On the total restrained domination edge critical graphs

K.M. Koh[a], Zeinab Maleki[b] and Behnaz Omoomi[b]

[a] Department of Mathematics
National University of Singapore
Singapore 117543, Singapore

[b] Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, 84156-83111, Iran

Abstract

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a *total restrained dominating set* of G if every vertex in V has a neighbor in D and every vertex in $V - D$ has a neighbor in $V - D$. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* of G. In this paper, we define the concept of *total restrained domination edge critical graphs*, find a lower bound for the total restrained domination number of graphs, and constructively characterize trees having their total restrained domination numbers achieving the lower bound.

Key Words: Domination; Total restrained domination number; Total restrained domination edge critical graphs; Matching; Edge cover; Trees.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n(G)$ and size $|E| = m(G)$. If there is no confusion, then we omit G in these notations and call G an (n, m)-*graph*. The *degree* of a vertex v in G is the number of vertices adjacent to v, and denoted by $deg_G(v)$. A vertex with no neighbor in G

is called an *isolated vertex*. A vertex of degree one in G is called an *end vertex*, the vertex adjacent to and the edge incident to an end vertex are called a *support vertex* and a *tail*, respectively. An edge is called a *strong edge* if it is not a tail. A path P in G is called an *end path* of G if P contains an end vertex of G and the degree of each vertex of P in G except end vertices is 2.

A set $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ has a neighbor in D. The cardinality of a minimum dominating set of G is the *domination number* of G and denoted by $\gamma(G)$ (see [5, 6]). If, in addition, the induced subgraph $\langle D \rangle$ has no isolated vertex, then D is called a *total dominating set* (TDS). The cardinality of a minimum total dominating set of G is called the *total domination number* and denoted by $\gamma_t(G)$. The total domination in graphs was introduced by Cockayne et al. in [1] (see also [3, 6, 9]).

Throughout this paper, we assume that G contains no isolated vertices. A set $D \subseteq V$ is a *total restrained dominating set* of G (TRDS) if D is a TDS of G and also the induced subgraph $\langle V - D \rangle$ has no isolated vertex. Note that the set V is a TRDS of G. The cardinality of a minimum total restrained dominating set of G is called the *total restrained domination number* of G and denoted by $\gamma_{tr}(G)$. We call a TRDS in graph G of cardinality $\gamma_{tr}(G)$ a $\gamma_{tr}(G)$ *- set*. The concept of total restrained domination was introduced by De-Xiang Ma et al. in [7].

A graph G is said to be *total restrained domination edge critical* if for every strong edge e in G, $\gamma_{tr}(G - e) > \gamma_{tr}(G)$. For simplicity, we call such G a γ_{tr}*-edge critical graph*. In this paper, we first characterize γ_{tr}-edge critical paths, cycles and caterpillars and find necessary and sufficient conditions for a graph to be γ_{tr}-edge critical. We then proceed to find a lower bound and an upper bound of $\gamma_{tr}(G)$ for γ_{tr}-edge critical graphs G, and hence derive a lower bound of $\gamma_{tr}(G)$ for all (n, m)-graphs G. Finally we characterize the trees which have their total restrained domination number achieving the lower bound. For unexplained terms and symbols, see [10].

2 Known results

In this section, we state some known results which are useful for proving our main theorems.

Proposition A. [2] *Let D be a TRDS of a graph G of order n, $n \geq 3$. Then every end vertex and every support vertex of G are in D.*

Proposition B. [7] *For every integer n, $n \geq 2$,*

(i) $\gamma_{tr}(K_n) = \begin{cases} 3 & \text{if } n = 3, \\ 2 & \text{if } n \neq 3; \end{cases}$

(ii) $\gamma_{tr}(K_{p,q}) = \begin{cases} p+q & \text{if } \min\{p,q\} = 1, \\ 2 & \text{if } \min\{p,q\} \neq 1; \end{cases}$

(iii) $\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor$;

(iv) $\gamma_{tr}(C_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor$.

A tree T is called a *caterpillar* if the resulting subgraph of T obtained by deleting all its end vertices is a path. We call this path the *spine* of the caterpillar. Let T be a caterpillar with spine $v_1 ... v_s$ and let $\{u_0 = v_1, u_1, ..., u_{k+1} = v_s\}$ be the ordered set of vertices in $\{v_1, ..., v_s\}$ with $deg_T(u_i) > 2$, for each i, $1 \leq i \leq k$. We denote the number of internal vertices in (u_i, u_{i+1})-path by z_i, $0 \leq i \leq k$, and one of the end vertices adjacent to u_i, $0 \leq i \leq k+1$, by a_i.

Proposition C. [2] *For every caterpillar T of order n, $n \geq 3$, $\gamma_{tr}(T) =$*

$$n - 2\sum_{i=1}^{k} \left\lfloor \frac{z_i + 2}{4} \right\rfloor.$$

Let G be a graph. A set $M \subseteq E$ is called a *matching* if no two edges in M are adjacent. The cardinality of a maximum matching in G is denoted by $\alpha'(G)$. A set $L \subseteq E$ is called an *edge cover* of G if every vertex of G is incident to some edge of L. The cardinality of a minimum edge cover is called the *edge cover number* of G and denoted by $\beta'(G)$. Obviously, the edge cover number of a graph is equal to the sum of the edge cover numbers of its components. The well known Gallai identity relating $\alpha'(G)$ and $\beta'(G)$ is stated below.

Theorem A. [10] *If G is a graph of order n without isolated vertices, then $\alpha'(G) + \beta'(G) = n$.*

3 γ_{tr}-edge critical graphs

In this section, we first characterize γ_{tr}-edge critical paths, cycles and caterpillars and provide necessary and sufficient conditions for a graph to be γ_{tr}-edge critical. We then proceed to derive a lower bound and an upper bound for the total restrained domination number of γ_{tr}-edge critical graphs.

It is obvious that every TRDS of a spanning subgraph H of graph G is also a TRDS of G. Thus we have:

Observation 1. *If H is a spanning subgraph of a graph G, then $\gamma_{tr}(H) \geq \gamma_{tr}(G)$.*

This observation implies that the $\gamma_{tr}(G)$ is nondecreasing if we delete an edge of G.

Definition. *A graph G is a γ_{tr}-**edge critical** graph if for every strong edge e of G, $\gamma_{tr}(G-e) > \gamma_{tr}(G)$.*

It is clear that every graph G contains a γ_{tr}-edge critical spanning subgraph H with $\gamma_{tr}(H) = \gamma_{tr}(G)$. This is seen by removing edges in succession, whenever possible, without diminishing the total restrained domination number.

Remark 1. *The difference $\gamma_{tr}(G-e) - \gamma_{tr}(G)$ can be arbitrary large. For example, in the graph of Figure 1, $\gamma_{tr}(G) = k+3$ while $\gamma_{tr}(G-e) = 2k+4$, for $k \geq 1$. Note that $D = A_1 \cup A_2$ is a $\gamma_{tr}(G)$-set and $D' = A_1 \cup A_2 \cup B_1 \cup B_2$ is a $\gamma_{tr}(G-e)$-set, where e is the dotted edge denoted in graph G.*

Suppose that G is a graph with components $G_1, G_2, ..., G_k$ and for each i, $1 \leq i \leq k$, D_i is a TRDS of G_i. Then the union of D_i is a TRDS of G. Thus, we have:

Observation 2. *If G is a graph with components $G_1, G_2, ..., G_k$, then*

$$\gamma_{tr}(G) = \sum_{i=1}^{k} \gamma_{tr}(G_i).$$

By Observation 2, the following observation is immediate.

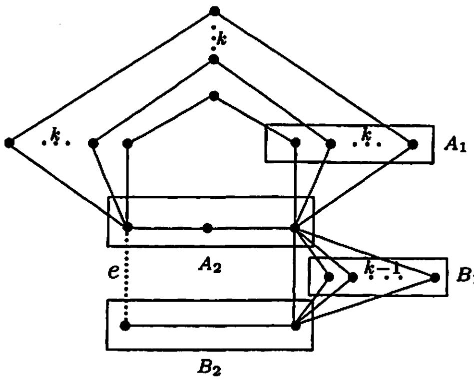

Figure 1: Graph G, where $\gamma_{tr}(G) = k+3$ and $\gamma_{tr}(G-e) = 2k+4$.

Observation 3. *A graph G is γ_{tr}-edge critical if and only if each component of G is γ_{tr}-edge critical.*

Theorem 1.

(i) *The path P_n, $n \geq 2$, is γ_{tr}-edge critical if and only if $n \equiv 2$ or $3 \pmod 4$.*

(ii) *The cycle C_n, $n \geq 3$, is γ_{tr}-edge critical if and only if $n \equiv 0$ or $1 \pmod 4$.*

(iii) *The caterpillar T is γ_{tr}-edge critical if and only if for each i, $0 \leq i \leq k$, $z_i \equiv 2$ or $3 \pmod 4$* (see page 2 for the definition of z_i).

Proof. (i) Consider the path P_n of order n and assume that $n \equiv 0$ *or* 1 (mod4). Let e be an edge adjacent to a tail. Then $P_n - e$ is a graph with two components P_2 and P_{n-2}. By Proposition B(iii) and Observation 2,

$$\begin{aligned}\gamma_{tr}(P_n - e) &= \gamma_{tr}(P_2) + \gamma_{tr}(P_{n-2}) \\ &= 2 + (n-2) - 2\left\lfloor \frac{(n-2)-2}{4} \right\rfloor \\ &= n - 2\left\lfloor \frac{n-4}{4} \right\rfloor.\end{aligned}$$

As $n \equiv 0$ *or* $1 \pmod 4$, we have $\lfloor \frac{n-4}{4} \rfloor = \lfloor \frac{n-2}{4} \rfloor$, and so

$$\gamma_{tr}(P_n - e) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor = \gamma_{tr}(P_n).$$

Thus, if $n \equiv 0$ *or* $1 \pmod 4$, then P_n is not γ_{tr}-edge critical.
Now suppose that $n \equiv 2$ *or* $3 \pmod 4$. Let e be a strong edge of P_n. Then $P_n - e$ is a graph with two components P_{n_1} and P_{n_2}, such that $n_1 + n_2 = n$. By Proposition B(iii) and Observation 2,

$$\begin{aligned}
\gamma_{tr}(P_n - e) &= \gamma_{tr}(P_{n_1}) + \gamma_{tr}(P_{n_2}) \\
&= n_1 - 2\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + n_2 - 2\left\lfloor \frac{n_2 - 2}{4} \right\rfloor \\
&= n - 2\left(\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor\right).
\end{aligned}$$

Assume that at least one of n_1 or n_2 is congruent to 0 or 1 modulo 4 (say, $n_1 \equiv 0$ *or* $1 \pmod 4$, and so $\lfloor \frac{n_1-2}{4} \rfloor = \lfloor \frac{n_1-4}{4} \rfloor$). Then

$$\begin{aligned}
\left\lfloor \frac{n_1 - 2}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor &= \left\lfloor \frac{n_1 - 4}{4} \right\rfloor + \left\lfloor \frac{n_2 - 2}{4} \right\rfloor \\
&\leq \frac{n_1 - 4}{4} + \frac{n_2 - 2}{4} \\
&= \frac{n - 2 - 4}{4} = \frac{n - 2}{4} - 1 \\
&< \left\lfloor \frac{n - 2}{4} \right\rfloor,
\end{aligned}$$

and so $n - 2(\lfloor \frac{n_1-2}{4} \rfloor + \lfloor \frac{n_2-2}{4} \rfloor) > n - 2\lfloor \frac{n-2}{4} \rfloor$; i.e., $\gamma_{tr}(P_n - e) > \gamma_{tr}(P_n)$.
Assume now that n_1 and n_2 are congruent to 3 modulo 4. In this case, $\lfloor \frac{n_1-2}{4} \rfloor + \lfloor \frac{n_2-2}{4} \rfloor = \lfloor \frac{n-2}{4} \rfloor - 1$, and it can be easily observed that $\lfloor \frac{n_1-2}{4} \rfloor + \lfloor \frac{n_2-2}{4} \rfloor < \lfloor \frac{n-2}{4} \rfloor$; i.e., $\gamma_{tr}(P_n - e) > \gamma_{tr}(P_n)$.

(ii) As $C_n - e$ is P_n for any edge e in C_n, by Proposition B(iv), C_n is γ_{tr}-edge critical if and only if

$$n - 2\left\lfloor \frac{n - 2}{4} \right\rfloor = \gamma_{tr}(P_n) > \gamma_{tr}(C_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor.$$

The inequality above holds if and only if $\lfloor \frac{n-2}{4} \rfloor < \lfloor \frac{n}{4} \rfloor$, i.e., $n \equiv 0$ *or* $1 \pmod 4$.

(iii) By Proposition A, it can be seen that the caterpillar T is a γ_{tr}-edge critical graph if and only if the (a_i, a_{i+1})-paths are γ_{tr}-edge critical, for each i, $0 \leq i \leq k$. By the first part above, the latter holds if and only if $z_i + 4 \equiv 2$ *or* $3\,(\text{mod}4)$. Thus, T is γ_{tr}-edge critical if and only if for each i, $0 \leq i \leq k$, $z_i \equiv 2$ *or* $3\,(\text{mod}4)$.

■

Theorem 2. *Let G be a graph. Then G is γ_{tr}-edge critical if and only if every $\gamma_{tr}(G)$-set D satisfies each of the following conditions:*
(1) Every component of $\langle D \rangle$ and $\langle V - D \rangle$ is a star.
(2) Every vertex in $V - D$ has exactly one neighbor in D.

Note. *Condition (2) implies that the number of edges between D and $V - D$ is equal to $n - \gamma_{tr}(G)$.*

Proof. Suppose that G is a γ_{tr}-edge critical graph and D is a $\gamma_{tr}(G)$-set. (1) If $\langle D \rangle$ or $\langle V - D \rangle$ has a strong edge, then D is a TRDS for the graph obtained from G by deleting the strong edge. This contradicts the fact that G is γ_{tr}-edge critical. Thus, every component of $\langle D \rangle$ and $\langle V - D \rangle$ is a star. (2) Every vertex in $V - D$ is dominated by some vertex in D. If a vertex v in $V - D$ has more than one neighbor in D, say u_1 and u_2, then D is a TRDS of the graph $G - u_1v$, a contradiction. Thus, condition (2) holds.

We now prove the sufficiency by contradiction. Assume that every $\gamma_{tr}(G)$-set satisfies the two conditions, but G is not γ_{tr}-edge critical. Let H be a γ_{tr}-edge critical proper spanning subgraph of G such that $\gamma_{tr}(H) = \gamma_{tr}(G)$. Suppose that D is a $\gamma_{tr}(H)$-set. By the above necessity conditions, D satisfies conditions (1) and (2) in H. Observe that D is also a $\gamma_{tr}(G)$-set, but now D no longer satisfies the conditions in G, as G contains at least one edge not in H. This contradiction shows that G is γ_{tr}-edge critical. ■

Corollary 1. *Let G be an (n, m)-graph. If G is γ_{tr}-edge critical, then*

$$\frac{3n}{2} - m \leq \gamma_{tr}(G) \leq 2n - m - 2.$$

Proof. Let D be a $\gamma_{tr}(G)$-set. By Theorem 2, the number of edges with one end in D and another one in $V - D$ is equal to $n - \gamma_{tr}(G)$. As $\langle D \rangle$ and $\langle V - D \rangle$ are forests, the number of edges in $\langle D \rangle$ and $\langle V - D \rangle$ does not exceed $|D| - 1$ and $|V - D| - 1$, respectively. Thus,

$$\begin{aligned} m &\leq (|D| - 1) + (|V - D| - 1) + (n - \gamma_{tr}(G)) \\ &= (\gamma_{tr}(G) - 1) + (n - \gamma_{tr}(G) - 1) + (n - \gamma_{tr}(G)) \\ &= 2n - \gamma_{tr}(G) - 2, \end{aligned}$$

and so

$$\gamma_{tr}(G) \leq 2n - m - 2.$$

On the other hand, as the degree of every vertex in $\langle D \rangle$ and $\langle V - D \rangle$ is at least one, we have

$$\begin{aligned} m &\geq \frac{|D|}{2} + \frac{|V - D|}{2} + n - \gamma_{tr}(G) \\ &= \frac{\gamma_{tr}(G)}{2} + \frac{n - \gamma_{tr}(G)}{2} + n - \gamma_{tr}(G) \\ &= \frac{3n}{2} - \gamma_{tr}(G), \end{aligned}$$

i.e.,

$$\frac{3n}{2} - m \leq \gamma_{tr}(G).$$

■

4 Total restrained domination number of graphs

In this section, we find some bounds for the total restrained domination number of graphs.

Lemma 1. *Let D be a $\gamma_{tr}(G)$-set of a γ_{tr}-edge critical graph G. If k and k' are the numbers of components in $\langle D \rangle$ and $\langle V - D \rangle$, respectively, then*

$$\gamma(G) \leq k + k' \leq \alpha'(G).$$

Proof. By Theorem 2, every component of $\langle D \rangle$ and $\langle V - D \rangle$ is a star. Let A be the set of the centers of these stars. Then A is a dominating set of G and $|A| = k + k'$. Hence

$$\gamma(G) \leq |A| = k + k'.$$

Form a set $B \subseteq E$ by selecting an edge from each component of $\langle D \rangle$ and $\langle V - D \rangle$. Then B is a matching of G, and so by above inequality

$$\gamma(G) \leq k + k' = |B| \leq \alpha'(G).$$

■

Remark 2. *Suppose that G is a graph and D is a subset of V such that each component of $\langle D\rangle$ and $\langle V-D\rangle$ is a star. Denote the set of edges between D and $V-D$ by $F_D(G)$ and let $f_D(G)=|F_D(G)|$. Now we construct a bipartite multigraph G_D^* with partite sets X and Y from G with respect to D as follows. Every vertex in X corresponds to a component of $\langle D\rangle$ and every vertex in Y corresponds to a component of $\langle V-D\rangle$. Let k and k' be the numbers of components in $\langle D\rangle$ and $\langle V-D\rangle$, respectively; so $|X|=k$ and $|Y|=k'$. Corresponding to every edge in G joining a component of $\langle D\rangle$ and a component of $\langle V-D\rangle$, there is an edge in G_D^* joining the two vertices corresponding to the components (note that G_D^* may contain multiple edges). Then G_D^* is an (n^*,m^*)-multigraph, where $n^*=n(G_D^*)=k+k'$ and $m^*=m(G_D^*)=f_D(G)$.*

Referring to the notations in Remark 2, we have:

Lemma 2.

$$m(G_D^*)=n(G_D^*)-(n(G)-m(G)).$$

Proof. We prove the equality by induction on $f_D(G)$. Assume $f_D(G)=0$. Then G is a forest with $k+k'$ components, and so $m(G)=n(G)-(k+k')$. Hence $n(G)-m(G)=k+k'=n(G_D^*)-m(G_D^*)$.
Assume that $f_D(G)>0$ and the equality holds for every graph H with $f_D(H)<f_D(G)$. Suppose that H is a graph obtained from G by deleting an edge of $F_D(G)$. Then $f_D(H)=f_D(G)-1<f_D(G)$, and by the induction hypothesis, $m(H_D^*)=n(H_D^*)-(n(H)-m(H))$. Since $m(H_D^*)=m(G_D^*)-1$, $n(H_D^*)=n(G_D^*)$, $m(H)=m(G)-1$ and $n(H)=n(G)$, we have $m(G_D^*)=n(G_D^*)-(n(G)-m(G))$, as desired. ■

Theorem 3. *For every γ_{tr}-edge critical (n,m)-graph G,*

$$\beta'(G)+n-m\leq\gamma_{tr}(G)\leq 2n-m-\gamma(G).$$

Proof. Let D be a $\gamma_{tr}(G)$-set and G_D^* be the corresponding (n^*,m^*)-multigraph constructed from G as described in Remark 2. By Theorem 2, $m^*=f_D(G)=n-\gamma_{tr}(G)$, and by Lemma 2, $n^*-(n-m)=m^*$. Hence

$$k+k'-(n-m)=n^*-(n-m)=m^*=n-\gamma_{tr}(G),$$

and so

$$\gamma_{tr}(G)=n-(k+k')+(n-m).$$

This equality and the inequalities in Lemma 1 imply that

$$n-\alpha'(G)+(n-m)\leq\gamma_{tr}(G)\leq n-\gamma(G)+(n-m).$$

Now, by Theorem A, we have

$$\beta'(G) + n - m \le \gamma_{tr}(G) \le 2n - m - \gamma(G).$$

■

Remark 3. *The above bounds are sharp, as stars are γ_{tr}-edge critical graphs and their γ_{tr} achieve both lower and upper bounds above.*

Corollary 2. *If G is an (n,m)-graph, then $\gamma_{tr}(G) \ge \beta'(G) + n - m$.*

Proof. Suppose that H is a γ_{tr}-edge critical spanning subgraph of G such that $\gamma_{tr}(H) = \gamma_{tr}(G)$. Since H is a spanning subgraph of G, each edge cover of H is an edge cover of G, so $\beta'(G) \le \beta'(H)$. Hence by Theorem 3,

$$\beta'(G) + n - m \le \beta'(H) + n(H) - m(H) \le \gamma_{tr}(H) = \gamma_{tr}(G).$$

■

Remark 4. *In [2] it is proved that if G is an (n,m)-graph, then*

$$\gamma_{tr}(G) \ge \frac{3n}{2} - m;$$

and in [4] it is proved that if T is a tree of order n, then

$$\gamma_{tr}(T) \ge \left\lfloor \frac{n+2}{2} \right\rfloor.$$

Since for every graph G of order n, $\frac{n}{2} \le \lfloor \frac{n+1}{2} \rfloor \le \beta'(G)$, the lower bound obtained in Corollary 2 is sharper than the above two.

Theorem 4. *If G is an (n,m)-graph such that $\gamma_{tr}(G) = \beta'(G) + n - m$, then G is γ_{tr}-edge critical.*

Proof. We prove the statement by contradiction. Suppose that G is not γ_{tr}-edge critical. Then there is an edge, say e, such that $\gamma_{tr}(G-e) = \gamma_{tr}(G)$. By Corollary 2 and the hypothesis,

$$\begin{aligned} \gamma_{tr}(G) &= \beta'(G) + n - m \le \beta'(G-e) + n - m \\ &= \beta'(G-e) + n(G-e) - (m(G-e) + 1) \\ &\le \gamma_{tr}(G-e) - 1 = \gamma_{tr}(G) - 1, \end{aligned}$$

a contradiction. ■

Remark 5. *For every integer $k > 0$ there exists a graph G such that $\gamma_{tr}(G) - \beta'(G) = k + 1$. For instance, in the graph G of Figure 2, the set $D = \bigcup_{i=1}^{2k+1} A_i$ is a $\gamma_{tr}(G)$-set with $|D| = 5k + 2$ and the set of bold edges is an edge cover of size $4k + 1$. Moreover, note that graph G is γ_{tr}-edge critical. So this example shows that the converse of Theorem 4 is not true.*

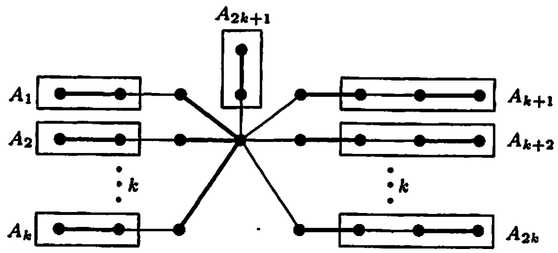

Figure 2: Graph G, where $\gamma_{tr}(G) - \beta'(G) = k + 1$.

5 Characterization of trees with minimum γ_{tr}

It follows from Corollary 2 that if T is a tree, then $\gamma_{tr}(T) \geq \beta'(T) + 1$. In this final section, we characterize all trees T such that $\gamma_{tr}(T) = \beta'(T) + 1$. We first present some useful lemmas.

Lemma 3. *Suppose that T and T' are two trees such that for some integer k, $\gamma_{tr}(T') \leq \gamma_{tr}(T) + k$ and $\beta'(T) \leq \beta'(T') - k$. If $\gamma_{tr}(T) = \beta'(T) + 1$, then $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) + k$.*

Proof. By Corollary 2 and the hypothesis, we have

$$\begin{aligned}\beta'(T') + 1 &\leq \gamma_{tr}(T') \leq \gamma_{tr}(T) + k \\ &= (\beta'(T) + 1) + k = (\beta'(T) + k) + 1 \leq \beta'(T') + 1.\end{aligned}$$

Hence $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) + k$. ∎

Lemma 4. *Suppose that T and T' are two trees such that for some integer k, $\gamma_{tr}(T') \leq \gamma_{tr}(T) - k$ and $\beta'(T) \leq \beta'(T') + k$. If $\gamma_{tr}(T) = \beta'(T) + 1$, then $\gamma_{tr}(T') = \beta'(T') + 1$ and $\gamma_{tr}(T') = \gamma_{tr}(T) - k$.*

Proof. By Corollary 2 and the hypothesis, we have

$$\begin{aligned}\beta'(T')+1 &\le \gamma_{tr}(T') \le \gamma_{tr}(T)-k \\ &= (\beta'(T)+1)-k = (\beta'(T)-k)+1 \le \beta'(T')+1.\end{aligned}$$

Hence $\gamma_{tr}(T') = \beta'(T')+1$ and $\gamma_{tr}(T') = \gamma_{tr}(T)-k$. ■

Lemma 5. *Let T be a tree with $\gamma_{tr}(T) = \beta'(T)+1$ and P be an end path with k vertices in T. If D is a $\gamma_{tr}(T)$-set such that $D' = D - V(P)$ is a TRDS for $T' = T - V(P)$, then at most $\lfloor \frac{k+1}{2} \rfloor$ vertices of P belong to D.*

Proof. Suppose that this is not true; i.e., D contains at least $\lfloor \frac{k+1}{2} \rfloor + 1$ vertices of P. By Corollary 2,

$$\beta'(T')+1 \le \gamma_{tr}(T').$$

Since $D' = D - V(P)$ is a TRDS of T',

$$\gamma_{tr}(T') \le |D'| \le |D| - (\left\lfloor \frac{k+1}{2} \right\rfloor + 1) = \gamma_{tr}(T) - \left\lfloor \frac{k+1}{2} \right\rfloor - 1.$$

The union of an edge cover of P and an edge cover of T' is an edge cover of T and $\beta'(P) = \lfloor \frac{k+1}{2} \rfloor$. Thus

$$\beta'(T) \le \beta'(T') + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Now we have

$$\begin{aligned}\beta'(T')+1 &\le \gamma_{tr}(T') \le \gamma_{tr}(T) - \left\lfloor \frac{k+1}{2} \right\rfloor - 1 \\ &= \beta'(T)+1-\left\lfloor \frac{k+1}{2} \right\rfloor - 1 = \beta'(T) - \left\lfloor \frac{k+1}{2} \right\rfloor \\ &\le (\beta'(T') + \lfloor \tfrac{k+1}{2} \rfloor) - \lfloor \tfrac{k+1}{2} \rfloor = \beta'(T'),\end{aligned}$$

a contradiction. This shows that D contains at most $\lfloor \frac{k+1}{2} \rfloor$ vertices of P. ■

Now we construct a family Φ of trees recursively as follows:

(i) Let P_2 be in Φ.
(ii) Let $T \in \Phi$ and D be a $\gamma_{tr}(T)$-set. Then $T' \in \Phi$ if T' is a tree constructed from T by performing one of the following operations.

O_1. Add a new vertex t to T and join t to a support vertex in T. Let $D' := D \cup \{t\}$.

O_2. Add a new path $abcd$ to T and join vertex a to a vertex s in D. Let $D' := D \cup \{c, d\}$.

O_3. Let $abcd$ be an end path in T such that $a \notin D$ and $b, c, d \in D$. Add a new path tx to T, and join t to vertex a. Let $D' := (D - \{b\}) \cup \{t, x\}$.

O_4. Let $abcd$ be an end path in T such that $a \notin D$. Add a new path txy to T and join t to a. Let $D' := D \cup \{x, y\}$.

In the following lemma, we show that D' is a $\gamma_{tr}(T')$-set and hence Φ can be constructed recursively.

Lemma 6. *Let T be a tree such that $\gamma_{tr}(T) = \beta'(T)+1$ and T' constructed from T by one of the operations above. Then $\gamma_{tr}(T') = \beta'(T') + 1$ and D' is a $\gamma_{tr}(T')$-set.*

Proof. We first show that if we perform each of the operations above, then T and T' satisfy the hypothesis of Lemma 3 for some k. Hence we can conclude that $\gamma_{tr}(T') = \beta'(T') + 1$. To see this, let M' be an edge cover of T'.

Operation O_1. By Proposition A, we have every support vertex is in D, so it is obvious that D' is a TRDS of T'. Thus $\gamma_{tr}(T') \le |D'| = \gamma_{tr}(T) + 1$. Suppose that M is obtained from M' by deleting the edge incident to t (note that each edge incident to an end vertex belongs to M'). The set M is an edge cover for T; so $\beta'(T) \le \beta'(T') - 1$. In this case, $k = 1$ and we are done.

Operation O_2. Similarly, for this operation, we have $\gamma_{tr}(T') \le |D'| = \gamma_{tr}(T) + 2$. Suppose that M is obtained from M' by deleting the edges incident to the vertices b and d. Since b and d are not adjacent, there are at least two such edges. Moreover if edge as belongs to M, then we substitute as with an edge of T incident to s to get an edge cover for T. So $\beta'(T) \le \beta'(T') - 2$. Hence, in this case, $k = 2$ and we are done.

Operation O_3. For this operation, we have $k = 1$, and the argument is similar to the above.

Operation O_4. Similarly, $\gamma_{tr}(T') \le \gamma_{tr}(T) + 2$. If $at, ab \in M'$, then we can substitute at with tx and get a new edge cover of T'. Hence by symmetry of edges ab and at, without loss of generality we may assume $at \notin M'$. Thus $tx \in M'$, also we know that $xy \in M'$, and so $M' - \{tx, xy\}$ is an edge cover for T of size $\beta'(T')-2$. Hence $\beta'(T) \leq \beta'(T')-2$ and we have $k=2$, and the desired result can be obtained.

For the second part of the lemma, it is seen that in each case D' is a TRDS of T'. Moreover, in each case for chosen k, we have $|D'| = \gamma_{tr}(T)+k$. On the other hand, by Lemma 3, $\gamma_{tr}(T') = \gamma_{tr}(T)+k$. Thus $|D'| = \gamma_{tr}(T')$ and so D' is a $\gamma_{tr}(T')$-set. ∎

Theorem 5. *The set Φ is the set of all trees T with $\gamma_{tr}(T) = \beta'(T)+1$.*

Proof. Obviously $\gamma_{tr}(P_2) = 2 = \beta'(P_2)+1$. Thus by Lemma 6 and using the induction on the number of the operations, for every tree T in Φ, we have $\gamma_{tr}(T) = \beta'(T)+1$.

We now show that every tree T of order n with $\gamma_{tr}(T) = \beta'(T)+1$ is contained in Φ. Our proof is by induction on n. For $n=2$, we have $T=P_2$, and $P_2 \in \Phi$. Suppose that $n \geq 3$ and the statement is true for all trees of order less than n. Our strategy is to find some proper subtree of T, say T', that satisfies the hypothesis of Lemma 4. Hence $\gamma_{tr}(T') = \beta'(T')+1$ and by the induction hypothesis, T' belongs to Φ. Moreover, we find T' such that T can be constructed from T' by performing one of the operations $O_1, \ldots, O_4$, and conclude that $T \in \Phi$.

Thus, let T be a tree of order $n \geq 3$ with $\gamma_{tr}(T) = \beta'(T)+1$. Note that, by Theorem 4, T is γ_{tr}-edge critical. Suppose that D is a $\gamma_{tr}(T)$-set, P is the longest path in T and c is a support vertex in P.

If $deg_T(c) > 2$, then c is adjacent to two end vertices, say t and d. By Proposition A, the vertices t, d and c are in D. Since $D' = D - \{t\}$ is a TRDS in $T' = T - \{t\}$, $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 1$. On the other hand, the union of an edge cover of $T' = T - \{t\}$ and edge ct is an edge cover of T, so $\beta'(T) \leq \beta'(T')+1$. On the other hand, the union of a $\gamma_{tr}(T')$-set and $\{t\}$ is a TRDS of T. Thus $\gamma_{tr}(T) \leq \gamma_{tr}(T')+1$, and so $\gamma_{tr}(T) = \gamma_{tr}(T')+1$. Therefore the tree T' is a desired subtree of T from which T can be constructed by O_1.

Assume now that $deg_T(c) = 2$. Then c is adjacent to an end vertex, say d and vertex, say b. If $deg_T(b) = 1$, then $T = P_3$ and $P_3 \in \Phi$. Assume that $deg_T(b) \geq 2$. Then we have the following two cases to consider.

Case 1. $deg_T(b) > 2$.

In this case, b has a neighbor not in P, say t. By our choice of P, it is obvious that the length (say l) of the longest path $bt\ldots$ beginning with bt is at most two. By Proposition A, the vertices c, d, t and the neighbors of t other than b (if there exist) are in D. Thus, for $l=1$ and $l=2$, $\gamma_{tr}(T - bc) = \gamma_{tr}(T)$, which contradicts that T is γ_{tr}-edge critical.

Case 2. $deg_T(b) = 2$.

In this case, let the neighbors of b be vertices a and c. If $deg_T(a) = 1$, then $T = P_4$, while $\gamma_{tr}(P_4) \neq \beta'(P_4) + 1$. Thus, we consider the following two subcases.

Case 2.1. $deg_T(a) > 2$.
Assume that t is a neighbor of a not in P. Let l be the length of longest path $at\ldots$ beginning with at. Then, by the choice of P, it is obvious that $l \leq 3$. The following three cases can happen.

Case 2.1.1. $l = 1$.
By Proposition A, the vertices a, c and d are in D, so b is also in D. This is a contradiction for, by Theorem 2, every component of $\langle D \rangle$ is a star.

Case 2.1.2. $l = 2$.
If x is an end vertex adjacent to t, then by Proposition A, vertices c, d, t and x are in D. If $b \in D$, then a has two neighbors in D, which, by Theorem 2, contradicts that T is a γ_{tr}-edge critical graph. Hence $b \in V - D$, and since it should not be an isolated vertex in $\langle V - D \rangle$, we have $a \notin D$. In this case, let $T' = T - \{t, x\}$. It can be seen that T can be constructed from T' by performing O_3. Moreover it can be easily checked that the union of an edge cover of T' and the edge tx is an edge cover of T; so $\beta'(T) \leq \beta'(T') + 1$. On the other hand, $(D \cup \{b\}) - \{x, t\}$ is a TRDS of T' (note that $deg_T(a) > 2$ and T is γ_{tr}-edge critical, hence by Theorem 2 all neighbors of a except t are in $V - D$); so $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 1$, and we are done in this case.

Case 2.1.3. $l = 3$
Let $atxy$ be a longest path beginning with at of length 3. Note that the path obtained by substituting the subpath $atxy$ with subpath $abcd$ in P is also a longest path in T. So by symmetry, we may assume that $deg_T(t) = 2$ and $deg_T(x) = 2$. By Proposition A, the vertices c, d, x and y are in D. If b and t both belong to D, then a has two neighbors in D which, by Theorem 2, contradicts that T is γ_{tr}-edge critical. Hence at least one of b and t is in $V - D$, say $t \in V - D$. Since there is no isolated vertex in $\langle V - D \rangle$ and $x \in D$, we have $a \notin D$. In this case, let $T' = T - \{t, x, y\}$. Then T can be constructed from T' by performing O_4. Moreover, it can be easily seen that the union of an edge cover of T' and the set $\{tx, xy\}$ is an edge cover of T; so $\beta'(T) \leq \beta'(T') + 2$. On the other hand, $D - \{x, y\}$ is a TRDS of T' and so $\gamma_{tr}(T') \leq \gamma_{tr}(T) - 2$, and we are done in this case.

Case 2.2. $deg_T(a) = 2$.
In this case, we denote the neighbors of a by b and s. By Proposition A, vertices c and d should be in D.
If $b \notin D$, then since $\langle V - D \rangle$ contains no isolated vertex, $a \notin D$ and $s \in D$ to dominate a. In this case, let $T' = T - \{a, b, c, d\}$. Then T can be constructed from T' by performing O_2. It can be easily shown that T and T' satisfy the conditions of Lemma 4 for $k = 2$.

If $b \in D$, then $a \in V - D$, because, by Theorem 2, every component of $\langle D \rangle$ is a star. Since there is no isolated vertex in $\langle V - D \rangle$, $a \in V - D$ implies that $s \notin D$. If $deg_T(s) > 2$, let $T' = T - \{a, b, c, d\}$, then the set $D - \{b, c, d\}$ is a $\gamma_{tr}(T')$-set (note that, by Theorem 2, all neighbors of s except one are in $V - D$), while D contains three vertices of the end path $abcd$ in T. This contradicts Lemma 5. Thus $deg_T(s) \not> 2$. However, in the case that $deg_T(s) = 1$, we have $T = P_5$, while $\gamma_{tr}(P_5) \neq \beta'(P_5) + 1$. Hence $deg_T(s) = 2$. Furthermore since $a \notin D$, the only other neighbor of s is in D. So $(D - \{b\}) \cup \{s\}$ is also a $\gamma_{tr}(T)$-set which does not contain b. We are done so long as $b \notin D$. ■

References

[1] *E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi. Total domination in graphs.* Networks **10**, (1980), 211-219.

[2] *J. Cyman and J. Raczek. On the total restrained domination number of a graph.* Astrulasian Journal of combinatorics **36**, (2006), 91-100.

[3] *S. Gravier. Total dominatin number of grid graphs.* Discrete Applied Mathematics **121**, (2002), 119-128.

[4] *J.H. Hattingh, E. Jonck, E.J. Joubert and A.R. Plummer. Total restrained domination in trees.* Discrete Mathematics **307**, (2007), 1643-1650.

[5] *T.W. Haynes, S.T. Hedetniemi and P.J. Slater. Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1997.

[6] *T.W. Haynes, S.T. Hedetniemi and P.J. Slater. Domination in Graphs.* Marcel Dekker Inc., New York, 1997.

[7] *D. Ma, X. Chen and L. Sun. On total restrained domination in graphs.* Czechoslovak Mathematical Journal **55**, (2005), 165-173.

[8] *J. Raczek and J. Cyman. Total restrained domination numbers of trees.* Discrete Mathematics **308**, (2008), 44-50.

[9] *E. Shan, L. Kang and M.A. Henning. A characterization of trees with equal total domination and paired-domination numbers.* Astrulasian Journal of combinatorics **30**, (2004), 31-39.

[10] *D.B. West. Introduction to Graph Theory.* 2nd Eddition, Prentice Hall, Upper Saddle River, NJ, 2001.