

Signed edge k -subdomination numbers in graphs

Abdollah Khodkar
Department of Mathematics
University of West Georgia
Carrollton, GA 30118
akhodkar@westga.edu

R. Saei and S.M. Sheikholeslami^{†*}
Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
[†]s.m.sheikholeslami@azaruniv.edu

Abstract

The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end-vertex with e . Let f be a function on $E(G)$, the edge set of G , into the set $\{-1, 1\}$. If $\sum_{x \in N[e]} f(x) \geq 1$ for at least k edges e of G , then f is called a signed edge k -subdominating function of G . The minimum of the values $\sum_{e \in E(G)} f(e)$, taken over all signed edge k -subdominating function f of G , is called the signed edge k -subdomination number of G and is denoted by $\gamma'_{k_s}(G)$. In this note we initiate the study of the signed edge k -subdomination in graphs and present some (sharp) bounds for this parameter.

Keywords: signed edge k -subdominating function; signed edge k -subdomination number; signed edge dominating function; signed domination number

*Corresponding author

1 Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [8] for terminology and notation which are not defined here. The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The *line graph* of a graph G , written $L(G)$, is the graph whose vertices are the edges of G , with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G . It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e . Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \rightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. If $S = N_G[e]$ for some $e \in E$, then we denote $f(S)$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of all edges incident to vertex v . A function $f : E(G) \rightarrow \{-1, 1\}$ is called a *signed edge k -subdominating function (SEkSDF)* of G , if $f[e] \geq 1$ for at least k edges e of G . The minimum of the values $f(E(G))$, taken over all signed edge k -subdominating functions f of G , is called the *signed edge k -subdomination number* of G and is denoted by $\gamma'_{ks}(G)$. The signed edge k -subdominating function f of G with $f(E(G)) = \gamma'_{ks}(G)$ is called $\gamma'_{ks}(G)$ -*function*. For any signed edge k -subdominating function f of G we define $P = \{e \in E(G) \mid f(e) = 1\}$, $M = \{e \in E(G) \mid f(e) = -1\}$ and $X = \{e \in E(G) \mid f[e] \geq 1\}$.

If $k = m$, where m is the size of a graph, then the signed edge k -subdomination number is called the *signed edge domination number*. The signed edge domination number was introduced by Xu in [9] and denoted by $\gamma'_s(G)$. This parameter has been studied by several authors [4, 5, 6, 9, 10, 12, 13, 14].

If $k = \lceil m/2 \rceil$, then the signed edge k -subdomination number is called the *signed edge majority domination number*. This parameter was introduced by Karami et al. in [7] and denoted by $\gamma'_{sm}(G)$.

An opinion function on a graph G is a function $f : V(G) \rightarrow \{-1, 1\}$. By the vote of a vertex v we mean $\sum_{w \in N[v]} f(w)$. A *k -subdominating function* (see [1]) of a graph G is an opinion function for which the votes of at least k vertices are positive. The *k -subdomination number* of G , denoted by $\gamma_{ks}(G)$, is the minimum of the values of $\sum_{v \in V(G)} f(v)$, taken over all k -subdominating functions f of G . The k -subdominating function f of G with $f(V(G)) = \gamma_{ks}(G)$ is called $\gamma_{ks}(G)$ -*function*.

The following table shows the notation introduced above.

For $e \in E(G)$, $N_G[e] = N_G(e) \cup \{e\}$
For $S \subseteq E(G)$, $f(S) = \sum_{e \in S} f(e)$
For $e \in E(G)$, $f[e] = \sum_{e' \in N[e]} f(e')$
For $S = N_G[e]$, $f(S) = f[e]$
For $v \in V(G)$, $E(v) =$ the set of all edges incident to vertex v
For $v \in V(G)$, $f(v) = \sum_{e \in E(v)} f(e)$
$P = \{e \in E(G) \mid f(e) = 1\}$
$M = \{e \in E(G) \mid f(e) = -1\}$
$X = \{e \in E(G) \mid f[e] \geq 1\}$
$\gamma'_s(G) =$ the signed edge domination number of G
$\gamma'_{sm}(G) =$ the signed edge majority domination number of G
$\gamma'_{ks}(G) =$ the signed edge k -subdomination number of G
$\gamma_{ks}(G) =$ the k -subdomination number of G

Table of notation

In this note, we initiate the study of the signed edge k -subdomination in graphs and present some (sharp) bounds for this parameter. Note that the signed edge k -subdomination number is a generalization of the signed edge domination number introduced by Xu in [9] and the signed edge majority domination number introduced by Karami et al. in [5]. Here are some well-known results on $\gamma'_s(G)$, $\gamma'_{sm}(G)$ and $\gamma_{ks}(G)$.

Theorem A. [4]) For any tree T of order $n \geq 2$, $\gamma'_s(T) \geq 1$ with equality if and only if T has no vertex of even degree and $\ell(v) \geq \lfloor (\deg(v) - 1)/2 \rfloor$ for every vertex v , where $\ell(v)$ denotes the number of pendant edges at vertex v . In addition, if $\gamma'_s(T) = 1$ and f is a $\gamma'_s(T)$ -function, then $f(v) = 1$ for every vertex of degree greater than one.

Theorem B. ([11]) Let G be a graph with $\delta(G) \geq 1$. Then $\gamma'_s(G) \geq |V(G)| - |E(G)|$ and this bound is sharp.

Define \mathcal{G}_0 to be the collection of all simple connected graphs of order $n \geq 2$ in which the degree of each vertex is odd and $\ell(v) \geq (\deg(v) - 1)/2$ for every vertex v .

Theorem C. ([5]) Let G be a simple connected graph of order $n \geq 2$ and size m . Then $\gamma'_s(G) = n - m$ if and only if $G \in \mathcal{G}_0$. Furthermore, if $\gamma'_s(G) = n - m$ and f is a $\gamma'_s(G)$ -function, then

1. $f(e) = 1$ for each non-pendant edge $e \in E(G)$;
2. $f(v) = 1$ for each vertex v of degree greater than 1;

3. $f[e] = 1$ for each edge $e \in E(G)$.

Theorem D. ([10]) For any positive integer m , define

$$\Psi(m) = \min\{\gamma'_s(G) \mid G \text{ is a graph of size } m\}.$$

Then

$$\Psi(m) = 2\left\lceil \frac{1}{3} \left\lceil \frac{\sqrt{24m+25} + 6m + 5}{6} \right\rceil \right\rceil - m.$$

Theorem E. ([7]) Let Ψ be as in Theorem D. Then

1. $m \geq \Psi(m)$ for every positive integer m , and
2. $\Psi(a) + \Psi(b) \geq \Psi(a + b)$ for each pair of positive integers a and b .

Theorem F. ([7]) Let G be a simple graph of order $n \geq 3$ and size m . Then

$$\gamma'_{sm}(G) \geq \Psi(t) - (m - t)$$

for some integer $\lceil \frac{m}{2} \rceil \leq t \leq m$. Furthermore, this bound is sharp for $t = \lceil \frac{m}{2} \rceil$.

Theorem G. ([1]) For $n \geq 2$ and $1 \leq k \leq n$, $\gamma_{ks}(P_n) = 2\lfloor (2k+4)/3 \rfloor - n$.

Theorem H. ([2]) If $n \geq 3$ and $1 \leq k \leq n - 1$, then

$$\gamma_{ks}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n - 1 \text{ and } k \equiv 1 \pmod{3} \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

Theorem I. ([1]) For any connected graph G of order n and any positive integer $k \leq \lceil \frac{n}{2} \rceil$,

$$\gamma_{ks}(G) \leq \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem J. ([3]) For any connected graph G of order n and any positive integer k with $\frac{n}{2} < k \leq n$,

$$\gamma_{ks}(G) \leq 2\left\lceil \frac{k}{n-k+1} \right\rceil (n-k+1) - n.$$

The proof of the following theorem is straightforward and therefore omitted.

Theorem 1. For any graph G of order $n \geq 2$ which has no isolates,

$$\gamma'_{ks}(G) = \gamma_{ks}(L(G)).$$

Theorems 1, G, H, I and J lead to:

Corollary 2. For $n \geq 3$ and $1 \leq k \leq n - 1$,

$$\gamma'_{ks}(P_n) = 2\lfloor(2k + 4)/3\rfloor - n + 1.$$

Corollary 3. For $n \geq 3$ and $1 \leq k \leq n - 1$,

$$\gamma'_{ks}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n - 1 \text{ and } k \equiv 1 \pmod{3} \\ 2\lfloor\frac{2k+4}{3}\rfloor - n & \text{otherwise.} \end{cases}$$

Corollary 4. For any connected graph G of size m and positive integer $k \leq \lceil\frac{m}{2}\rceil$,

$$\gamma'_{ks}(G) \leq \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even.} \end{cases}$$

These bounds are sharp for stars.

Corollary 5. For any connected graph G of size m and any positive integer k with $\frac{m}{2} < k \leq m$,

$$\gamma'_{ks}(G) \leq 2\lceil\frac{k}{m - k + 1}\rceil(m - k + 1) - m.$$

2 Lower bounds on the SEkSDNs of graphs

In this section we generalize Theorem F to the signed edge k -subdomination number. Then we present a lower bound for $\gamma'_{ks}(G)$ in terms of the size, the minimum degree and the maximum degree of G . We find a sharp lower bound for $\gamma'_{ks}(T)$ in terms of k , the order and the size of tree T . This generalizes the lower bound given in Theorem A to the SEkSDNs of trees. Finally, we find a lower bound for $\gamma'_{ks}(G)$ in terms of k , the order and the size of G , generalizing Theorem B. We show that this bound is sharp for $k \geq 7$ and odd.

Theorem 6. For any simple graph G of order $n \geq 3$, size m and integer $1 \leq k \leq m$,

$$\gamma'_{ks}(G) \geq \Psi(t) - (m - t),$$

for some integer $k \leq t \leq m$. Furthermore, this bound is sharp when $t = k$.

Proof. The statement holds for all simple graphs of size $m = 1, 2, 3$. Now assume $m \geq 4$. Let, to the contrary, G be a simple graph of size $m \geq 4$ such that $\gamma'_{ks}(G) < \Psi(t) - (m - t)$ for every integer $k \leq t \leq m$. Choose such a graph G with as few edges as possible for which $\omega(G) + |T(G)|$ is

maximum, where $\omega(G)$ denotes the number of components of G and $T(G) = \{u \in V(G) \mid \deg(u) \leq 2\}$. Without loss of generality we may assume G has no isolated vertices. Let f be a $\gamma'_{ks}(G)$ -function. Let $G_1, \dots, G_{\omega(G)}$ be the connected components of G . If $G_i \simeq K_2$ for each $1 \leq i \leq \omega(G)$, then obviously

$$\gamma'_{ks}(G) = k - (m - k) \geq \Psi(k) - (m - k).$$

Let G have a connected component H of size at least 2. First we prove that $E(H) \subseteq X$. Hence, $f|_H$ is actually a γ'_s -function on H for every connected component H with $|E(H)| \geq 3$ of G . Then we use a simple counting argument and Lemma E to complete the proof.

Claim 1. $E(H) \cap M \subseteq X$.

Let $e \in E(H) \cap M$. Suppose that, to the contrary, $e \notin X$. Assume G' is obtained from $G - e$ by adding a new component u_0v_0 . Define $g : E(G') \rightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G') \setminus \{u_0v_0\}$. Obviously, g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G . Thus $e \in X$.

Claim 2. For every non-pendant edge $e = uv \in E(H) \cap M$ we have $\deg(u) = \deg(v) = 2$.

If $f(u) \geq 1$ (the case $f(v) \geq 1$ is similar) and G' is obtained from $G - e$ by adding a pendant edge uv' , then obviously $g : E(G') \rightarrow \{-1, 1\}$, which is defined by $g(uv') = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$, is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$. This contradicts the assumptions on G . Hence, $f(u) = f(v) = 0$. Therefore $\deg(u)$ and $\deg(v)$ are even. Let $\deg(u) \geq 4$ (the case $\deg(v) \geq 4$ is similar). Then there is a $+1$ edge $e' = uw$ at u . Assume G' is obtained from $G - \{e, e'\}$ by adding a new vertex z and two new edges vz and wz . Define $g : E(G') \rightarrow \{-1, 1\}$ by $g(vz) = -1$, $g(wz) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, e'\}$. Then g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Hence, $\deg(u) = \deg(v) = 2$.

Claim 3. Let $e = uv \in E(H) \cap M$ be a non-pendant edge and $uu', vv' \in E(G)$. Then $uu', vv' \in X$.

Let, to the contrary, $uu' \notin X$ (the case $vv' \notin X$ is similar). Since $e \in X$, $f(uu') = f(vv') = 1$. Suppose that $\deg(u') = 1$ and G' is obtained from $G - \{e, uu'\}$ by adding a pendant edge vv_1 and a new component u_0v_0 . Define $g : E(G') \rightarrow \{-1, 1\}$ by $g(vv_1) = -1$, $g(u_0v_0) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu'\}$. Then g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $\deg(u') \geq 2$. Similarly, we can see that $\deg(v') \geq 2$.

First let $u' = v'$. Since $uu' \notin X$, we have $vv' \notin X$. Suppose that there exists a -1 pendant edge $u'z$ at u' . By Claim 1, $u'z \in X$, which implies that $f(u') \geq 1$. Let G' be the graph obtained from $G - \{e\}$ by adding

a new component u_0v_0 . Define $g : E(G') \rightarrow \{-1, 1\}$ by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$. Obviously, g is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore, there is no -1 pendant edge at $u' = v'$. If there exists a -1 non-pendant edge at u' , then an argument similar to that described in Claim 2 shows that $\deg(u') = 2$, a contradiction. Thus every edge at u' is a $+1$ edge. This forces $uu' \in X$, a contradiction.

Now let $u' \neq v'$. Since we have assumed $uu' \notin X$, it follows that $f(u') \leq 1$. If there is a -1 pendant edge $u'w$ at u' , then by Claim 1 we have $u'w \in X$ and hence, $f(u') = f[u'w] \geq 1$. If there is a -1 non-pendant edge at u' , then $\deg(u') = 2$ by Claim 2 and hence, $f(u') = 0$. It follows that $f(u') = 0, 1$.

When $f(u') = 1$, define G' to be the graph obtained from $G - \{e\}$ by adding a new component u_0v_0 . Then $g : E(G') \rightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e\}$ is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $f(u') = 0$ and hence, there exists a -1 edge $u'u''$ at u' . If $\deg(u'') = 1$, define G' to be the graph obtained from $G - \{u'u''\}$ by adding a new component u_0v_0 . Then $g : E(G') \rightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{u'u''\}$ is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Hence, $\deg(u'') = 2$ (see Claim 2). Let G' be obtained from $G - \{e, uu', u'u''\}$ by adding a new component u_0v_0 and two new edges $u''z, zv$. Then $g : E(G') \rightarrow \{-1, 1\}$ defined by $g(u_0v_0) = -1$, $g(u''z) = -1$, $g(zv) = 1$ and $g(x) = f(x)$ if $x \in E(G) \setminus \{e, uu', u'u''\}$ is an SEkSDF of G' with $g(E(G')) = f(E(G))$ and $\omega(G') + |T(G')| > \omega(G) + |T(G)|$, a contradiction. Therefore $uu' \in X$, a contradiction.

Claim 4. $E(H) \cap P \subseteq X$.

Let $e = uv \in E(H) \cap P$. If there is a -1 non-pendant edge at u or at v , then by Claim 3 we have $e \in X$. If there exists a -1 pendant edge e' at u , then $e' \in X$ by Claim 1 and hence, $f(u) = f[e'] \geq 1$. If all the edges at u are $+1$ edges, then $f(u) \geq 1$. Similarly, if there is no -1 non-pendant edge at v , we see that $f(v) \geq 1$. Hence, $e \in X$.

Let G_1, G_2, \dots, G_s be the connected components of G for which $E(G_i) \subseteq X$. Thus, $f|_{G_i}$ is a γ'_s -function on G_i for each $1 \leq i \leq s$. Now by Claims 1 and 3, $X \cap [\cup_{i=s+1}^w(G) E(G_i)] = \emptyset$.

Let $|E(G_i)| = m_i$ for each $1 \leq i \leq w(G)$. Then $|X| = \sum_{i=1}^s m_i \geq k$ and

$\sum_{i=s+1}^{w(G)} m_i \leq m - k$. Then by Lemma E,

$$\begin{aligned} \gamma'_{ks}(G) &= \sum_{i=1}^s \gamma'_s(G_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \sum_{i=1}^s \Psi(m_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \Psi(\sum_{i=1}^s m_i) - \sum_{i=s+1}^{w(G)} m_i \\ &\geq \Psi(t) - (m - t) \end{aligned}$$

where $t = \sum_{i=1}^s m_i \geq k$.

In order to prove that the lower bound is sharp when $t = k$, let H_1 be a graph of size k with $\gamma'_s(H) = \Psi(k)$ (see [10]) and let H_2 be a graph of size $m - k$ such that $V(H_1) \cap V(H_2) = \emptyset$. Suppose $G = H_1 \cup H_2$ and f is a $\gamma'_s(H_1)$ -function. Then $g : E(G) \rightarrow \{-1, 1\}$ defined by $g(e) = f(e)$ if $e \in E(H_1)$ and $g(e) = -1$ if $e \in E(H_2)$, is an SEKSDF of G with $g(E(G)) = \Psi(k) - (m - k)$. This completes the proof. \square

Theorem 7. Let G be a simple graph of size m , minimum degree δ , maximum degree Δ and no isolates. Then

$$\gamma'_{ks}(G) \geq \frac{2k\delta}{2\Delta - 1} - m.$$

Proof. Let (d_1, \dots, d_n) be the degree sequence of G where $d_1 \leq d_2 \leq \dots \leq d_n$. Assume g is a $\gamma'_{ks}(G)$ -function of G and let $g(e) \geq 1$ for k distinct edges e in $\{e_{j_1} = u_{j_1} v_{j_1}, \dots, e_{j_k} = u_{j_k} v_{j_k}\}$. Define $f : E(G) \rightarrow \{0, 1\}$ by $f(e) = (g(e) + 1)/2$ for each $e \in E(G)$. We have

$$\begin{aligned} \sum_{i=1}^k f(N_G[e_{j_i}]) &= \sum_{i=1}^k \frac{g(N_G[e_{j_i}]) + \deg(u_{j_i}) + \deg(v_{j_i}) - 1}{2} \\ &\geq \sum_{i=1}^k \frac{\deg(u_{j_i}) + \deg(v_{j_i})}{2} \\ &\geq k\delta. \end{aligned} \quad (1)$$

On the other hand,

$$\begin{aligned} \sum_{i=1}^k f(N_G[e_{j_i}]) &\leq \sum_{e \in E} f(N_G[e]) \\ &= \sum_{e=uv \in E} (\deg(u) + \deg(v) - 1)f(e) \\ &\leq \sum_{e \in E} (2\Delta - 1)f(e) \\ &= (2\Delta - 1)f(E(G)). \end{aligned} \quad (2)$$

By (1) and (2), $f(E(G)) \geq \frac{k\delta}{2\Delta - 1}$. Since $g(E(G)) = 2f(E(G)) - m$,

$$\gamma'_{ks}(G) = g(E(G)) \geq \frac{2k\delta}{2\Delta - 1} - m,$$

as desired. \square

As an immediate consequence of Theorem 7 we have:

Corollary 8. Let G be an r -regular graph of size m and $r \geq 1$. Then $\gamma'_{ks}(G) \geq \frac{2rk}{2r-1} - m$. Furthermore, this bound is sharp when $r = 1$.

Now we prove that for any tree of size $m \geq 2$ and any integer $1 \leq k \leq m-1$, $\gamma'_{ks}(T) \geq k - m + 2$. This generalizes the lower bound for signed edge domination numbers for trees given in Theorem A to the signed edge k -subdomination numbers for trees.

Theorem 9. If T is a tree of size $m \geq 2$ and k is an integer, $1 \leq k \leq m-1$, then $\gamma'_{ks}(T) \geq 2\lceil k/2 \rceil - m + 2$. Furthermore, these bounds are sharp for each value of k .

Proof. First, by induction on m , we prove that $\gamma'_{ks}(T) \geq k - m + 2$, where $1 \leq k \leq m-1$. The statement holds for all trees of size $m = 2, 3, 4$. Assume T is an arbitrary tree of size $m \geq 5$ and that the statement holds for all trees of smaller sizes. Let f be a γ'_{ks} -function for T . If $M = \{e \in E(T) \mid f(e) = -1\} = \emptyset$, then obviously the theorem is true. Let $M \neq \emptyset$.

Case 1. There is a non-pendant edge $e = uv \in E(T)$ for which $f(e) = -1$. Let T_1 and T_2 be the connected components of $T - e$ with $u \in T_1$. Then, $\gamma'_{ks}(T) = f(E(T_1)) - 1 + f(E(T_2))$. We consider two subcases.

Subcase 1.1 For $i = 1, 2$, $X \cap E(T_i) \neq \emptyset$, where $X = \{e \in E(T) \mid f[e] \geq 1\}$. Let $|X \cap E(T_1)| = k_1$ and $|X \cap E(T_2)| = k_2$. Then for $i = 1, 2$, the function f , restricted to T_i is an $\text{SE}k_i\text{SDF}$ for T_i . Hence, $\gamma'_{k_i s}(T_i) \leq f(E(T_i))$ for $i = 1, 2$. First let $E(T_i) \subseteq X$ for $i = 1, 2$. Then since $k \leq m-1$, it follows that $e \notin X$ and by Theorem A,

$$\gamma'_{ks}(T) = f(E(T)) = f(E(T_1)) - 1 + f(E(T_2)) \geq 1 - 1 + 1 \geq k - m + 2.$$

Now without loss of generality we assume $E(T_1) \not\subseteq X$. If $E(T_2) \subseteq X$, then by the inductive hypothesis and Theorem A, $\gamma'_{k_1 s}(T_1) \geq k_1 - |E(T_1)| + 2$ and $\gamma'_{k_2 s}(T_2) \geq 1$. If $e \notin X$, then obviously $\gamma'_{ks}(T) \geq k - m + 3$. If $e \in X$, then

$$\begin{aligned} \gamma'_{ks}(T) &= f(E(T)) = f(E(T_1)) - 1 + f(E(T_2)) \\ &\geq k_1 - |E(T_1)| + 2 - 1 + 1 \geq k - m + 2. \end{aligned}$$

The case $E(T_2) \not\subseteq X$ is similar.

Subcase 1.2 $X \cap E(T_1) = \emptyset$ (the case $X \cap E(T_2) = \emptyset$ is similar). First let $e \notin X$. Then $|X \cap E(T_2)| = k_2 \geq k$. We claim that f assigns -1 to all edges of T_1 . If $E(T_1) \cap P \neq \emptyset$, where $P = \{e \in E(T) \mid f(e) = 1\}$, then we define $g : E(T) \rightarrow \{-1, +1\}$ by $g(e) = -1$ if $e \in E(T_1)$ and $g(e) = f(e)$ if $e \in E(T) \setminus E(T_1)$. Then g is a $\text{SE}k\text{SDF}$ of T of weight less than f , a

contradiction. This proves our claim. Now by the inductive hypothesis on $T' = T_2 + uv$ we have

$$\begin{aligned}\gamma'_{ks}(T) &= f(E(T_1)) + f(E(T')) \\ &\geq -|E(T_1)| + k_2 - |E(T')| + 2 \geq k - m + 2.\end{aligned}$$

Now assume $e \in X$. First let $f(v) \geq 2$. Then f , restricted to T' , is an SEkSDF for T' . If $k = |E(T')|$, since $f(v) \geq 2$, by Theorem A,

$$\gamma'_{ks}(T) \geq f(E(T')) - |E(T_1)| \geq (k - |E(T')|) + 2 - |E(T_1)| = k - m + 2.$$

If $k < |E(T')|$, by the inductive hypothesis,

$$\gamma'_{ks}(T) \geq f(E(T')) - |E(T_1)| \geq k - |E(T')| + 2 - (|E(T_1)|) \geq k - m + 2.$$

Now suppose that $f(v) = 1$. Then f , restricted to T' , is an SEkSDF for T' and there exists at least one $+1$ edge at u in T_1 (note that $e \in X$). By Theorem A or the inductive hypothesis

$$\begin{aligned}\gamma'_{ks}(T) &\geq f(E(T')) - |E(T_1)| + 2 \\ &\geq (k - |E(T')| + 1) - |E(T_1)| + 2 = k - m + 3.\end{aligned}$$

Finally, let $f(v) \leq 0$. Then either there exists an edge at v not in X or there exists a vertex $w \in V(T_2)$ other than v for which $f(w) \geq 2$. Since $e \in X$, $f(u) \geq -f(v)$, hence in T_1 there are at least $-f(v)+1$, $+1$ edges at u . Assume $E(v) \cap M = \{uv, vv_1, \dots, vv_s\}$ and $E(u) \cap P = \{uu_1, uu_2, \dots, uu_r\}$, where $r \geq s + 1$. Define $g : E(T) \rightarrow \{-1, +1\}$ by $g(uv) = g(vv_i) = 1$, $g(uu_{s+1}) = g(uu_i) = -1$, for $i = 1, \dots, s$ and $g(e) = f(e)$ if $e \in E(T) \setminus \{uu_{s+1}, uv, vv_i, uu_i \mid 1 \leq i \leq s\}$. Obviously, g is a $\gamma'_{ks}(T)$ -function and $g(v) \geq 1$. Now by an argument similar to that described above (for the cases $f(v) \geq 2$ or $f(v) = 1$) we have $\gamma'_{ks}(T) = g(E(T)) \geq k - m + 2$.

Case 2. The only edges e for which $f(e) = -1$ are pendant edges.

First suppose that there exists a pendant edge $e = uv \in M \cap X^c$ with $\deg(u) = 1$. Then f , restricted to $T - u$, is an SEkSDF for $T - u$. If $k \leq m - 2$, then by the inductive hypothesis

$$\gamma'_{ks}(T) = f(E(T - u)) - 1 \geq k - (m - 1) + 2 - 1 = k - m + 2.$$

If $k = m - 1$, then obviously there exists a vertex $w \in V(T - \{u, v\})$ such that $f(w) \geq 2$. Now the result follows by Theorem A. Thus we may assume $M \subseteq X$. This implies that for each non leaf vertex $v \in V(T)$, $f(v) \geq 1$. Thus f is an SEDF for T and the result follows by Theorem A. This proves the first statement.

If k is odd, then $2|P| \neq k + 2$ and so $\gamma'_{ks}(G) = 2|P| - m \neq k - m + 2$. Hence, $\gamma'_{ks}(T) \geq k - m + 3$.

To prove sharpness, let T be the tree obtained from the star $K_{1,k+1}$, with vertex set $\{v, v_1, \dots, v_{k+1}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k+1\}$, by adding $m-k-1$ pendant edges $v_1u_1, \dots, v_1u_{m-k-1}$. Obviously, $\gamma'_{ks}(T) = k-m+2$ for k even and $\gamma'_{ks}(T) = k-m+3$ for k odd. This completes the proof. \square

Now we find a lower bound for the SEkSDNs of simple connected graphs. This is a generalization of Theorem B.

Theorem 10. Let G be a simple connected graph of order $n \geq 3$, size m and $1 \leq k \leq m-1$. Then

$$\gamma'_{ks}(G) \geq n + k + 1 - 2m.$$

Furthermore, the bound is sharp for each odd $k \geq 7$.

Proof. The proof is by induction on m . Obviously, the statement is true for $m = 2, 3$. Assume the statement is true for all simple connected graphs of size less than m , where $m \geq 4$. Let G be a simple connected graph of size m and let f be a $\gamma'_{ks}(G)$ -function. we distinguish two cases.

Case 1 There is a non-pendant edge $e = uv \in E(G)$ for which $f(e) = -1$. First let e not be a bridge. If $e \notin X = \{e \in E(T) \mid f[e] \geq 1\}$, then f is an SEkSDF of $G - e$. If $k \leq m-2$, then by the inductive hypothesis

$$\begin{aligned} \gamma'_{ks}(G) &= f(E(G)) = f(E(G-e)) - 1 \\ &\geq n + k + 1 - 2(m-1) - 1 = n + k + 2 - 2m. \end{aligned}$$

If $k = m-1$, then f is an SEDF of $G - e$ and by Theorem B we have

$$\begin{aligned} \gamma'_{ks}(G) &= f(E(G)) = f(E(G-e)) - 1 \\ &\geq n - (m-1) - 1 = n - m = n + k + 1 - 2m. \end{aligned}$$

Let $e \in X$. If f , restricted to $G - e$, is an SEDF of $G - e$, then the result follows as above. Otherwise f , restricted to $G - e$, is an SE(k-1)SDF of $G - e$ and the result follows by the inductive hypothesis.

Assume e is a bridge and G_1 and G_2 are the connected components of $G - e$. An argument similar to that described in the proof of Case 1 of Theorem 9 shows that $\gamma'_{ks}(G) \geq n + k + 1 - 2m$. Note that for this case we apply Theorem C instead of Theorem A.

Case 2 The only edges e for which $f(e) = -1$ are pendant edges.

First suppose that there exists a pendant edge $e = uv \in M \cap X^c$ with $\deg(u) = 1$. Then the result follows by an argument similar to that described in Case 2 of the proof of Theorem 9 and Theorem C. Now assume $M \subseteq X$. If G is a tree, then the statement is true by Theorem 9. Now let G have a cycle C . By assumption $f(e) = 1$ for every $e \in E(C)$. First assume

$f(e') = 1$ for every edge e' with an end-vertex in $V(C)$. Then $f|_{G-e}$ is an SE(k-1)SDF of $G - e$ for every $e \in E(C)$. So we have

$$\begin{aligned} f(E(G)) &= f|_{G-e}(E(G - e)) + 1 \\ &\geq n + (k - 1) + 1 - 2(m - 1) + 1 \\ &= n + k - 2m + 3. \end{aligned}$$

Now assume there is an edge $e' = uu'$ with $f(e') = -1$ and $u \in V(C)$. Hence e' is a pendant edge by assumption. Let $e = uv \in E(C)$. If for every edge e'' at v , $f(e'') = 1$, then f restricted to $G_1 = G - \{e, e'\}$ is an SE(k-2)SDF of G_1 . So we have

$$\begin{aligned} f(E(G)) &= f|_{G_1}(E(G_1)) \\ &\geq (n - 1) + (k - 2) + 1 - 2(m - 2) \\ &\geq n + k + 2 - 2m. \end{aligned}$$

Finally, let there be an edge e'' at v with $f(e'') = -1$. Note that by assumption e'' is a pendant edge. Hence, f restricted to $G_2 = G - \{e, e', e''\}$ is an SE(k-3)SDF of G_2 . So we have

$$\begin{aligned} f(E(G)) &= f|_{G_2}(E(G_2)) - 1 \\ &\geq (n - 2) + (k - 3) + 1 - 2(m - 3) - 1 \\ &\geq n + k + 1 - 2m. \end{aligned}$$

To prove sharpness, we consider two cases.

- $k \geq 7$ is odd and $k = m - 1$. Let G be obtained from star $K_{1, k-3}$ with vertex set $\{v, v_1, \dots, v_{k-3}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-3\}$ by adding three pendant edges $v_1v'_1, v_2v'_2, v_3v'_3$ and an edge v_1v_2 . Define $f : V(G) \rightarrow \{-1, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an SEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.
- $k \geq 7$ is odd and $k \leq m - 2$. Let G be obtained from star $K_{1, k-2}$ with vertex set $\{v, v_1, \dots, v_{k-2}\}$ and edge set $\{vv_i \mid 1 \leq i \leq k-2\}$ by adding pendant edges $v_1v'_1, v_2v'_2, v_3v'_3$ for $j = 1, 2, \dots, m - k - 1$ and v_1v_2 . Define $f : V(G) \rightarrow \{-1, 1\}$ by $f(v_1v_2) = 1$, $f(vv_i) = 1$ if $1 \leq i \leq \frac{k-1}{2}$ and $f(e) = -1$ otherwise. Then f is an SEkSDF of G with $f(E(G)) = n + k + 1 - 2m$.

This completes the proof. □

Acknowledgement. The authors would like to thank the referee whose suggestions were most helpful in writing the final version of this paper.

References

- [1] E.J. Cockayne and C. Mynhardt, *On a generalization of signed dominating functions of graphs*, *Ars Combin.* **43** (1996), 235–245.
- [2] J.H. Hattingh, M.A. Henning and E. Ungerer, *Partial signed domination in graphs*, *Ars Combin.* **48** (1998), 33–42.
- [3] L. Kang, C. Dang, M. Cai and E. Shan, *Upper bounds for the k -subdomination number of graphs*, *Discrete Math.* **247** (2002), 229–234.
- [4] H. Karami, A. Khodkar and S.M. Sheikholeslami, *Signed edge domination numbers in trees*, *Ars Combin.* (to appear).
- [5] H. Karami, A. Khodkar and S.M. Sheikholeslami, *Some notes on signed edge domination in graphs*, *Graphs and Combin.* (to appear).
- [6] H. Karami, A. Khodkar, and S.M. Sheikholeslami, *An improved upper bound for signed edge domination numbers in graphs*, *Utilitas Mathematica* (to appear).
- [7] H. Karami, A. Khodkar and S.M. Sheikholeslami, *Signed edge majority domination numbers in graphs*, *Australas. J. Combin.* (to appear).
- [8] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.
- [9] B. Xu, *On signed edge domination numbers of graphs*, *Discrete Math.* **239** (2001), 179–189.
- [10] B. Xu, *On lower bounds of signed edge domination numbers in graphs*, *J. East China Jiaotong Univ.* **1** (2004), 110–114 (In Chinese).
- [11] B. Xu, *On edge domination numbers of graphs*, *Discrete Math.* **294**, (2005) 311–316.
- [12] B. Xu, *Two classes of edge domination in graphs*, *Discrete Applied Math.* **154** (2006), 1541–1546.
- [13] F. Xueliang, Y. Yuansheng and J. Baoqi, *A note on the signed edge domination number in graphs*, *Discrete Math.* (to appear).
- [14] B. Zelinca, *On signed edge domination numbers of trees*, *Mathematica Bohemica* **127** (2002), 49–55.