Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U(3,q),\ q = 3,4,5,7$

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Key words: block design, projective plane, unital, unitary group, primitive group.
AMS 2000 Subject Classification: 05B05, 05E20, 05E30

Abstract

Let G be a finite permutation group acting primitively on sets Ω_1 and Ω_2. We describe a construction of a 1-design with the block set Ω_1 and the point set Ω_2, having G as an automorphism group. Applying this method, we construct a unital $2-(q^3+1, q+1, 1)$, and a semi-symmetric design $(q^4-q^3+q^2, q^2-q, (1))$ from the unitary group $U(3,q),\ q = 3,4,5,7$. From the unital and the semi-symmetric design we build a projective plane $PG(2,q^2)$. Further, we describe other combinatorial structures constructed from these unitary groups.

1 Introduction

An incidence structure is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks and $\mathcal{I}$ is called an incidence relation. If $|\mathcal{P}| = |\mathcal{B}|$ then the incidence structure is called symmetric. The incidence matrix of an incidence structure is a $b \times v$ matrix $[m_{ij}]$ where b and v are the number of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and block x_i are incident, and $m_{ij} = 0$ otherwise. An isomorphism from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A $t-(v,k,\lambda)$ design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,

3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

A $2-(v,k,\lambda)$ design is called a block design. A $2-(v,k,\lambda)$ design is called quasi-symmetric if the number of points in the intersection of any two blocks takes only two values. A symmetric $2-(v,k,1)$ design is called a projective plane.

A semi-symmetric $(v,k,(\lambda))$ design is a finite incidence structure with v points and v blocks satisfying:

1. every point (block) is incident with exactly k blocks (points),

2. every pair of points (blocks) are incident with 0 or λ blocks (points).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a finite incidence structure. $\mathcal{G}$ is a graph if each element of $\mathcal{E}$ is incident with exactly two elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of $\mathcal{E}$ are called edges. Two vertices u and v are called adjacent or neighbors if they are incident with the same edge. The number of neighbors of a vertex v is called the degree of v. If all the vertices of the graph $\mathcal{G}$ have the same degree k, then $\mathcal{G}$ is called k-regular. Define a square $\{0,1\}-$matrix $A = (a_{uv})$ labelled with the vertices of $\mathcal{G}$ in such a way that $a_{uv} = 1$ if and only if the vertices u and v are adjacent. The matrix A is called the adjacency matrix of the graph $\mathcal{G}$. An automorphism of a graph is any permutation of the vertices preserving adjacency. The set of all automorphisms forms the full automorphism group of the graph.

A graph $\mathcal{G}$ is called a strongly regular graph with parameters (n,k,λ,μ), and denoted by $SRG(n,k,\lambda,\mu)$, if $\mathcal{G}$ is $k-$regular graph with n vertices and if any two adjacent vertices have λ common neighbors and any two non-adjacent vertices have μ common neighbors.

Let x and y $(x<y)$ be the two cardinalities of block intersections in a quasi-symmetric design $\mathcal{D}$. The block graph of $\mathcal{D}$ has as vertices the blocks of $\mathcal{D}$ and two vertices are adjacent if and only if they intersect in y points. The block graph of a quasi-symmetric $2-(v,k,\lambda)$ design is strongly regular. In a $2-(v,k,1)$ design which is not a projective plane two blocks intersect in 0 or 1 points, therefore the block graph of this design is strongly regular (see [2]).

Let G be a simple group and H be a maximal subgroup of G. The conjugacy class of H in G is denoted by $ccl_G(H)$. Obviously $N_G(H) = H$, so $|ccl_G(H)| = [G:H]$. Denote the elements of $ccl_G(H)$ by $H^{g_1}, H^{g_2}, \ldots, H^{g_j}$, $j = [G:H]$, $g_i \in G$ for $i = 1,\ldots,j$.

In this paper we consider combinatorial structures constructed from the unitary group $U(3,q)$, $q = 3,4,5,7$. We define incidence structures on the elements of conjugacy classes of maximal subgroups, i.e. points and blocks are labelled by elements of conjugacy classes of maximal subgroups of simple groups $U(3,q)$. The construction used in this paper is a generalization of the construction described in [9].

The paper is arranged as follows: in Section 2 we explain the method of construction. In the sections 3, 4, 5, and 6 we describe the construction of designs and strongly regular graphs from the groups $U(3,q)$, $q = 3,4,5,7$. In particular, from the unitary group $U(3,q)$, $q = 3,4,5,7$, we construct a unital $2-(q^3+1, q+1, 1)$ and a semi-symmetric design $(q^4-q^3+q^2, q^2-q, (1))$, which we use to build a projective plane $PG(2,q^2)$. In Section 7 we summarize the results.

Generators of the groups $U(3,q)$, $q = 3,4,5,7$, and their maximal subgroups are available on the Internet:

http://brauer.maths.qmul.ac.uk/Atlas/clas/.

For basic definitions and group theoretical notation used in this paper and we refer the reader to [3] and [12].

2 The construction

The following construction of symmetric 1-designs and regular graphs is presented in [9]:

Theorem 1 *Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If $\mathcal{B} = \{\Delta g : g \in G\}$ and, given $\delta \in \Delta$, $\mathcal{E} = \{\{\alpha,\delta\}g : g \in G\}$, then $\mathcal{D} = (\Omega, \mathcal{B})$ forms a symmetric $1-(n, |\Delta|, |\Delta|)$ design. Further, if Δ is a self-paired orbit of G_α then $\Gamma(\Omega, \mathcal{E})$ is a regular connected graph of valency $|\Delta|$, $\mathcal{D}$ is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.*

The construction described in Theorem 1 produces symmetric 1-designs admitting a primitive action of a group G, such that a stabilizer of a point and a stabilizer of a block are conjugate in G. The following generalization of the above construction allows us to construct primitive 1-designs which are not necessarily symmetric, and stabilizers of a point and a block are not necessarily conjugate:

Theorem 2 *Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$, $\delta \in \Omega_2$, and let $\Delta_2 = \delta G_\alpha$ be the G_α-orbit of $\delta \in \Omega_2$ and $\Delta_1 = \alpha G_\delta$ be the G_δ-orbit of $\alpha \in \Omega_1$. If $\Delta_2 \neq \Omega_2$ and*

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G,\alpha,\delta) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, |\Delta_1|)$ design with m blocks, and G acts as an automorphism group, primitive on points and blocks of the design.

Proof It is clear that the number of points $v = n$, since the point set is $\mathcal{P} = \Omega_2$, and also that each element of $\mathcal{B}$ consists of $k = |\Delta_2|$ elements of Ω_2.

Since Δ_2 is a G_α-orbit, we have $G_\alpha \subseteq G_{\Delta_2}$, where G_{Δ_2} is the setwise stabilizer of Δ_2. Since G is primitive on Ω_1, G_α is a maximal subgroup of G, and therefore $G_{\Delta_2} = G_\alpha$. The number of blocks is

$$b = |\Delta_2 G| = \frac{|G|}{|G_{\Delta_2}|} = \frac{|G|}{|G_\alpha|} = |\Omega_1| = m.$$

Since G acts transitively on Ω_1 and Ω_2 the constructed structure is a 1-design, hence $bk = vr$, where each point is incident with r blocks. Therefore

$$|\Omega_1|\ |\Delta_2| = |\Omega_2|\ r,$$

and consequently

$$\frac{|G|}{|G_\alpha|}\ \frac{|G_\alpha|}{|(G_\alpha)_\delta|} = \frac{|G|}{|G_\delta|}\ r.$$

It follows that

$$r = \frac{|G_\delta|}{|(G_\alpha)_\delta|} = \frac{|G_\delta|}{|(G_\delta)_\alpha|} = |\alpha G_\delta| = |\Delta_1|.$$

□

We can interpret the construction of a design from Theorem 2 in the following way:

- the point set is $\Omega_2 = \delta G$, and the block set is $\Omega_1 = \alpha G$,
- the block $\alpha g'$ is incident with the set of points $\{\delta g : g \in G_\alpha g'\}$.

Let a point $\delta g \in \Omega_2$ be incident with a block $\alpha g' \in \Omega_1$. Then $g \in G_\alpha g'$, hence there exists $\overline{g} \in G_\alpha$ such that $g = \overline{g} g'$. Therefore,

$$G_{\alpha g'} \cap G_{\delta g} = G_{\alpha g'} \cap G_{\delta \overline{g} g'} = G_\alpha^{g'} \cap G_{\delta \overline{g}}^{g'} = (G_\alpha \cap G_{\delta \overline{g}})^{g'} =$$

$$(G_\alpha \cap G_\delta^{\overline{g}})^{g'} = (G_\alpha^{\overline{g}^{-1}} \cap G_\delta)^{\overline{g} g'} = (G_\alpha \cap G_\delta)^{\overline{g} g'} = (G_\alpha \cap G_\delta)^{g}.$$

If a point $\delta g \in \Omega_2$ is incident with the block $\alpha \in \Omega_1$, then $G_\alpha \cap G_{\delta g} = (G_\alpha \cap G_\delta)^g$. If the set $\{G_\alpha \cap G_{\delta g} \mid g \in G\}$ contains $Orb(G_\alpha, \Omega_2)$ G_α-conjugacy classes, where $Orb(G_\alpha, \Omega_2)$ is the number of G_α-orbits on Ω_2, then each conjugacy class corresponds to one G_α-orbit, and the incidence relation in the design $\mathcal{D}(G, \alpha, \delta)$ can be defined as follows:

- the block $\alpha g'$ is incident with the point δg if and only if $G_{\alpha g'} \cap G_{\delta g}$ is conjugate to $G_\alpha \cap G_\delta$.

Similarly, if the set $\{G_\alpha \cap G_{\delta g} \mid g \in G\}$ contains $Orb(G_\alpha, \Omega_2)$ isomorphism classes, then the incidence in the design $\mathcal{D}(G, \alpha, \delta)$ can be defined as follows:

- the block $\alpha g'$ is incident with the point δg if and only if $G_{\alpha g'} \cap G_{\delta g} \cong G_\alpha \cap G_\delta$,

In the construction of the design $\mathcal{D}(G, \alpha, \delta)$ described in Theorem 2, instead of taking a single G_α-orbit, we can take Δ_2 to be any union of G_α-orbits. In fact, this construction gives us all designs on which the group G acts primitively on points and blocks:

Corollary 1 *If the group G acts primitively on the points and the blocks of a 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem 2, where Δ_2 is a union of G_α-orbits. The set Δ_1 of blocks incident with the point δ is a union of G_δ-orbits.*

Proof Let α be any block of the design $\mathcal{D}$. G acts transitively on the block set $\mathcal{B}$ of the design $\mathcal{D}$, hence $\mathcal{B} = \alpha G$. Since G acts primitively on $\mathcal{B}$, the stabilizer G_α is a maximal subgroup of G. G_α fixes α, so α is a union of G_α-orbits. In a similar way one concludes that Δ_1 is a union of G_δ-orbits. □

Let G be a simple group, and let H and K be maximal subgroups of G. Then $|ccl_G(H)| = [G : H]$, $|ccl_G(K)| = [G : K]$, and G acts primitively on $ccl_G(H)$ and $ccl_G(K)$ by conjugation. The stabilizers of H^x and K^y, $x, y \in G$, are H^x and K^y, respectively. We can construct a 1-design as follows:

- the point set is $ccl_G(K)$, and the block set is $ccl_G(H)$,
- the block H^{h_i} is incident with K^{k_j} if and only if $H^{h_i} \cap K^{k_j} \cong H \cap K$.

In this article we use the above described construction of designs from conjugacy classes of maximal subgroups of a simple group to obtain designs from the unitary groups $U(3, q)$, $q = 3, 4, 5, 7$.

3 Constructions from $U(3, 5)$

We consider structures constructed from the unitary group $U(3, 5)$, the classical simple group of order 126000. The group $U(3, 5)$ has eight distinct conjugacy classes of maximal subgroups: $H_1 \cong Z_2.A_5.Z_2$, $H_2 \cong H_3 \cong H_4 \cong A_6.Z_2$, $H_5 \cong (E_{25} : Z_5) : Z_8$, and $H_6 \cong H_7 \cong H_8 \cong A_7$.

3.1 The projective plane $PG(2, 25)$ constructed from the group $U(3, 5)$

Let G be a group isomorphic to the unitary group $U(3,5)$, and $H_1 \cong Z_2.A_5.Z_2$, $H_5 \cong (E_{25} : Z_5) : Z_8$ be maximal subgroups of G. The cardinality of the conjugacy class $ccl_G(H_1)$ is 525, and the cardinality of the conjugacy class $ccl_G(H_5)$ is 126. Using GAP ([11]), one can check that $H_1^x \cap H_5^y \cong Z_5 : Z_8$ or Z_2 for all $x, y, \in G$. Further, for every H_1^x,

$$|\{H_5^y \mid y \in G,\ H_1^x \cap H_5^y \cong Z_5 : Z_8\}| = 6.$$

Let us define sets $S_i = \{H_1^{g_j} \in ccl_G(H_1) \mid H_1^{g_j} \cap H_5^{h_i} \cong Z_5 : Z_8\}$, $1 \leq i \leq 126$. For every $1 \leq i, j \leq 126$, $i \neq j$, the set $S_i \cap S_j$ has exactly one element. That proves that the incidence structure $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathcal{I}_1)$ where $\mathcal{P}_1 = \{P_1, \ldots, P_{126}\}$, $\mathcal{B}_1 = \{x_1, \ldots, x_{525}\}$, and

$$(P_i, x_j) \in \mathcal{I}_1 \Leftrightarrow (H_1^{g_j} \cap H_5^{h_i} \cong Z_5 : Z_8)$$

is a block design $2-(126, 6, 1)$. Let us denote the design obtained in this way by $\mathcal{D}(U(3,5), H_5, H_1; Z_5 : Z_8)$. The full automorphism group of $\mathcal{D}_1$ is isomorphic to the group $Aut(U(3,5)) \cong U(3,5) : S_3$, of order 756000.

The intersection of two distinct elements of $ccl_G(H_1)$ is isomorphic to Z_2, Z_5, or $Z_3 : E_4$. One can check that $\mathcal{S} = \mathcal{D}(U(3,5), H_1, H_1; Z_3 : E_4)$ is a semi-symmetric design $(525, 20, (1))$, and $Aut(\mathcal{S}) \cong Aut(U(3,5)) \cong U(3,5) : S_3$.

Let M_1 and M be the incidence matrices of $\mathcal{D}_1$ and $\mathcal{S}$, respectively, and I_{126} be the identity matrix of order 126. Then the matrix

$$P = \begin{bmatrix} I_{126} & M_1^T \\ M_1 & \mathrm{M} \end{bmatrix}$$

is the incidence matrix of the Desarguesian projective plane $PG(2, 25)$, i.e., a symmetric $2-(651, 26, 1)$ design. $Aut(PG(2,25)) \cong P\Gamma L(3, 25)$, of order 304668000000. P is a symmetric matrix, and therefore the projective plane admits a unitary polarity (for the definition see e.g. [7]). The absolute points and blocks are the conjugates of H_5, and the non-absolute points and blocks are the conjugates on H_1. $\mathcal{D}_1$ is the Hermitian unital in $PG(2, 25)$.

$\mathcal{D}_1$ is a block design with blocks intersection sizes 1 and 0, and its block graph is a strongly regular graph with parameters (525,144,48,36). Denote this graph by $\mathcal{G}_1$. It can be obtained directly from the conjugates of H_1. The adjacency matrix of the graph $\mathcal{G}_1$ is the matrix $A_1 = (a_{ij}^{(1)})$ defined as follows:

$$a_{ij}^{(1)} = \begin{cases} 1, & \text{if } H_1^{g_i} \cap H_1^{g_j} \cong Z_5, \\ 0, & \text{otherwise.} \end{cases}$$

Let us denote this graph by $\mathcal{G}(U(3,5), H_1; A_5)$. The group $Aut(\mathcal{G}_1)$ is isomorphic to $U(3,5) : S_3$.

3.2 Construction of block designs 2-(50,14,13) and 2-(126,36,14)

The cardinality of $ccl_G(H_3)$ is 175, the cardinality of $ccl_G(H_8)$ is 50, and $H_3^x \cap H_8^y \cong Z_5 : Z_4$, A_5, or A_6 for all $x, y, \in G$. The design $\mathcal{D}_2 = \mathcal{D}(U(3,5), H_8, H_3; A_5 \text{ or } A_6)$ is a 2-design with parameters (50,14,13), and $Aut(\mathcal{D}_2) \cong U(3,5) : Z_2$. $\mathcal{D}_2$ is a derived design of the Higman design $2-(176,50,14)$ (see [10]). The designs $\mathcal{D}(U(3,5), H_6, H_2; A_5 \text{ or } A_6)$ and $\mathcal{D}(U(3,5), H_7, H_4; A_5 \text{ or } A_6)$ are isomorphic to $\mathcal{D}_2$.

The cardinality of $ccl_G(H_5)$ is 126, the cardinality of $ccl_G(H_4)$ is 175, and $H_4^x \cap H_5^y \cong Z_8$ or $Z_5 : Z_4$ for all $x, y, \in G$. $\mathcal{D}_3 = \mathcal{D}(U(3,5), H_5, H_4; Z_5 : Z_4)$ is a 2-(126,36,14) design, and $Aut(\mathcal{D}_3) \cong U(3,5) : Z_2$. $\mathcal{D}_3$ is a residual design of the Higman design $2-(176,50,14)$ (see [10]). The designs $\mathcal{D}(U(3,5), H_5, H_3; Z_5 : Z_4)$ and $\mathcal{D}(U(3,5), H_5, H_2; Z_5 : Z_4)$ are isomorphic to $\mathcal{D}_3$.

3.3 Construction of strongly regular graphs $(50,7,0,1)$ and $(175,72,20,36)$

The cardinality of the conjugacy class $ccl_G(H_6)$ is 50. For $H_6^x \neq H_6^y$, $H_6^x \cap H_6^y \cong A_5$ or A_6. $\mathcal{G}_2 = \mathcal{G}(U(3,5), H_6; A_6)$ is a $SRG(50,7,0,1)$. $Aut(\mathcal{G}_2) \cong U(3,5) : Z_2$, of order 252000. $\mathcal{G}_2$ is the unique strongly regular graph with these parameters, i.e., the Hoffman-Singleton graph (see [1]). The graphs $\mathcal{G}(U(3,5), H_7; A_6)$ and $\mathcal{G}(U(3,5), H_8; A_6)$ are isomorphic to $\mathcal{G}_2$.

The cardinality of the conjugacy class $ccl_G(H_2)$ is 175. The intersection of two distinct elements of $ccl_G(H_2)$ is isomorphic to Q_8, D_{10}, or A_5. $\mathcal{G}_3 = \mathcal{G}(U(3,5), H_2; D_{10})$ is a $SRG(175,72,20,36)$. $Aut(\mathcal{G}_3) \cong U(3,5) : Z_2$, of order 252000. $\mathcal{G}_3$ is the graph whose vertices are edges of the Hoffman-Singleton graph $\mathcal{G}_2$, two vertices being adjacent if their distance is two (see [10]). The graphs $\mathcal{G}(U(3,5), H_3; D_{10})$ and $\mathcal{G}(U(3,5), H_4; D_{10})$ are isomorphic to $\mathcal{G}_3$.

The graph $\mathcal{G}_3$ can be constructed from the designs $\mathcal{D}_2$ and $\mathcal{D}_3$. Any two blocks of $\mathcal{D}_2$ intersect in 3, 4, or 8 points. The graph which has as its vertices the blocks of $\mathcal{D}_2$, two vertices being adjacent if and only if the corresponding blocks intersect in three points, is isomorphic to $\mathcal{G}_3$. Denote this graph by $\mathcal{G}(\mathcal{D}_2, \{3,4,8\}; 3)$. The graph $\mathcal{G}(\mathcal{D}_3, \{6,10,11\}; 11)$ is also isomorphic to $\mathcal{G}_3$.

4 The projective plane $PG(2,9)$ constructed from the group $U(3,3)$

The unitary group $U(3,3)$ is the simple group of order 6048, and it has four distinct conjugacy classes of maximal subgroups: $K_1 \cong L(2,7)$, $K_2 \cong Z_4.S_4$, $K_3 \cong (Z_4 \times Z_4) : S_3$, and $K_4 \cong ((Z_3 \times Z_3) : Z_3) : Z_8$.

$\mathcal{D}(U(3,3), K_4, K_2; Z_3 : Z_8)$ is a block design $2-(28,4,1)$, with the full automorphism group isomorphic to $Aut(U(3,3)) \cong U(3,3) : Z_2$ (see [4]).

The cardinality of $ccl_{U(3,3)}(K_2)$ is 63, since $[U(3,3) : K_2] = 63$. The intersection of two distinct elements from $ccl_{U(3,3)}(K_2)$ is isomorphic to Z_3, Z_4 or $Z_4 \times Z_4$. $\mathcal{D}(U(3,3), K_2, K_2; Z_4 \times Z_4)$ is a semi-symmetric design $(63,6,(1))$ with the full automorphism group isomorphic to $Aut(U(3,3)) \cong U(3,3) : Z_2$.

Let M_1 and M be the incidence matrices of the constructed block design $2-(28,4,1)$ and the semi-symmetric design $(63,6,(1))$, respectively, and I_{28} be the identity matrix of order 28. Then the matrix

$$P = \begin{bmatrix} I_{28} & M_1^T \\ M_1 & \mathrm{M} \end{bmatrix}$$

is the incidence matrix of the Desarguesian projective plane $PG(2,9)$, and $Aut(PG(2,9)) \cong P\Gamma L(3,9)$. P is a symmetric matrix, hence the design $2-(28,4,1)$ is the Hermitian unital in $PG(2,9)$. For other structures constructed from $U(3,3)$ see [4] and [5].

5 The projective plane $PG(2,16)$ constructed from the group $U(3,4)$

In [6] we have constructed, from the group $U(3,4)$, a 2-design with parameters $(65,5,1)$ and a semi-symmetric design $(208,12,(1))$, both having $Aut(U(3,4)) \cong U(3,4) : Z_4$ as the full automorphism group. From these structures we have constructed the Desarguesian projective plane $PG(2,16)$, $Aut(PG(2,16)) \cong P\Gamma L(3,16)$, in the same way as $PG(2,25)$ is constructed from $U(3,5)$ in Section 3, and $PG(2,9)$ is constructed from $U(3,3)$ in Section 4. The block design $2-(65,5,1)$ is the Hermitian unital in $PG(2,16)$. For construction of $PG(2,16)$ and the other structures from the group $U(3,4)$ see [6].

6 The projective plane $PG(2,49)$ constructed from the group $U(3,7)$

The unitary group $U(3,7)$ is the simple group of order 5663616. $L_1 \cong (E_{49} : Z_7) : Z_{48}$ and $L_2 \cong Z_2.(L(2,7) \times Z_4).Z_2$ are maximal subgroups of $U(3,7)$, $[U(3,7) : L_1] = 344$ and $[U(3,7) : L_2] = 2107$. One can check that $L_1^x \cap L_2^y \cong Z_7 : Z_{48}$ or Z_8 for all $x, y, \in U(3,7)$, and $\mathcal{D}(U(3,7), L_1, L_2; Z_7 : Z_{48})$ is a block design $2-(344,8,1)$ with the full automorphism group isomorphic to the group $AutU(3,7) \cong U(3,7) : Z_2$. The intersection of two distinct elements of $ccl_{U(3,7)}(L_2)$ is isomorphic to Z_7, Z_8, or $Z_8 \times Z_8$. $\mathcal{D}(U(3,7), L_2, L_2; Z_4 \times Z_4)$ is a semi-symmetric design with parameters $(2107, 42, (1))$.

Let M_1 and M be the incidence matrices of the block design $2-(344,8,1)$ and the semi-symmetric design $(2107, 42, (1))$, respectively, and I_{344} be the identity matrix of order 344. Then the matrix

$$P = \begin{bmatrix} I_{344} & M_1^T \\ M_1 & \text{M} \end{bmatrix}$$

is the incidence matrix of the Desarguesian projective plane $PG(2,49)$ with $Aut(PG(2,49)) \cong P\Gamma L(3,49)$. The matrix P is a symmetric matrix, so the design $2-(344,8,1)$ is the Hermitian unital in $PG(2,49)$.

7 Conclusions

In Table 1 and Table 2 we give the list of the constructed structures and describe their full automorphism groups. For other combinatorial structures constructed from $U(3,3)$ and $U(3,4)$ we refer the reader to [4], [5], and [6].

Table 1: The structures constructed from $U(3,5)$

Combinatorial structure	Order of the full automorphism group	Structure of the full automorphism group
2-(126,6,1) design	756000	$U(3,5) : S_3$
(525,20,(1)) design	756000	$U(3,5) : S_3$
$PG(2,25)$	304668000000	$P\Gamma L(3,25)$
$SRG(525,144,48,36)$	756000	$U(3,5) : S_3$
2-(50,14,13) design	252000	$U(3,5) : Z_2$
2-(126,36,14)design	252000	$U(3,5) : Z_2$
$SRG(50,7,0,1)$	252000	$U(3,5) : Z_2$
$SRG(175,72,20,36)$	252000	$U(3,5) : Z_2$

Table 2: The structures constructed from $U(3,3)$, $U(3,4)$, and $U(3,7)$

Combinatorial structure	Order of the full automorphism group	Structure of the full automorphism group
2-(28,4,1) design	12096	$U(3,3):Z_2$
(63,6,(1)) design	12096	$U(3,3):Z_2$
$PG(2,9)$	84913920	$P\Gamma L(3,9)$
2-(65,5,1) design	249600	$U(3,4):Z_4$
(208,12,(1)) design	249600	$U(3,4):Z_4$
$PG(2,16)$	34217164800	$P\Gamma L(3,16)$
2-(344,8,1) design	11327232	$U(3,7):Z_2$
(2107,42,(1)) design	11327232	$U(3,7):Z_2$
$PG(2,49)$	66437613849600	$P\Gamma L(3,49)$

The results obtained in this article lead us to the conjecture that from any group $U(3,q)$, defining incidence structures on conjugacy classes of maximal subgroups, one can construct a Hermitian unital $2-(q^3+1,q+1,1)$ and a semi-symmetric design $(q^4-q^3+q^2,q^2-q,(1))$ having $Aut(U(3,q))$ as an automorphism group, which can be used to build a Desarguesian projective plane $PG(2,q^2)$ (in the way presented in this article).

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