Prime labeling on Knödel graphs $W_{3,n}$ *

Kh. Md. Mominul Haque[1,2], Lin Xiaohui[1],
Yang Yuansheng[1]†, Zhao Pingzhong[1]
1. Department of Computer Science and Engineering
Dalian University of Technology
Dalian, 116024, P. R. China
2. Department of Computer Science and Engineering
Shahjalal University of Science and Technology
Sylhet-3114 , Bangladesh

Abstract

A graph G with vertex set V is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge xy in E, the labels assigned to x and y are relatively prime or coprime. In this paper, we show that the Knödel graph $W_{3,n}$ is prime for $n \leq 130$.
Keywords: *Prime labeling; Prime graph; Knödel graph*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A graph G with vertex set V is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge xy in E, the labels assigned to x and y are relatively prime or coprime. A graph is called prime if it has a prime labeling. This concept was originated by Roger Entringer and introduced in [21] by Tout, Dabboucy and Howalla. Roger Entringer conjectured that all trees are prime. Hung-Lin Fu and

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†corresponding author's email : yangys@dlut.edu.cn

Kun-Ching Huang [7] proved that every tree with $n \leq 15$ vertices is prime, Pikhurko [15, 16] extended this result up to $n \leq 50$. Salmasian [17] has shown that the set of vertices of every n-vertex tree ($n \geq 50$) can be labeled by n elements chosen from 1,2,...,$4n$ such that every two adjacent vertices have coprime labels. Seoud, Diab, and Elsakhawi [18] have shown that following graphs are prime: Fans, Helms, Flowers, Stars, $K_{2,n}$ and $K_{3,n}$ unless $n = 3$ or 7. They have also shown that $P_n + \overline{K_m}$ ($m \geq 3$) is not prime. Carlson [1] proved that generalized Books and C_m-Snakes are prime graphs. Vilfred, Somasundaram, and Nicholas [22] have conjectured that the grid $P_m \times P_n$ is prime when n is prime and $n > m$. This conjecture was proved by Sundaram, Ponraj, and Somasundaram [20]. In the same article they also showed that $P_n \times P_n$ is prime when n is prime. Seoud and Yussef [19] have proved that following graphs are prime: $S_n^{(m)}$; $C_n \bigodot P_m$; $P_n + \overline{K_2}$ if and only if $n = 2$ or n is odd. They also proved that following graphs are not prime: $C_m + C_n$; $C_n^2 (n \geq 2)$; P_n^2 for $n = 6$ and $n \geq 8$; and *Möbius* ladder M_n for even n. The authors [14] proved that generalized Petersen graph $P(n, 1)$ is prime for even $n \leq 2500$. We refer the readers to the dynamic survey by Gallian [8].

The *Knödel graphs* $W_{\Delta,n}$, are regular graphs of even order n and degree $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. They were introduced as the topology underlying a time optimal algorithm for gossiping among n nodes in 1975 by *Knödel* [13], and were formally defined in [6]. Since then, they have been widely studied as interconnection networks, mainly because of their good properties in terms of broadcasting and gossiping. In particular, *Knödel graphs* of order 2^k and $2^k - 2$ of degree k and $k - 1$ respectively are popular interconnection networks [5, 12, 13]. These three families are commonly presented as good topologies for multi-computer networks. Some combinatorial properties of *Knödel graphs* such as dimensionality are presented in [2, 9]. [10] gives a logarithmic algorithm to find a minimum path in *Knödel graphs*, while [11] presents an upper bound on the dominating number of *Knödel graphs*. [3] presents a polynomial algorithm to recognize *Knödel graphs*.

The *Knödel graph*, on $n \geq 2$ vertices (n even) and of maximum degree $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ is denoted by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the cupels (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$. For every j, $0 \leq j \leq n/2 - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, j + 2^k - 1 \bmod n/2)$, for $k = 0, 1, \ldots, \Delta - 1$ [6].

For $W_{\Delta,n}$, let v_{2j} represent vertex (1,j) and v_{2j-1} represent vertex (2,j). In this paper, the vertex labels are read modulo n unless specified otherwise. From the definition of the Knödel graph, for $\Delta = 3$ and even $n \geq 8$, we have

$$
\begin{aligned}
V(W_{3,n}) &= \{v_i : 0 \le i \le n-1\}, \\
E(W_{3,n}) &= \{v_i v_{i+1} : 0 \le i \le n-1\} \cup \{v_{2i} v_{2i+5} : 0 \le i \le n/2-1\}.
\end{aligned}
$$

Figure 1.1 shows Knödel graph $W_{3,12}$.

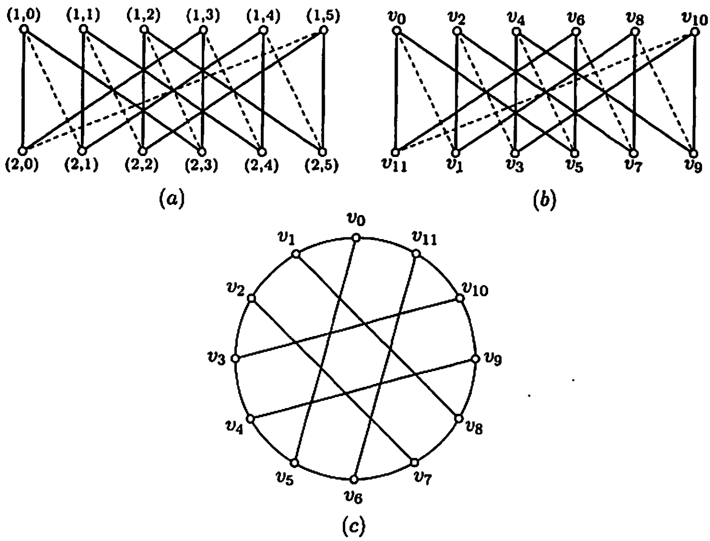


Figure 1.1. The Knödel graph $W_{3,12}$

Now, we consider the prime labeling of $W_{3,n}$.

2 Prime labeling of $W_{3,n}$

Let

$$\mathcal{N}_i = \{n : n+i \text{ is prime}\}, \qquad \mathcal{N} = \bigcup_{-6 \le i \le 5} (\mathcal{N}_{2i+1} \cap \mathcal{N}_{2i+5}).$$

We will prove the following Theorem by Lemmas 2.3 - 2.14.

Theorem 2.1. $W_{3,n}$ is prime for even $n \in \mathcal{N}$.

Observation 2.2. $f(u)$ and $f(v)$ are coprime if they satisfy any one of the following conditions:

(1) $f(u) = 1$ or $f(v) = 1$,

(2) $f(u) + f(v)$ is prime,

(3) $|f(u) - f(v)| = 1$,
(4) $|f(u) - 2f(v)| = 1$,
(5) $|f(u) - 2f(v)| = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ and $f(v) \not\equiv 0 \bmod p_i$ $(1 \le i \le k)$,
(6) $|f(u) - f(v)| = p^t$ is a prime power and $f(u) \not\equiv 0 \bmod p$.

Lemma 2.3. $W(3, n)$ is prime for even $n \in \mathcal{N}_1 \cap \mathcal{N}_5$.
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \bmod 2 = 0, 0 \le i \le n-1, \\ n + 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \bmod 2 = 1, 0 \le i \le n-1. \end{cases}$$

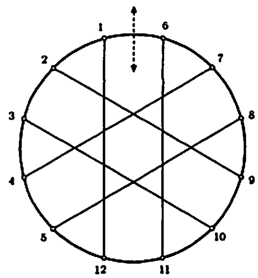


Figure 2.1. $W(3, 12)$ for $12 \in \mathcal{N}_1 \cap \mathcal{N}_5$

Figure 2.1 shows a prime labeling of $W(3, n)$ for even $n = 12 \in \mathcal{N}_1 \cap \mathcal{N}_5$. Now, we verify that f is a prime labeling.

For $0 \le i \le n-1$ and $0 \le (i+1) \bmod 6 \le 4$, by Observation 2.2(3), $f(v_i)$ and $f(v_{i+1})$ are coprime. For $0 \le i \le n-1$ and $(i+1) \bmod 6 = 5$, by Observation 2.2(2), $f(v_i)$ and $f(v_{i+1})$ are coprime.

For $0 \le i \le n/2 - 1$, by Observation 2.2(2), $f(v_{2i})$ and $f(v_{2i+5})$ are coprime.

Hence f is a prime labeling of $W(3, n)$ for even $n \in \mathcal{N}_1 \cap \mathcal{N}_5$. □

For Lemmas 2.4 - 2.14, we only define f, and leave for the readers to verify that f is a prime labeling of $W(3, n)$.

Lemma 2.4. $W(3, n)$ is prime for even $n \in \mathcal{N}_3 \cap \mathcal{N}_7$.
Proof. We define the function f as follows:
Let

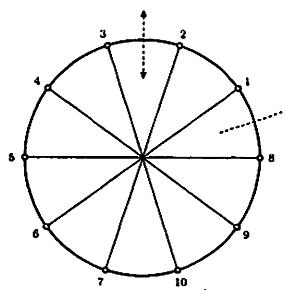
$$f(v_i) = \begin{cases} i + 3 - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \bmod 2 = 0, 0 \le i \le n-3, \\ n + 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \bmod 2 = 1, 0 \le i \le n-3, \\ 1, & i = n-2, \\ 2, & i = n-1. \end{cases}$$

In Figure 2.2(a), we show a prime labeling of $W(3,n)$, where $n = 10 \in \mathcal{N}_3 \cap \mathcal{N}_7$.

Lemma 2.5. $W(3,n)$ is prime for even $n \in \mathcal{N}_5 \bigcap \mathcal{N}_9$.
Proof. We define the function f as follows:
Let

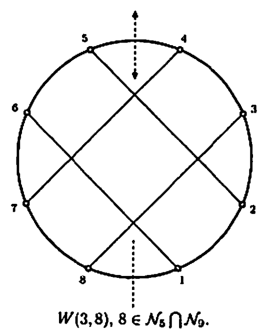
$$f(v_i) = \begin{cases} i + 5 - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \leq i \leq n-5, \\ n + 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \leq i \leq n-5, \\ 1, & i = n-4, \\ 2, & i = n-3, \\ 3, & i = n-2, \\ 4, & i = n-1. \end{cases}$$

In Figure 2.2(b), we show a prime labeling of $W(3,n)$, where $n = 8 \in \mathcal{N}_5 \cap \mathcal{N}_9$.



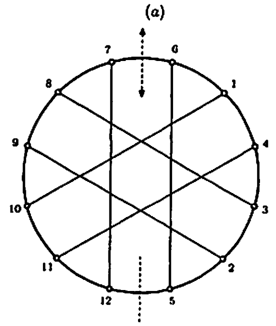
$W(3,10)$, $10 \in \mathcal{N}_3 \bigcap \mathcal{N}_7$.

(a)



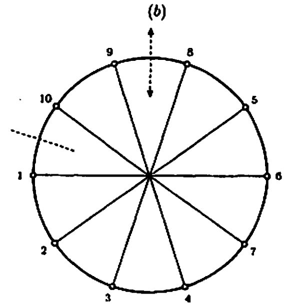
$W(3,8)$, $8 \in \mathcal{N}_5 \bigcap \mathcal{N}_9$.

(b)



$W(3,12)$, $12 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11}$.

(c)



$W(3,10)$, $10 \in \mathcal{N}_9 \bigcap \mathcal{N}_{13}$.

(d)

Figure 2.2.

Lemma 2.6. $W(3,n)$ is prime for even $n \in \mathcal{N}_7 \cap \mathcal{N}_{11}$.
Proof. We define the function f as follows:

Let

$$f(v_i) = \begin{cases} i+7-\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-7, \\ n+5-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-7, \\ 5, & i = n-6, \\ 2, & i = n-5, \\ 3, & i = n-4, \\ 4, & i = n-3, \\ 1, & i = n-2, \\ 6, & i = n-1. \end{cases}$$

In Figure 2.2(c), we show a prime labeling of $W(3,n)$, where $n = 12 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$.

Lemma 2.7. $W(3,n)$ is prime for even $n \in \mathcal{N}_9 \cap \mathcal{N}_{13}$.
Proof. We define the function f as follows:
Let

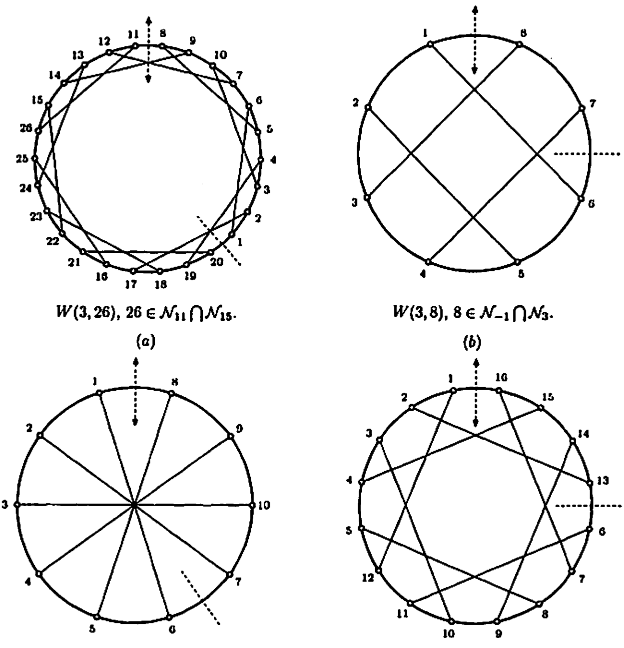
$$f(v_i) = \begin{cases} i+9-\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-9, \\ n+5-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-9, \\ 1, & i = n-8, \\ 2, & i = n-7, \\ 3, & i = n-6, \\ 4, & i = n-5, \\ 7, & i = n-4, \\ 6, & i = n-3, \\ 5, & i = n-2, \\ 8, & i = n-1. \end{cases}$$

In Figure 2.2(d), we show a prime labeling of $W(3,n)$, where $n = 10 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$.

Lemma 2.8. $W(3,n)$ is prime for even $n \in \mathcal{N}_{11} \bigcap \mathcal{N}_{15}$.
Proof.
In $\mathcal{N}_{11} \cap \mathcal{N}_{15}$, there is only one integer smaller than 26, namely 8. Since $8 \in \mathcal{N}_5 \cap \mathcal{N}_9$, by Lemma 2.5, $W(3,8)$ is prime. Hence, we only consider even $n \ge 26$. And define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+11-\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-11, \\ n+5-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-11, \\ 1, & i = n-10, \\ 2, & i = n-9, \\ 3, & i = n-8, \\ 4, & i = n-7, \\ 5, & i = n-6, \\ 6, & i = n-5, \\ 7, & i = n-4, \\ 10, & i = n-3, \\ 9, & i = n-2, \\ 8, & i = n-1. \end{cases}$$

In Figure 2.3(a), we show a prime labeling of $W(3,n)$, where $n = 26 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$.



$W(3,26)$, $26 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$.
(a)

$W(3,8)$, $8 \in \mathcal{N}_{-1} \cap \mathcal{N}_{3}$.
(b)

$W(3,10)$, $10 \in \mathcal{N}_{-3} \cap \mathcal{N}_{1}$.
(c)

$W(3,16)$, $16 \in \mathcal{N}_{-3} \cap \mathcal{N}_{1}$.
(d)

Figure 2.3.

Lemma 2.9. $W(3,n)$ is prime for even $n \in \mathcal{N}_{-1} \cap \mathcal{N}_3$.
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-3, \\ n + 3 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-3, \\ n-1, & i = n-2, \\ n, & i = n-1. \end{cases}$$

In Figure 2.3(b), we show a prime labeling of $W(3,n)$, where $n = 8 \in \mathcal{N}_{-1} \cap \mathcal{N}_3$.

Lemma 2.10. $W(3,n)$ is prime for even $n \in \mathcal{N}_{-3} \cap \mathcal{N}_1$.
Proof. We define the function f as follows:
Case 1. $n \not\equiv 6, 16, 40, 52 \pmod{60}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-5, \\ n + 1 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-5, \\ n-3, & i = n-4, \\ n, & i = n-3, \\ n-1, & i = n-2, \\ n-2, & i = n-1. \end{cases}$$

Case 2. $n \equiv 6, 16, 40, 52 \pmod{60}$. Let

$$f(v_i)=\begin{cases} i-\lfloor\frac{i+1}{12}\rfloor\times 6+1, & \lfloor\frac{i+1}{6}\rfloor \mod 2=0, 0\le i\le n-5,\\ n+1-i+\lfloor\frac{i+1}{12}\rfloor\times 6, & \lfloor\frac{i+1}{6}\rfloor \mod 2=1, 0\le i\le n-5,\\ n-3, & i=n-4,\\ n-2, & i=n-3,\\ n-1, & i=n-2,\\ n, & i=n-1.\end{cases}$$

In Figure 2.3(c), we show a prime labeling of $W(3,n)$, where $n=10\in\mathcal{N}_{-3}\cap\mathcal{N}_1$ and $n\not\equiv 6,16,40,52 \pmod{60}$. In Figure 2.3(d), we show a prime labeling of $W(3,n)$, where $n=16\in\mathcal{N}_{-3}\cap\mathcal{N}_1$ and $n\equiv 16 \pmod{60}$.

Lemma 2.11. $W(3,n)$ is prime for even $n\in\mathcal{N}_{-5}\bigcap\mathcal{N}_{-1}$.
Proof. We define the function f as follows:
Case 1. $n\not\equiv 12,24,48 \pmod{60}$. Let

$$f(v_i)=\begin{cases} i-\lfloor\frac{i+1}{12}\rfloor\times 6+1, & \lfloor\frac{i+1}{6}\rfloor \mod 2=0, 0\le i\le n-7,\\ n-i+\lfloor\frac{i+1}{12}\rfloor\times 6-1, & \lfloor\frac{i+1}{6}\rfloor \mod 2=1, 0\le i\le n-7,\\ n-5, & i=n-6,\\ n-4, & i=n-5,\\ n-3, & i=n-4,\\ n-2, & i=n-3,\\ n-1, & i=n-2,\\ n, & i=n-1.\end{cases}$$

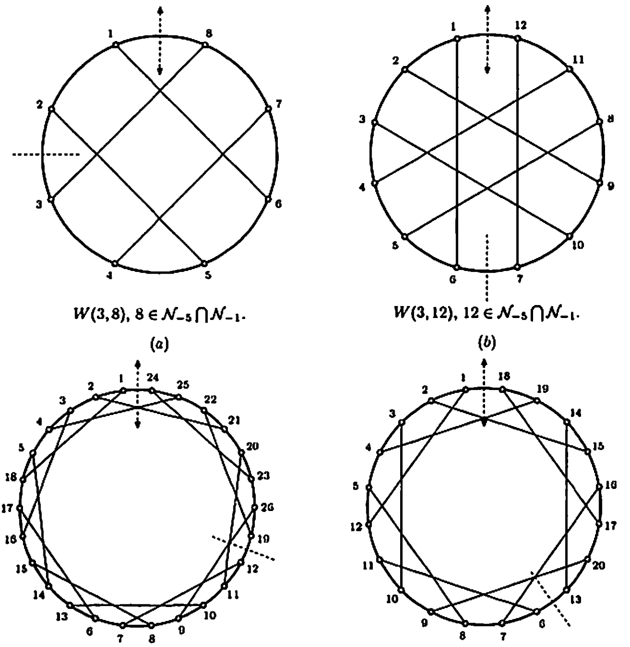
Case 2. $n\equiv 12,24,48 \pmod{60}$. Let

$$f(v_i)=\begin{cases} i-\lfloor\frac{i+1}{12}\rfloor\times 6+1, & \lfloor\frac{i+1}{6}\rfloor \mod 2=0, 0\le i\le n-7,\\ n-i+\lfloor\frac{i+1}{12}\rfloor\times 6-1, & \lfloor\frac{i+1}{6}\rfloor \mod 2=1, 0\le i\le n-7,\\ n-5, & i=n-6,\\ n-2, & i=n-5,\\ n-3, & i=n-4,\\ n-4, & i=n-3,\\ n-1, & i=n-2,\\ n, & i=n-1.\end{cases}$$

In Figure 2.4(a), we show a prime labeling of $W(3,n)$, where $n=8\in\mathcal{N}_{-5}\cap\mathcal{N}_{-1}$ and $n\not\equiv 12,24,48 \pmod{60}$. In Figure 2.4(b), we show a prime labeling of $W(3,n)$, where $n=12\in\mathcal{N}_{-5}\cap\mathcal{N}_{-1}$ and $n\equiv 12 \pmod{60}$.

Lemma 2.12. $W(n,3)$ is prime for even $n\in\mathcal{N}_{-7}\cap\mathcal{N}_{-3}$.
Proof.
In $\mathcal{N}_{-7}\cap\mathcal{N}_{-3}$, there are only two integers smaller than 20, namely 10,14. Since $10\in\mathcal{N}_{-3}\cap\mathcal{N}_1$, by Lemma 2.10, $W(3,10)$ is prime. Since $14\in\mathcal{N}_{-1}\bigcap\mathcal{N}_3$, by Lemma 2.9, $W(3,14)$ is prime. Hence, we only consider even $n\ge 20$. And define the function f as follows:

Case 1. $n\not\equiv 20,44,56 \pmod{60}$. Let



$W(3,8)$, $8 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1}$.
(a)

$W(3,12)$, $12 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1}$.
(b)

$W(3,26)$, $26 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3}$.
(c)

$W(3,20)$, $20 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3}$.
(d)

Figure 2.4.

$$
f(v_i) = \begin{cases}
i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-9, \\
n - 3 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-9, \\
n-7, & i = n-8, \\
n, & i = n-7, \\
n-3, & i = n-6, \\
n-6, & i = n-5, \\
n-5, & i = n-4, \\
n-4, & i = n-3, \\
n-1, & i = n-2, \\
n-2, & i = n-1.
\end{cases}
$$

Case 2. $n \equiv 20, 44 \pmod{60}$. Let

$$
f(v_i) = \begin{cases}
i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-9, \\
n - 3 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-9, \\
n-7, & i = n-8, \\
n, & i = n-7, \\
n-3, & i = n-6, \\
n-4, & i = n-5, \\
n-5, & i = n-4, \\
n-6, & i = n-3, \\
n-1, & i = n-2, \\
n-2, & i = n-1.
\end{cases}
$$

Case 3. $n \equiv 56 \pmod{60}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \leq i \leq n-9, \\ n-3-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \leq i \leq n-9, \\ n-7, & i = n-8, \\ n, & i = n-7, \\ n-5, & i = n-6, \\ n-4, & i = n-5, \\ n-1, & i = n-4, \\ n-2, & i = n-3, \\ n-3, & i = n-2, \\ n-6, & i = n-1. \end{cases}$$

In Figure 2.4(c), we show a prime labeling of $W(3,n)$, where $n = 26 \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3}$ and $n \not\equiv 20, 44, 56 \pmod{60}$. In Figure 2.4(d), we show a prime labeling of $W(3,n)$, where $n = 20 \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3}$ and $n \equiv 20 \pmod{60}$.

Lemma 2.13. $W(3,n)$ is prime for even $n \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5}$.
Proof.
In $\mathcal{N}_{-9} \cap \mathcal{N}_{-5}$, there are only two integers smaller than 16, namely 10,12. Since $10 \in \mathcal{N}_{-3} \cap \mathcal{N}_{1}$, by Lemma 2.10, $W(3,10)$ is prime. Since $12 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$, by Lemma 2.11, $W(3,12)$ is prime. Hence, we only consider even $n \geq 16$. And define the function f as follows:
Case 1. $n \not\equiv 16, 28, 22, 52 \pmod{60}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \leq i \leq n-11, \\ n-5-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \leq i \leq n-11, \\ n-9, & i = n-10, \\ n-8, & i = n-9, \\ n-7, & i = n-8, \\ n-6, & i = n-7, \\ n-5, & i = n-6, \\ n-4, & i = n-5, \\ n-3, & i = n-4, \\ n-2, & i = n-3, \\ n-1, & i = n-2, \\ n, & i = n-1, \end{cases}$$

Case 2. $n \equiv 16 \pmod{60}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \leq i \leq n-11, \\ n-5-i+\lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \leq i \leq n-11, \\ n-9, & i = n-10, \\ n-8, & i = n-9, \\ n-5, & i = n-8, \\ n-4, & i = n-7, \\ n-3, & i = n-6, \\ n-6, & i = n-5, \\ n-7, & i = n-4, \\ n-2, & i = n-3, \\ n-1, & i = n-2, \\ n, & i = n-1. \end{cases}$$

Case 3. $n \equiv 28 \pmod{60}$. Let

$$
f(v_i) = \begin{cases}
i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-11, \\
n - 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-11, \\
n-9, & i = n-10, \\
n-8, & i = n-9, \\
n-7, & i = n-8, \\
n-2, & i = n-7, \\
n-5, & i = n-6, \\
n-4, & i = n-5, \\
n-3, & i = n-4, \\
n-6, & i = n-3, \\
n-1, & i = n-2, \\
n, & i = n-1.
\end{cases}
$$

Case 4. $n \equiv 22, 52 \pmod{60}$.

Case 4.1. $n \equiv 0 \pmod 7$. Let

$$
f(v_i) = \begin{cases}
i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-11, \\
n - 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 10 \le i \le n-11, \\
n-9, & i = n-10, \\
n-2, & i = n-9, \\
n-5, & i = n-8, \\
n-4, & i = n-7, \\
n-3, & i = n-6, \\
n-8, & i = n-5, \\
n-7, & i = n-4, \\
n-6, & i = n-3, \\
n-1, & i = n-2, \\
n, & i = n-1.
\end{cases}
$$

Case 4.2. $n \not\equiv 0 \pmod 7$. Let

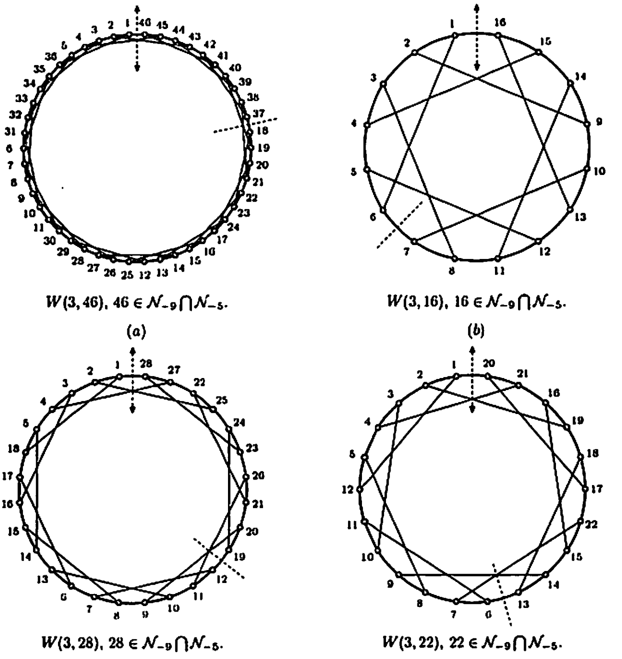
$$
f(v_i) = \begin{cases}
i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \le i \le n-11, \\
n - 5 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \le i \le n-11, \\
n-9, & i = n-10, \\
n-8, & i = n-9, \\
n-7, & i = n-8, \\
n, & i = n-7, \\
n-5, & i = n-6, \\
n-4, & i = n-5, \\
n-3, & i = n-4, \\
n-6, & i = n-3, \\
n-1, & i = n-2, \\
n-2, & i = n-1.
\end{cases}
$$

In Figure 2.5(a), we show a prime labeling of $W(3,n)$, where $n = 46 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5}$ and $n \not\equiv 16, 28, 22, 52 \pmod{60}$. In Figure 2.5(b), we show a prime labeling of $W(3,n)$, where $n = 16 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5}$ and $n \equiv 16 \pmod{60}$. In Figure 2.5(c), we show a prime labeling of $W(3,n)$, where $n = 28 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5}$ and $n \equiv 28 \pmod{60}$. In Figure 2.5(d) we show a prime labeling of $W(3,n)$, where $n = 22 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5}$, $n \equiv 22 \pmod{60}$ and $n \not\equiv 0 \pmod 7$.

Lemma 2.14. *$W(3,n)$ is prime for even $n \in \mathcal{N}_{-11} \bigcap \mathcal{N}_{-7}$.*

Proof.

In $\mathcal{N}_{-11} \cap \mathcal{N}_{-7}$, there are only two integers smaller than 18, namely 12,14. Since $12 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$, by Lemma 2.11, $W(3,12)$ is prime. Since



$W(3,46)$, $46 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5}$.
(a)

$W(3,16)$, $16 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5}$.
(b)

$W(3,28)$, $28 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5}$.
(c)

$W(3,22)$, $22 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5}$.
(d)

Figure 2.5.

$14 \in \mathcal{N}_{-1} \bigcap \mathcal{N}_3$, by Lemma 2.9, $W(3,14)$ is prime. Hence, we only consider even $n \geq 18$. And define the function f as follows:

Case 1. $n \equiv 18$ (mod 30). Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6 + 1, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 0 \leq i \leq n-13, \\ n - 7 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 0 \leq i \leq n-13, \\ n-11, & i = n-12, \\ n-10, & i = n-11, \\ n-7, & i = n-10, \\ n-6, & i = n-9, \\ n-1, & i = n-8, \\ n, & i = n-7, \\ n-5, & i = n-6, \\ n-8, & i = n-5, \\ n-9, & i = n-4, \\ n-4, & i = n-3, \\ n-3, & i = n-2, \\ n-2, & i = n-1. \end{cases}$$

Case 2. $n \equiv 24$ (mod 30). Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 1 \leq i \leq n-13, \\ n - 6 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 1 \leq i \leq n-13, \\ n-11, & i = n-12, \\ n-4, & i = n-11, \\ n-7, & i = n-10, \\ n-6, & i = n-9, \\ n-5, & i = n-8, \\ n, & i = n-7, \\ n-1, & i = n-6, \\ n-10, & i = n-5, \\ n-9, & i = n-4, \\ n-8, & i = n-3, \\ n-3, & i = n-2, \\ n-2, & i = n-1. \end{cases}$$

Case 3. $n \equiv 0 \pmod{30}$.

Case 3.1. $n \not\equiv 1 \pmod{7}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 1 \leq i \leq n-13, \\ n - 6 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 1 \leq i \leq n-13, \\ n-11, & i = n-12, \\ n-10, & i = n-11, \\ n-9, & i = n-10, \\ n-4, & i = n-9, \\ n-7, & i = n-8, \\ n, & i = n-7, \\ n-1, & i = n-6, \\ n-8, & i = n-5, \\ n-3, & i = n-4, \\ n-2, & i = n-3, \\ n-5, & i = n-2, \\ n-6, & i = n-1. \end{cases}$$

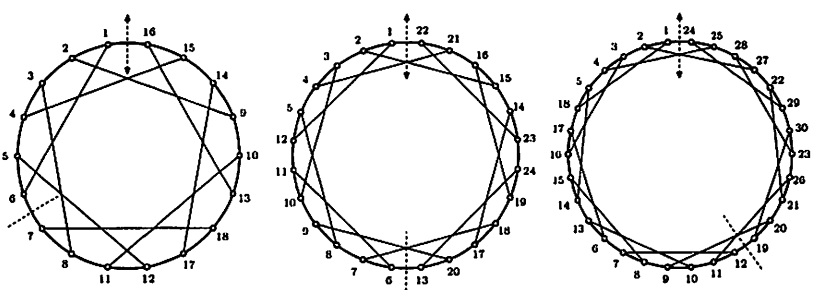
Case 3.2. $n \equiv 1 \pmod{7}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 1 \leq i \leq n-13, \\ n - 6 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 1 \leq i \leq n-13, \\ n-11, & i = n-12, \\ n-10, & i = n-11, \\ n-9, & i = n-10, \\ n-4, & i = n-9, \\ n-7, & i = n-8, \\ n, & i = n-7, \\ n-1, & i = n-6, \\ n-2, & i = n-5, \\ n-3, & i = n-4, \\ n-8, & i = n-3, \\ n-5, & i = n-2, \\ n-6, & i = n-1. \end{cases}$$

Case 4. $n \not\equiv 0, 18, 24 \pmod{30}$. Let

$$f(v_i) = \begin{cases} i - \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 0, 1 \le i \le n-13, \\ n - 6 - i + \lfloor \frac{i+1}{12} \rfloor \times 6, & \lfloor \frac{i+1}{6} \rfloor \mod 2 = 1, 1 \le i \le n-13, \\ n-11, & i = n-12, \\ n-10, & i = n-11, \\ n-9, & i = n-10, \\ n-8, & i = n-9, \\ n-7, & i = n-8, \\ n-6, & i = n-7, \\ n-5, & i = n-6, \\ n-4, & i = n-5, \\ n-3, & i = n-4, \\ n-2, & i = n-3, \\ n-1, & i = n-2, \\ n, & i = n-1. \end{cases}$$

In Figure 2.6(a), we show a prime labeling of $W(3,n)$, where $n = 18 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$ and $n \equiv 18 \pmod{30}$. In Figure 2.6(b), we show a prime labeling of $W(3,n)$, where $n = 24 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$ and $n \equiv 24 \pmod{30}$. In Figure 2.6(c), we show a prime labeling of $W(3,n)$, where $n = 30 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$, $n \equiv 0 \pmod{30}$ and $n \not\equiv 1 \pmod 7$.



$W(3,18)$, $18 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$. (a)

$W(3,24)$, $24 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$. (b)

$W(3,30)$, $30 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$. (c)

Figure 2.6.

Table 2.1.

$n \in \mathcal{N}_i \cap \mathcal{N}_{i+4}$	$n \in \mathcal{N}_i \cap \mathcal{N}_{i+4}$	$n \in \mathcal{N}_i \cap \mathcal{N}_{i+4}$	$n \in \mathcal{N}_i \cap \mathcal{N}_{i+4}$	$n \in \mathcal{N}_i \cap \mathcal{N}_{i+4}$
$2 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$28 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$	$54 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7}$	$80 \in \mathcal{N}_{-1} \cap \mathcal{N}_3$	$106 \in \mathcal{N}_3 \cap \mathcal{N}_7$
$4 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$30 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$	$56 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$	$82 \in \mathcal{N}_{-3} \cap \mathcal{N}_1$	$108 \in \mathcal{N}_1 \cap \mathcal{N}_5$
$6 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$32 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$58 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$	$84 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$	$110 \in \mathcal{N}_{-1} \cap \mathcal{N}_3$
$8 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$34 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$60 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$	$86 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$	$112 \in \mathcal{N}_{-3} \cap \mathcal{N}_1$
$10 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$36 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$62 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$88 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$	$114 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$
$12 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$38 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$64 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$90 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$	$116 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$
$14 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$40 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$66 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$92 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$118 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$
$16 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$42 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$68 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$	$94 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$120 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$
$18 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$44 \in \mathcal{N}_{-1} \cap \mathcal{N}_{-3}$	$70 \in \mathcal{N}_9 \cap \mathcal{N}_{13}$	$96 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$122 \in \mathcal{N}_5 \cap \mathcal{N}_9$
$20 \in \mathcal{N}_{-1} \cap \mathcal{N}_3$	$46 \in \mathcal{N}_{-3} \cap \mathcal{N}_1$	$72 \in \mathcal{N}_7 \cap \mathcal{N}_{11}$	$98 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$124 \in \mathcal{N}_3 \cap \mathcal{N}_7$
$22 \in \mathcal{N}_{-3} \cap \mathcal{N}_1$	$48 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$	$74 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$100 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$126 \in \mathcal{N}_1 \cap \mathcal{N}_5$
$24 \in \mathcal{N}_{-5} \cap \mathcal{N}_{-1}$	$50 \in \mathcal{N}_{-7} \cap \mathcal{N}_{-3}$	$76 \in \mathcal{N}_3 \cap \mathcal{N}_7$	$102 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$128 \in \mathcal{N}_{-1} \cap \mathcal{N}_3$
$26 \in \mathcal{N}_{11} \cap \mathcal{N}_{15}$	$52 \in \mathcal{N}_{-9} \cap \mathcal{N}_{-5}$	$78 \in \mathcal{N}_1 \cap \mathcal{N}_5$	$104 \in \mathcal{N}_5 \cap \mathcal{N}_9$	$130 \in \mathcal{N}_{-3} \cap \mathcal{N}_1$

By Lemmas 2.3 - 2.14, we have Theorem 2.2 for any even $n \leq 130$. Furthermore, we have the following conjecture

Conjecture 2.15. $W(n, 3)$ is prime for all even n.

Since $n \in \mathcal{N}$ for any even $n \leq 130$, by Theorem 2.2 and Table 2.1, we have Conjecture 2.15 holds for any even $n \leq 130$.

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