binom{\lambda i+a}{n-j}\binom{\lambda i-c}{j}}{j!\binom{\lambda i-a}{j}\binom{\lambda i+c}{n-j}}\right] = \prod_{k=0}^{n}\frac{(a-c)_k(1-a-c+n-2k)_k}{(a-\lambda k)_n(-c-\lambda k)_n} \tag{32a}$$

$$\times \lambda^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(1-n+\lambda i+\lambda j). \tag{32b}$$

§4.3. Reformulating the binomial coefficients in terms of shifted factorials

$$\frac{\binom{x_i+y_j+j}{x_i-y_j-a-j}}{\binom{x_i+y_j}{x_i-y_j-a}\binom{x_i+c+j}{x_i-a-c-j}} = \frac{\binom{a+2c+2j}{a+2j+2y_j}}{\binom{a+2c}{a+2y_j}\binom{x_i+c}{x_i-a-c}} \times \frac{(1+x_i+y_j)_j(a-x_i+y_j)_j}{(1+c+x_i)_j(a+c-x_i)_j}$$

we derive from Proposition 7 the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\frac{\binom{x_i+y_j+j}{x_i-y_j-a-j}}{\binom{x_i+y_j}{x_i-y_j-a}\binom{x_i+c+j}{x_i-a-c-j}}\right] = \prod_{0\le i<j\le n}(x_i-x_j)(1-a+x_i+x_j) \tag{33a}$$

$$\times\prod_{k=0}^{n}\frac{(c-y_k)_k(a+c+y_k+k)_k}{(1+c+x_k)_n(a+c-x_k)_n}\frac{\binom{a+2c+2k}{a+2y_k+2k}}{\binom{a+2c}{a+2y_k}\binom{x_k+c}{x_k-a-c}} \tag{33b}$$

which reduces, for $x_i = bi$ and $y_j = 0$, to the following determinantal identity:

$$\det_{0\le i,j\le n}\left[\frac{1}{i!}\binom{bi+j}{2j}\binom{bi+c+j}{2j+2c}^{-1}\right] \tag{34a}$$

$$= (-b)^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(1+bi+bj)\prod_{k=0}^{n}\frac{(2k+2c)!(c)_{2k}(bk-c-n)!}{(2k)!(bk+c+n)!}. \tag{34b}$$

§4.4. According to Corollary 8, the binomial relation

$$\binom{x_i+y_j+j}{c+x_i-y_j-j}\binom{c+x_i-y_j}{x_i+y_j} = (-1)^j\frac{(1+x_i+y_j)_j(y_j-x_i-c)_j}{(2y_j+2j-c)!(c-2y_j)!}$$

leads to the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\binom{x_i+y_j+j}{c+x_i-y_j-j}\binom{c+x_i-y_j}{x_i+y_j}\right] \tag{35a}$$

$$= \prod_{k=0}^{n}\frac{(-1)^k}{(2k)!}\binom{2k}{c-2y_k}\prod_{0\le i<j\le n}(x_i-x_j)(1+c+x_i+x_j). \tag{35b}$$

For $x_i = a+bi$ and $y_j = 0$, this determinant identity recovers the result due to Amdeberhan and Zeilberger [3, Eq 1]:

$$\det_{0\le i,j\le n}\left[\frac{1}{i!}\binom{a+bi+j}{c+bi-j}\right] = \frac{b^{\binom{n+1}{2}}\prod_{k=0}^{n}(a+bk)!}{\prod_{k=0}^{n}(c+bk)!(a-c+2k)!} \tag{36a}$$

$$\times \prod_{0\le i<j\le n}(1+a+c+bi+bj). \tag{36b}$$

Similarly, letting $x_i = bi$ and $\alpha_{ij} = dj - i$, we recover from Corollary 8 the result due to Amdeberhan and Zeilberger [3, Eq 14]:

$$\det_{0\le i,j\le n}\left[j!\binom{a+bi+dj}{j}\binom{c-bi+dj}{j}\right] = (-b)^{\binom{n+1}{2}} \prod_{0\le i<j\le n} (a-c+bi+bj). \tag{37}$$

§4.5. By means of Corollary 9, the binomial relation

$$\binom{x_i+a-j}{x_i+j} = \frac{(-1)^j(x_i+a)!}{(x_i)!(a-2j)!(x_i+1)_j(-x_i-a)_j}$$

yields the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\binom{x_i+a-j}{x_i+j}\right] = \prod_{k=0}^{n}\frac{(x_k+a-n)!}{(x_k+n)!(a-2k)!} \tag{38a}$$

$$\times \prod_{0\le i<j\le n}(x_i-x_j)(a+1+x_i+x_j). \tag{38b}$$

For $x_i = c+bi$, the last determinant identity reduces to

$$\det_{0\le i,j\le n}\left[\frac{1}{i!}\binom{a+bi-j}{c+bi+j}\right] = (-b)^{\binom{n+1}{2}}\prod_{k=0}^{n}\frac{(a+bk-n)!}{(c+bk+n)!(a-c-2k)!} \tag{39a}$$

$$\times \prod_{0\le i<j\le n}(1+a+c+bi+bj) \tag{39b}$$

which is equivalent to another result due to Amdeberhan and Zeilberger [3, Eq 2]:

$$\det_{0\le i,j\le n}\left[\frac{1}{i!}\binom{a+bi+j}{c+bi-j}^{-1}\right] = (-b)^{\binom{n+1}{2}}\prod_{k=0}^{n}\frac{(a-c+2k)!(c+bk-n)!}{(a+bk+n)!} \tag{40a}$$

$$\times \prod_{0\le i<j\le n}(1+a+c+bi+bj). \tag{40b}$$

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