Hypercube-Anti-Ramsey Numbers of Q_5

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A rainbow coloring of the edges of a graph is a coloring such that no two edges of the graph have the same color. The anti-Ramsey number $f(G, H)$ is the maximum number of colors such that there is an H-anti-Ramsey edge coloring of G, that is, there exists no rainbow copy of the subgraph H of G in some coloring of the edges of the host graph G with $f(G, H)$ colors. In this note we exactly determine $f(Q_5, Q_2)$ and $f(Q_5, Q_3)$ where Q_n is the n-dimensional hypercube.

1. Introduction

Given a host graph G and a subgraph $H \subseteq G$, a coloring of the edges of G is called H-anti-Ramsey iff every copy of H in G has at least two edges of the same color. A rainbow coloring of the edges of a graph is a coloring such that no two edges of the graph have the same color. The anti-Ramsey number $f(G, H)$ is the maximum number of colors such that there is an H-anti-Ramsey edge coloring of G, that is, there exists no rainbow copy of H in some coloring of the edges of G with $f(G, H)$ colors.

The function $f(G, H)$ was introduced by Erdős, Simonovits and Sós [4]. In most of the papers on this function complete host graphs $G \cong K_n$ are considered [5,8,9]. Just recently, Montellano-Ballesteros and Neumann-Lara solved the case $f(K_n, C_k)$ where C_k is a cycle on k vertices [6]. There are some results for bipartite graphs G [2,3,7] and hypercubes $G \cong Q_n$ [1].

A hypercube Q_n consists of the 2^n vertices $(a_1, \ldots, a_n)$, $a_i \in \{0, 1\}$, $i = 1, \ldots, n$, such that two vertices are adjacent iff the corresponding sequences differ in exactly one position (see Figure 1 for $Q_2, \ldots, Q_5$).

In Figure 1 the hypercubes are drawn such that all vertices with the same number i of 1s, $i = 0, \ldots, n$, in the corresponding sequence are in layer V_i. In between vertex layers V_{i-1} and V_i there is the edge layer E_i, $i = 1, \ldots, n$. Such drawings are called Hasse diagrams of Q_n.

In [1] the following general bounds for $f(Q_n, Q_k)$ are proved:

$$n2^{n-1} - \left\lfloor \frac{n}{k}(2^{n-1} - k + 1) \right\rfloor \le f(Q_n, Q_k) \le n2^{n-1} \left(1 - \frac{n-k}{(n-1)k2^{k-2}}\right) \quad (1)$$

For $k = n - 1$ the exact value of $f(Q_n, Q_k)$ is determined in [1]:

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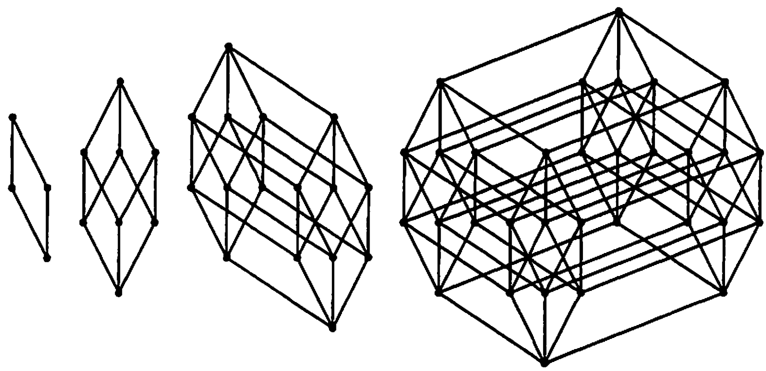

Figure 1. The hypercubes $Q_2, \ldots, Q_5$.

$$f(Q_n, Q_{n-1}) = \begin{cases} n2^{n-1} - 4 & \text{for } n = 3,4,5, \\ n2^{n-1} - 3 & \text{for } n \geq 6. \end{cases} \tag{2}$$

In this note we investigate $f(Q_5, Q_k)$. From (2) we have $f(Q_5, Q_4) = 76$. In the following we prove $f(Q_5, Q_3) = 68$ and $f(Q_5, Q_2) = 43$ such that the hypercube-anti-Ramsey numbers with Q_5 as host graph are completely determined. In Table 1 known values and bounds for $f(Q_n, Q_k)$ according to (1) and Proposition 1 are summarized.

	$k = 2$	3	4	5	6
$n = 3$	8				
4	18	28			
5	43	68	76		
6	99...102	132...172	149...187	189	

Table 1. $f(Q_n, Q_k)$.

2. $f(Q_n, Q_2)$

Choosing $k = 2$ in (1) we obtain

$$n2^{n-2} + \left\lceil \frac{n}{2} \right\rceil \leq f(Q_n, Q_2) \leq (n+1)2^{n-2}. \tag{3}$$

If $n = 3$ then lower and upper bound of (3) coincide: $f(Q_3, Q_2) = 8$. It is proved in [1] that also for $n = 4$ the lower bound of (3) is achieved: $f(Q_4, Q_2) = 18$. If $n = 5$ then $43 \leq f(Q_5, Q_2) \leq 48$ by (3). We show in the following that also in this case the lower bound is attained. We use the following observations in the proof:

(A) Each edge of Q_n is contained in $\binom{n-1}{k-1}$ distinct subgraphs Q_k.

(B) Each pair of edges of Q_n is contained in at most $\binom{n-2}{k-2}$ distinct subgraphs Q_k.

(C) A hypercube Q_n contains $2^{n-k}\binom{n}{k}$ copies of Q_k.

(D) A hypercube Q_n contains n pairs of vertex disjoint copies of Q_{n-1}.

Theorem 1. $f(Q_5, Q_2) = 43$.

Proof. The edge coloring that yields the lower bound of (3) is shown in Figure 2 for $n = 5$. In this coloring equally bold marked edges of Q_5 have the same color and different colors are assigned to different bold marks. All other edges are colored pairwise different and also different to the colors of the bold marked edges. All subgraphs Q_2 are nonrainbow since each Q_2 has its edges in 2 successive edge layers.

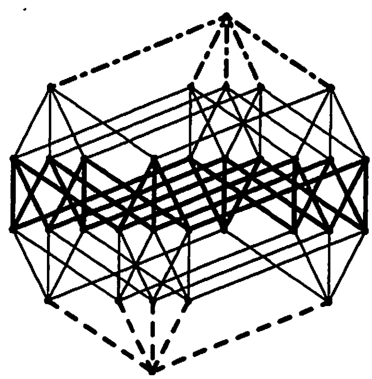

Figure 2. $f(Q_5, Q_2) \geq 43$.

In order to prove $f(Q_5, Q_2) \leq 43$ we assume the existence of an Q_2-anti-Ramsey edge coloring c of Q_5 with at least 44 colors, that is $|c(E(Q_5))| \geq 44$.

At first we show that there exists a pair of vertex disjoint copies of Q_4 in Q_5 such that their edges are colored with at least 36 colors. Otherwise, the edges of all 5 pairs (see (D)) of vertex disjoint copies of Q_4 are colored with at most 35 colors and therefore the edges of Q_5 with at most $\lfloor 35 \cdot 5/4 \rfloor = 43 < 44$ colors since each edge of Q_5 is counted 4 times by (A) and all 10 distinct copies of Q_4 in Q_5 (see (C)) are covered by the 5 pairs, which is a contradiction to the above assumption.

Therefore, there is a pair Q, Q' of vertex disjoint Q_4s in Q_5 whose edges are colored with a least 36 colors. Since $f(Q_4, Q_2) = 18$ we have $|c(E(Q))| = |c(E(Q'))| = 18$ with disjoint colors in Q and Q'.

Let $E = \{uu' : u \in V(Q), u' \in V(Q')\}$ the set of edges between Q and Q' and let $E_1 \subseteq E$ a set which contains exactly one edge of all of the at least

8 colors that are distinct from the colors of Q and Q' and let $E_2 = E \setminus E_1$. Moreover, let V_1 and V_2 be the sets of end-vertices of the edges of E_1 and E_2 in Q, respectively.

Because of $c(e) \neq c(e')$ for all $e \in E(Q)$, $e' \in E(Q')$ there exist no adjacent vertices in V_1 since otherwise a rainbow Q_2 would exist. If we consider a spanning cycle C_{16} in Q then we have at most 8 mutually nonadjacent vertices in C_{16} and therefore also in Q which implies that E_1 has at most 8 edges and therefore $|E_1| = |E_2| = 8$. Therefore, also the vertices of V_2 have pairwise even distance in C_{16} and thus also in Q since Q is bipartite.

Consider the subgraph of Q_5 of Figure 3 where $e_i \in E(Q)$, $e_i' \in E(Q')$, $h \in E_2$, $h_i \in E_1$. Since the cycles (e_1, h_1, e_1', h) and (e_2, h_2, e_2', h) are nonrainbow and $|c(\{e_1, h_1, e_1'\})| = |c(\{e_2, h_2, e_2'\})| = 3$ with $c(h_1) \neq c(h_2)$ we obtain $c(h) = c(e_1) = c(e_2)$ or $c(h) = c(e_1') = c(e_2')$ and therefore $c(e_1) = c(e_2) = c(e_3) = c(e_4)$ or $c(e_1') = c(e_2') = c(e_3') = c(e_4')$, that is, a monochromatic 4-star in Q or in Q'. Using this argument for each edge of E_2 we get 8 monochromatic 4-stars in Q and Q' together which are edge disjoint since the vertices of V_2 are mutually nonadjacent. In Q and Q' there are at most 4 monochromatic 4-stars each since otherwise there would be less than 18 colors in $E(Q)$ or $E(Q')$. This implies that there are exactly 4 monochromatic 4-stars in Q and Q' each.

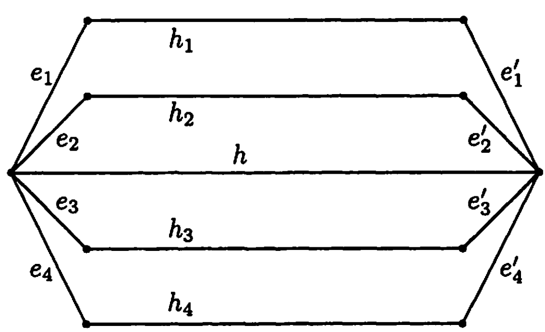

Figure 3. Subgraph of Q_5.

The centers of the 4-stars in $Q \cong Q_4$ are in V_2 and have mutually distance 2 or 4. There are $\binom{4}{2} = 6$ pairs of these stars. Since the center of a 4-star has distance 4 to at most one center of another 4-star in Q there exist at least 4 pairs of 4-stars whose centers have distance 2.

Each monochromatic 4-star in Q contains $\binom{4}{2} = 6$ monochromatic pairs of edges which yields 6 nonrainbow Q_2s. Since every pair of 4-stars whose centers have distance 2 have one nonrainbow Q_2 in common there are at most $4 \cdot 6 - 4 = 20$ nonrainbow Q_2s with two edges in one of the monochromatic 4-stars each.

Since Q has 32 edges there are 16 edges not contained in one of the 4-stars which must be colored with at least $18 - 4 = 14$ colors. Thus, among these 16 edges there is at most one monochromatic triple or there are at most three monochromatic pairs which yields at most 3 monochromatic Q_2s in Q. Edges having a color of one of the monochromatic 4-stars do not yield additional nonrainbow Q_2s since every Q_2 with one edge of a 4-star contains two edges of this star.

Therefore, the number of nonrainbow Q_2s in Q is at most 23. Since Q contains 24 subgraphs Q_2 (see (C)) there is at least one rainbow Q_2 in Q and thus also in Q_5 which contradicts the assumption that there is a Q_2-anti-Ramsey edge coloring of Q_5 with at least 44 colors. □

Choosing $n = 6$ in (3) we obtain $f(Q_6, Q_2) \geq 99$. Using the same idea as in the proof of Theorem 1 it is an easy task to prove $f(Q_6, Q_2) \leq 102$.

Proposition 1. $99 \leq f(Q_6, Q_2) \leq 102$.

3. $f(Q_5, Q_3)$

In this chapter we determine $f(Q_5, Q_3)$ by improving both the lower and upper bound of (1).

Theorem 2. $f(Q_5, Q_3) = 68$.

Proof of $f(Q_5, Q_3) \geq 68$. The edge coloring of Figure 4 shows the lower bound $f(Q_5, Q_3) \geq 68$: The bold marked edges are colored with 4 distinct colors and all other 64 edges with pairwise different new colors. All 40 subgraphs Q_3 contain 2 equally colored edges each. □

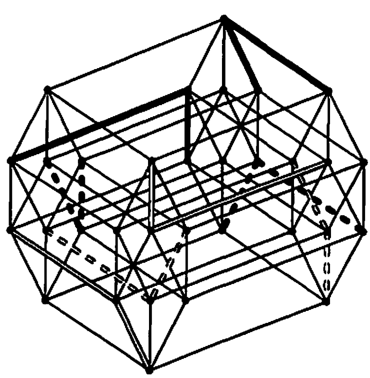

Figure 4. $f(Q_5, Q_3) \geq 68$.

To prove the upper bound we use the following notation and lemmas.

We consider an arbitrary set of $i + 1$ edges of Q_5 and count the Q_3s that contain at least two of these edges. The maximum number of Q_3s among all such sets is denoted by g_i.

Lemma 1. $g_i \leq 3(i+1)$.

Proof. According to (A) each edge of Q_5 is contained in 6 Q_3s. Therefore, these are at most $6(i+1)/2$ Q_3s that contain at least two of the $i+1$ edges. □

In the following we improve the upper bound for g_i of Lemma 1 for $i = 1, \ldots, 4$.

By (B) each pair of edges of Q_5 is contained in at most 3 distinct subgraphs Q_3 proving $g_1 \leq 3$ with equality if the edges are adjacent.

Lemma 2. $g_1 = 3$.

We determined the exact values of g_i for $i = 2$, 3, 4 by computer.

Lemma 3. $g_2 = 7$, $g_3 = 10$, $g_4 = 13$.

These values can also be determined by hand by a reasonable case analysis.

Proof of $f(Q_5, Q_3) \leq 68$. We assume the existence of a Q_3-anti-Ramsey edge coloring of Q_5 using the colors $1, \ldots, 69$.

Since the number of edges of Q_5 is 80 there are $p \leq 80 - 69 = 11$ colors, say $1, \ldots, p$, with at least 2 edges of these colors each. If $i_j + 1$, $j = 1, \ldots, p$, denotes the number of edges of color j then $i_1 + \ldots + i_p = 11$. According to the definition of g_i, the sum $g_{i_1} + \ldots + g_{i_p}$ is an upper bound for the number of nonrainbow Q_3s in Q_5. For all 56 partitions $i_1 + \ldots + i_p = 11$ in positive integers $i_1, \ldots, i_p$, $1 \leq p \leq 11$, the sum $g_{i_1} + \ldots + g_{i_p}$ is at most 39 which can be checked easily. To reduce the number of cases that must be considered one can use observations as the following: Since $g_1 + g_3 \leq g_2 + g_2$ partitions containing 1 and 3 must not be checked. Moreover, the upper bounds for g_{i+1} and $g_1 + g_i$, $i \geq 5$, are $3(i+2)$ each, thus it is enough to check partitions containing i and 1.

Since the number of Q_3s in Q_5 is 40 there is at least one rainbow Q_3 contradicting the assumption above. □

4. Concluding Remarks

It is conjectured in [1] that the lower bound $f(Q_n, Q_2) \geq n2^{n-2} + \lceil n/2 \rceil$ of (3) is attained for $n \geq 3$. We proved this conjecture for $n = 5$.

Using the idea of the proof of Theorem 1 for $n = 6$ results in an upper bound that exceeds the lower bound 99 of (3) by 3. Thus additional methods are necessary to prove the conjecture also for $n = 6$.

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