Lattices associated with subspaces in d-bounded distance-regular graphs*

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Abstract

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 3$. A regular strongly closed subgraph of Γ is said to be a subspace of Γ. For $0 \leq i < i+s \leq d-1$. Suppose Δ_1 and Δ_0 are subspaces with diameter i and $i+s$, respectively, and with $\Delta_1 \subseteq \Delta_0$. Let $\mathcal{L}(i,\, i+s;\, d)$ denote the set of all subspaces Δ' with diameters $\geq i$ such that $d(\Delta_0 \cap \Delta') = \Delta_1$ and $d(\Delta_0 + \Delta') = d(\Delta') + s$ in Γ including Γ. If we partial order $\mathcal{L}(i,\, i+s;\, d)$ by ordinary inclusion (resp. reverse inclusion), then $\mathcal{L}(i,\, i+s;\, d)$ is a poset, denoted by $\mathcal{L}_O(i,\, i+s;\, d)$ (resp. $\mathcal{L}_R(i,\, i+s;\, d)$). In the present paper we show that both $\mathcal{L}_O(i,\, i+s;\, d)$ and $\mathcal{L}_R(i,\, i+s;\, d)$ are atomic lattices, and classify their geometricity.

Keywords: Distance-regular graph; Subspaces; Geometric lattice.

1 Introduction

In this section We first recall some terminology and definitions about finite posets and lattices ([1, 2]), then introduce some concepts concerning d-bounded distance-regular graphs and our main results.

Let P be a poset. For $a, b \in P$, we say *a covers b*, denoted by $b <\cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If P has the minimum

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(resp. maximum) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b <\cdot a$.

A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an *atom* in P, we mean an element in P covering 0. We say P is *atomic* if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \forall a,\ b \in P$.

Now we shall introduce some concepts concerning d-bounded distance-regular graphs. Let $\Gamma = (X, R)$ be a connected regular graph. For vertices u and v in X, let $\partial(u, v)$ denote the *distance* between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $d = d(\Gamma)$. For vertices u and v at distance i, define

$$\begin{aligned} C(u,v) &= C_i(u,v) = \{w \mid \partial(u,w) = i-1, \partial(w,v) = 1\}, \\ A(u,v) &= A_i(u,v) = \{w \mid \partial(u,w) = i, \partial(w,v) = 1\}. \end{aligned}$$

For the cardinalities of these sets we use lower case letters $c_i(u, v)$ and $a_i(u, v)$.

A connected regular graph Γ with diameter d is said to be *distance-regular* if $c_i(u, v)$ and $a_i(u, v)$ depend only on i for all $1 \leq i \leq d$. The reader is referred to [3] for general theory of distance-regular graphs.

Recall that a subgraph induced on Δ of Γ is said to be *strongly closed* if $C(u, v) \cup A(u, v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. Suzuki ([9]) determined all the types of strongly closed subgraphs of a distance-regular graph.

A distance-regular graph Γ with diameter d is said to be *d-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$. For instance, the ordinary 5-gon is a 2-bounded distance-regular graph. But the ordinary 6-gon is not a 3-bounded distance-regular graph. Indeed, let $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 1$ be the ordinary 6-gon. Then it is clear that $1 \sim 2 \sim 3$ is strongly closed, but it is not regular. By [8, Theorem 1.3], [11, Theorem 4.3] and [10], all the following graphs are d-bounded distance-regular graphs: When $a_1 = 0$ and $a_2 \neq 0$, all distance-regular graphs with classical parameters $(3, b, \alpha, \beta)$; Hamming graph $H(d, q)$ $(d \geq 3, q \geq 2)$ with classical parameters $(d, b, \alpha, \beta) = (d, 1, 0, q-1)$; when $c_2 \geq 1$ and $a_1 \neq 0$, Hermitian forms graph $Her_{-b}(d)$ $(d \geq 3)$ with geometric parameter $(d, b, \alpha) = (d, -r, -1-r)$, where r is a prime power; when $c_2 \geq 1$ and $a_1 \neq 0$, dual polar graph ${}^2A_{2d-1}(-b)$ $(d \geq 3)$ with geometric parameter $(d, b, \alpha) = (d, -r, r(1+r)/1-r)$, where r is a prime power; when $a_1 = 0$, the dual polar graph $D_d(b)$ $(d \geq 4)$ with classical parameters $(d, b, \alpha, \beta) = (d, b, 0, 1)$, where b is a prime power; when $c_2 > 1$ and $a_1 \neq 0$, all distance-regular graphs with geometric parameter $(d, b, \alpha) = (d, -r, -(1+r)/2)$, where r is an odd prime power.

Weng ([11, 12]) used the term *weak-geodetically closed subgraphs* for strongly closed subgraphs, obtained the basic properties and characterized when a distance-regular graph is d-bounded. A regular strongly closed subgraph of Γ is said to be a *subspace* of Γ. For any two subspaces Δ_1 and Δ_2 of Γ, the intersection of all subspaces that contain Δ_1 and Δ_2 is called the *join* of Δ_1 and Δ_2, and denoted by $\Delta_1 + \Delta_2$.

The results on the lattices generated by subspaces in d-bounded distance-regular graph with diameter d can be found in Gao, Guo and Liu ([4]), Guo and Gao ([6]), Guo, Gao and Wang ([7]).

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 3$. For $0 \leq i < i+s \leq d-1$. Suppose Δ_1 and Δ_0 are subspaces with diameter i and $i+s$, respectively, and with $\Delta_1 \subseteq \Delta_0$. Let $\mathcal{L}(i, i+s; d)$ denote the set of all subspaces Δ' with diameters $\geq i$ such that $d(\Delta_0 \cap \Delta') = \Delta_1$ and $d(\Delta_0 + \Delta') = d(\Delta') + s$ in Γ including Γ. If we partial order $\mathcal{L}(i, i+s; d)$ by ordinary inclusion (resp. reverse inclusion), then $\mathcal{L}(i, i+s; d)$ is a poset, denoted by $\mathcal{L}_O(i, i+s; d)$ (resp. $\mathcal{L}_R(i, i+s; d)$). In this paper we show that both $\mathcal{L}_O(i, i+s; d)$ and $\mathcal{L}_R(i, i+s; d)$ are atomic lattices, and classify their geometricity. Our main results are the following.

Theorem 1.1. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. For $0 \leq i < i+s \leq d-1$, the following hold:*

(i) $\mathcal{L}_R(i, i+s; d)$ *is a finite atomic lattice.*

(ii) $\mathcal{L}_R(i, d-1; d)$ *is a finite geometric lattice.*

(iii) *For* $i+s \leq d-2$, $\mathcal{L}_R(i, i+s; d)$ *is a finite geometric lattice if and only if for any two elements* Δ' *and* Δ'',

$$d(\Delta') + d(\Delta'') - d(\Delta' \cap \Delta'') \begin{cases} = d(\Delta' + \Delta''), & \text{if } \Delta' + \Delta'' \in \mathcal{L}_R(i, i+s; d)\backslash\{\Gamma\}, \\ \leq d-s+1, & \text{otherwise}. \end{cases}$$

Theorem 1.2. *Let* Γ *be a d-bounded distance-regular graph with diameter* $d \geq 3$. *For* $0 \leq i < i+s \leq d-1$, *the following hold:*

(i) $\mathcal{L}_O(i, i+s; d)$ *is a finite atomic lattice.*

(ii) $\mathcal{L}_O(i, d-1; d)$ *is a finite geometric lattice.*

(iii) *For* $i+s \leq d-2$, $\mathcal{L}_O(i, i+s; d)$ *is a finite geometric lattice if and only if for any two elements* Δ' *and* Δ'',

$$d(\Delta') + d(\Delta'') - d(\Delta' \cap \Delta'') \geq \begin{cases} d(\Delta' + \Delta''), & \text{if } \Delta' + \Delta'' \in \mathcal{L}_O(i, i+s; d)\backslash\{\Gamma\}, \\ d-s+1, & \text{otherwise}. \end{cases}$$

2 Proofs of main results

In this section we discuss lattices in d-bounded distance-regular graphs. We begin with some useful Propositions.

Proposition 2.1. *([11, Lemma 2.6]) Let* Γ *be a d-bounded distance-regular graph with diameter d. Then we have* $b_i > b_{i+1}$ *where* $0 \leq i \leq d-1$.

Proposition 2.2. *([12, Lemmas 4.2, 4.5]) Let* Γ *be a d-bounded distance-regular graph with diameter d. Then the following hold:*

(i) *Let* Δ *be a subspace of* Γ *and* $0 \leq i \leq d(\Delta)$. *Then* Δ *is distance-regular with intersection numbers* $c_i(\Delta) = c_i$, $a_i(\Delta) = a_i$, $b_i(\Delta) = b_i - b_{d(\Delta)}$.

(ii) *For any vertices* x *and* y, *the subspace with diameter* $\partial(x, y)$ *containing* x, y *is unique.*

Proposition 2.3. *([4, Lemma 2.1]) Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. For $1 \leq i+1 \leq i+s \leq i+s+t \leq d$, suppose Δ and Δ' are two subspaces satisfying $\Delta \subseteq \Delta'$, $d(\Delta) = i$ and $d(\Delta') = i+s+t$. Then the number of the subspaces with diameter $i+s$ containing Δ and contained in Δ' is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t})\cdots(b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s})\cdots(b_{i+s-1} - b_{i+s})}.$$

Proposition 2.4. *([4, Lemma 2.8]) Let Γ be a d-bounded distance-regular graph with diameter d. Suppose Δ and Δ' are two subspaces. If $d(\Delta\cap\Delta') \neq \emptyset$, then $d(\Delta) + d(\Delta') \geq d(\Delta \cap \Delta') + d(\Delta + \Delta')$.*

Proposition 2.5. *([5, Lemma 2.8]) Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. For $0 \leq i \leq i+s$, $i+t \leq i+s+t \leq d$, let Δ and Δ' be two subspaces in Γ with diameter $i+s$ and $i+t$, respectively, such that $d(\Delta\cap\Delta') = i$ If $d(\Delta)+d(\Delta') = d(\Delta\cap\Delta')+d(\Delta+\Delta')$, then the following hold:*

(i) *For fixed $x, y \in \Delta\cap\Delta'$ with $\partial(x, y) = i$, for all vertices $u \in \Delta$ with $\partial(u, x) = l$, $\partial(u, y) = i+l$, $0 \leq l \leq s$, and for all vertices $v \in \Delta'$ with $\partial(x, v) = i+m$, $\partial(y, v) = m$, $0 \leq m \leq t$, we have $\partial(u, v) = i+l+m$.*

(ii) *For all subspaces Δ_1 containing $\Delta\cap\Delta'$ in Δ, and for all subspaces Δ_2 containing $\Delta\cap\Delta'$ in Δ', we have $d(\Delta_1)+d(\Delta_2) = d(\Delta_1\cap\Delta_2)+ d(\Delta_1+\Delta_2)$.*

Proof of Theorem 1.1.

(i) For any two elements Δ', $\Delta'' \in \mathcal{L}_R(i, i+s;\, d)$,

$$\Delta' \vee \Delta'' = +\{\Delta \in \mathcal{L}_R(i, i+s;\, d) | \Delta \subseteq \Delta'\cap\Delta''\},$$

$$\Delta' \wedge \Delta'' = \cap\{\Delta \in \mathcal{L}_R(i, i+s;\, d) | \Delta \supseteq \Delta' + \Delta''\}.$$

Therefore $\mathcal{L}_R(i, i+s;\, d)$ is a finite lattice.

Note that Γ is the minimum element. Let $P(j;\, d)$ be the set of all subspaces Δ' with diameter j such that $d(\Delta_0\cap\Delta') = \Delta_1$ and $d(\Delta_0+\Delta') = d(\Delta')+s$ in Γ, where $i \leq j \leq d-s$. Then $P(d-s;\, d)$ is the set of all atoms in $\mathcal{L}_R(i, i+s;\, d)$. In order to prove $\mathcal{L}_R(i, i+s;\, d)$ is atomic, it suffices to show that every element of $P(j;\, d)$ $(i \le j \le d-s)$ is a union of some atoms. The result is trivial for $j = d-s$. Suppose that the result is true for $j = d-s-l$. For $\Delta \in P(d-s-(l+1);\, d)$. Fixed $x, y \in \Delta_1$ with $\partial(x,y) = i$, by Proposition 2.2, $\Delta_1 = \{x\} + \{y\}$. Fixed a vertex $u \in \Delta_0$ such that $\partial(u,x) = s$, $\partial(u,y) = s+i$ and fixed a vertex $v \in \Delta$ such that $\partial(x,v) = d-s-l-1$, $\partial(y,v) = d-s-l-1-i$, by Proposition 2.2, $\{u\}+\{y\} = \Delta_0$, $\{x\}+\{v\} = \Delta$. By Propositions 2.5, $\partial(u,v) = d-l-1$. Fixed a vertex $w \in \Gamma$ such that $\partial(v,w) = l+1$, $\partial(u,w) = d$, then $\Delta \subseteq \{x\}+\{w\}$ and $d(\{x\}+\{w\}) = d-s$. By Proposition 2.4, $d((\{x\}+\{w\}) \cap \Delta_0) \le i$. Thus $(\{x\}+\{w\}) \cap \Delta_0 = \Delta_1$. By Propositions 2.3 and 2.1, the number of the subspaces with diameter $d-s-l$ containing Δ in $\{x\}+\{w\}$ is

$$e = \frac{b_{d-s-l-1} - b_{d-s}}{b_{d-s-l-1} - b_{d-s-l}} \ge 2.$$

By Propositions 2.5, there exist two different subspaces $\Delta', \Delta'' \in P(d-s-l;\, d)$ containing Δ. Suppose $\widetilde{\Delta} = \Delta' \vee \Delta''$. Then $d(\widetilde{\Delta}) = d-s-l-1$ or $d-s-l$. If $d(\widetilde{\Delta}) = d-s-l$, by Proposition 2.2 $\Delta' = \widetilde{\Delta} = \Delta''$, a contradiction. It follows that $d(\widetilde{\Delta}) = d-s-l-1$ and $\Delta = \widetilde{\Delta} = \Delta' \vee \Delta''$ by Proposition 2.2 again. By induction Δ is a union of some atoms. Therefore, $\mathcal{L}_R(i,\, i+s;\, d)$ is a finite atomic lattice.

(ii) It is obvious that $\mathcal{L}_R(i,\, d-1;\, d)$ is a geometric lattice.

(iii) For any $\Delta \in \mathcal{L}_R(i,\, i+s;\, d)$, we define

$$r_R(\Delta) = \begin{cases} 0, & \text{if } \Delta = \Gamma, \\ d-s+1-d(\Delta), & \text{otherwise.} \end{cases}$$

It is routine to check that r_R is the rank function on $\mathcal{L}_R(i,\, i+s;\, d)$.

For $i+s \le d-2$. Suppose that $\mathcal{L}_R(i,\, i+s;\, d)$ is a finite geometric lattice. Then for any two subspaces Δ' and Δ'',

$$r_R(\Delta' \vee \Delta'') + r_R(\Delta' \wedge \Delta'') \le r_R(\Delta') + r_R(\Delta'').$$

If $\Delta' + \Delta'' \in \mathcal{L}_R(i,\, i+s;\, d) \backslash \{\Gamma\}$, then by Proposition 2.5, $\Delta' \vee \Delta'' = \Delta' \cap \Delta''$ and $\Delta' \wedge \Delta'' = \Delta' + \Delta''$. It follows that

$$\begin{aligned} & r_R(\Delta' \vee \Delta'') + r_R(\Delta' \wedge \Delta'') \\ = \;& d-s+1-d(\Delta' \cap \Delta'') + d-s+1-d(\Delta' + \Delta'') \\ \le \;& r_R(\Delta') + r_R(\Delta'') \\ = \;& d-s+1-d(\Delta') + d-s+1-d(\Delta''), \end{aligned}$$

that is $d(\Delta') + d(\Delta'') \leq d(\Delta' \cap \Delta'') + d(\Delta' + \Delta'')$. By Proposition 2.4, $d(\Delta') + d(\Delta'') = d(\Delta' \cap \Delta'') + d(\Delta' + \Delta'')$.

If $\Delta' + \Delta'' = \Gamma$ or $\Delta' + \Delta'' \notin \mathcal{L}_R(i, i+s; d)$, then by Proposition 2.5, $\Delta' \vee \Delta'' = \Delta' \cap \Delta''$ and $\Delta' \wedge \Delta'' = \Gamma$. It follows that

$$\begin{aligned} & r_R(\Delta' \vee \Delta'') + r_R(\Delta' \wedge \Delta'') \\ = \ & d - s + 1 - d(\Delta' \cap \Delta'') \\ \leq \ & r_R(\Delta') + r_R(\Delta'') \\ = \ & d - s + 1 - d(\Delta') + d - s + 1 - d(\Delta''), \end{aligned}$$

that is $d(\Delta') + d(\Delta'') \leq d(\Delta' \cap \Delta'') + d - s + 1$.

Conversely, for any two subspaces $\Delta', \Delta'' \in \mathcal{L}_R(i, i+s; d)$,

$$\Delta' \wedge \Delta'' = \begin{cases} \Delta' + \Delta'', & \text{if } \Delta' + \Delta'' \in \mathcal{L}_R(i, i+s; d)\backslash\{\Gamma\}, \\ \Gamma, & \text{otherwise.} \end{cases}$$

It is routine to check that $\mathcal{L}_R(i, i+s; d)$ is a finite geometric lattice. □

Proof of Theorem 1.2.

(i) Clearly, $\mathcal{L}_O(i, i+s; d)$ is a finite lattice.

Note that Δ_1 is the minimum element. Let $P(j; d)$ be the set of all subspaces Δ' with diameter j such that $d(\Delta_0 \cap \Delta') = \Delta_1$ and $d(\Delta_0 + \Delta') = d(\Delta') + s$ in Γ, where $i \leq j \leq d - s$. Then $P(i+1; d)$ is the set of all atoms in $\mathcal{L}_O(i, i+s; d)$. In order to prove $\mathcal{L}_O(i, i+s; d)$ is atomic, it suffices to show that every element of $P(j)$ $(i+1 \leq j \leq d-s)$ is a union of some atoms. The result is trivial for $j = i+1$. Suppose that the result is true for $j = i+l$. For $\Delta \in P(i + (l+1))$. By Propositions 2.1 and 2.3, the number of subspaces with diameter $i+l$ containing Δ_1 and contained in Δ is

$$\frac{(b_i - b_{i+l+1}) \cdots (b_{i+l-1} - b_{i+l+1})}{(b_i - b_{i+l}) \cdots (b_{i+l-1} - b_{i+l})} \geq 2.$$

By Proposition 2.5, there exist two different subspaces $\Delta', \Delta'' \in P(i+l; d) \cap \Delta$. Let $\widetilde{\Delta} = \Delta' \vee \Delta''$. Then $d(\widetilde{\Delta}) = i+l$ or $i+l+1$. If $d(\widetilde{\Delta}) = i+l$, by Proposition 2.2 $\Delta' = \widetilde{\Delta} = \Delta''$, a contradiction. It follows that $d(\widetilde{\Delta}) = i+l+1$ and $\Delta = \widetilde{\Delta} = \Delta' \vee \Delta''$ by Proposition 2.2 again. By induction Δ is a union of some atoms. Hence $\mathcal{L}_O(i, i+s; d)$ is a finite atomic lattice.

(ii) It is clear that $\mathcal{L}_O(i, d-1; d)$ is a geometric lattice.

(iii) For any $\Delta \in \mathcal{L}_O(i, i+s; d)$, define

$$r_O(\Delta) = \begin{cases} d - s - i + 1, & \text{if } \Delta = \Gamma, \\ d(\Delta) - i, & \text{otherwise.} \end{cases}$$

It is routine to check that r_O is the rank function on $\mathcal{L}_O(i, i+s; d)$.

For $i + s \le d - 2$. Suppose that $\mathcal{L}_O(i, i+s; d)$ is a finite geometric lattice. Then for any two subspaces Δ' and Δ'',

$$r_O(\Delta' \vee \Delta'') + r_O(\Delta' \wedge \Delta'') \le r_O(\Delta') + r_O(\Delta'').$$

If $\Delta' + \Delta'' \in \mathcal{L}_O(i, i+s; d)\backslash\{\Gamma\}$, then by Proposition 2.5, $\Delta' \wedge \Delta'' = \Delta' \cap \Delta''$ and $\Delta' \vee \Delta'' = \Delta' + \Delta''$. It follows that

$$\begin{aligned} r_O(\Delta' \vee \Delta'') + r_O(\Delta' \wedge \Delta'') &= d(\Delta' + \Delta'') - i + d(\Delta' \cap \Delta'') - i \\ &\le r_O(\Delta') + r_O(\Delta'') \\ &= d(\Delta') - i + d(\Delta'') - i, \end{aligned}$$

that is $d(\Delta') + d(\Delta'') \ge d(\Delta' \cap \Delta'') + d(\Delta' + \Delta'')$.

If $\Delta' + \Delta'' = \Gamma$ or $\Delta' + \Delta'' \notin \mathcal{L}_O(i, i+s; d)$, then by Proposition 2.5, $\Delta' \wedge \Delta'' = \Delta' \cap \Delta''$ and $\Delta' \vee \Delta'' = \Gamma$. It follows that

$$\begin{aligned} r_O(\Delta' \vee \Delta'') + r_O(\Delta' \wedge \Delta'') &= d - s - i + 1 + d(\Delta' \cap \Delta'') - i \\ &\le r_O(\Delta') + r_O(\Delta'') \\ &= d(\Delta') - i + d(\Delta'') - i, \end{aligned}$$

that is $d(\Delta') + d(\Delta'') \ge d(\Delta' \cap \Delta'') + d - s + 1$.

Conversely, for any two subspaces $\Delta', \Delta'' \in \mathcal{L}_O(i, i+s; d)$,

$$\Delta' \vee \Delta'' = \begin{cases} \Delta' + \Delta'', & \text{if } \Delta' + \Delta'' \in \mathcal{L}_O(i, i+s; d)\backslash\{\Gamma\}, \\ \Gamma, & \text{otherwise.} \end{cases}$$

It is routine to check that $\mathcal{L}_O(i, i+s; d)$ is a finite geometric lattice. □

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