4-regular bipartite matching extendable graphs*

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Abstract

Let G be a simple connected graph containing a perfect matching. G is said to be BM-extendable (bipartite matching extendable) if every matching M which is a perfect matching of an induced bipartite subgraph of G extends to a perfect matching of G. The BM-extendable cubic graphs are known to be K_4 and $K_{3,3}$. In this paper, the 4-regular BM-extendable graphs are characterized. We show that the only 4-regular BM-extendable graphs are $K_{4,4}$ and $T_{4n}, n \geq 2$, where T_{4n} is the graph on $4n$ vertices $u_i, v_i, x_i, y_i, 1 \leq i \leq n$, such that $\{u_i, v_i, x_i, y_i\}$ is a clique and $x_iu_{i+1}, y_iv_{i+1} \in E(T_{4n})(\text{mod } n)$.

Keywords: Matching; Bipartite matching; Bipartite matching extendable graph

1 Introduction

Graphs considered in this paper are finite, simple and connected. Let $G = (V(G), E(G))$ be a graph. For $V' \subseteq V(G)$, we denote by $G[V']$ the subgraph induced by V'. For $M \subseteq E(G)$, set

$$V(M) = \{v \in V(G) : \text{ there is an } x \in V(G) \text{ such that } vx \in M\}.$$

$M \subseteq E(G)$ is a *matching* of G if $V(e) \cap V(f) = \emptyset$ for every two distinct edges $e, f \in M$. A matching M of G is *perfect* if $V(M) = V(G)$. Matching extendability is a significant topic in matching theory [6]. In 1980, Plummer

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[7] first proposed the notion of k-extendability: G is said to be *k-extendable* if every matching M with k edges extends to a perfect matching. There has been an extensive study on the characterizations of k-extendable graphs in the literature [1, 7, 8]. The notions of factor-critical and bicritical graphs play important roles in classical matching theory as well [6]. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for each vertex v of G, and *bicritical* if $G - u - v$ has a perfect matching for any pair of vertices u, v of G. In 1996, Favaron [3] introduced the k-factor-critical graphs: A graph G is said to be *k-factor-critical* if $G - T$ has a perfect matching for every $T \subset V(G)$ with $|T| = k$. Graphs which are k-factor-critical have a close relation to k-extendable graphs — a $2k$-factor-critical graph is k-extendable [3]; every connected, non-bipartite, k-extendable graph is k-factor-critical, where k is even [4]. Yu [12] and Lou [5] called a k-factor-critical graph k-critical and k-matchable, respectively. In 1998, Yuan [13] suggested a variant of k-extendability: A graph G is said to be *induced matching extendable* (IM-extendable in short) if every induced matching M extends to a perfect matching. Motivated by the study of k-extendable graphs, k-factor-critical graphs, and IM-extendable graphs, and by the fact that there are essential differences between matching problems of non-bipartite graphs and those of bipartite graphs, we investigate another variant — bipartite matching extendable graphs. We say that a matching M of a graph G is a bipartite matching if $G[V(M)]$ is a bipartite graph. We further say that G is bipartite-matching extendable (BM-extendable in short) if every bipartite matching M of G is included in a perfect matching of G.

The BM-extendability has close relations with other matching extendabilities:

$$\text{BM-extendable} \Longrightarrow \text{IM-extendable} \Longrightarrow \text{1-extendable} \Longrightarrow \text{elementary}.$$

A graph G is called *elementary* if the set of edges each of which lies in a perfect matching of G induce a connected subgraph. Moreover, we say that a graph G is *equilibrium decomposable* if G has a perfect matching, and there is a maximal barrier S of G with $|S| \geq 2$ such that all components of $G - S$ are factor-critical and the graph obtained from G by contracting each of these components into a single vertex is a complete bipartite graph. We showed that a BM-extendable graph is either bicritical or equilibrium decomposable. This indicates that the idea of BM-extendability can be traced back to those of factor-critical graphs, bicritical graphs, and Gallai-Edmonds's decomposable structure.

In our previous papers [10, 11], we proved that the recognition of BM-extendable graphs is hard in a computational complexity point of view; and we obtained the degree-type conditions for BM-extendable graphs, which implies that BM-extendable graphs would exist extensively in the class of comparatively dense graphs. On the other hand, the BM-extendable graphs are quite few in the class of low-degree graphs. We have shown that the only BM-extendable cubic graphs are K_4 and $K_{3,3}$. In this paper, we further characterize the 4-regular ones.

Let T_{4n} be a graph with $4n$ vertices $u_i, v_i, x_i, y_i, 1 \leq i \leq n, n \geq 2$, in which $\{u_i, v_i, x_i, y_i\}$ is a clique of T_{4n} and $x_i u_{i+1}, y_i v_{i+1} \in E(T_{4n}) (\text{mod } n)$ (see Figure 6).

Our main result is the following.

Theorem *A 4-regular graph G is BM-extendable if and only if G is isomorphic to $K_{4,4}$ or T_{4n}.*

Similar work has been done for IM-extendable graphs [9]: The only 4-regular claw-free connected IM-extendable graphs are $C_6{}^2$, $C_8{}^2$, and T_{4n}.

The proof of the theorem is organized as follows. In Section 2, we present some notations and theoretic tools needed in this paper. In the succeeding sections the structural features of a BM-extendable graph G are exhibited. Section 3 focuses on the classification of local structures in the neighbor set of each vertex. Section 4 is devoted to the interrelations of two types of neighbor sets. In Section 5, we make sure of the conclusion for triangle-free graphs. In Section 6, we finally show that the only 4-regular BM-extendable graphs are isomorphic to $K_{4,4}$ or T_{4n}.

2 Preliminaries

In this paper, we follow the graph-theoretic terminology and notation of [2, 6]. Let S be a subset of $V(G)$. The *neighbor set* of S, denoted by $N_G(S)$, consists of those vertices which are not in S but adjacent to some ones in S. If $S = \{v\}$, we write $N_G(v)$ for $N_G(\{v\})$. Let $d_G(v)$ denote the degree of vertex v in G. For simplicity, we may write $d(v)$ and $N(S)$ for $d_G(v)$ and $N_G(S)$ respectively. We say that v is a pendent vertex of G if $d(v) = 1$. For $V' \subseteq V(G)$, let $G - V'$ be the subgraph obtained from G by deleting all vertices in V' together with their incident edges, and $E(V') = E(G[V'])$ for convenience. For $E' \subseteq E(G)$, let $G - E'$ denote the spanning subgraph of G with edge set $E(G) \setminus E'$. If $E' = \{e\}$, we write $G - e$ for $G - \{e\}$. Let P_n, C_n, K_n and $\overline{K}_n$ denote the path, the cycle, the complete graph and the empty graph on n vertices respectively. Let $o(G)$ denote the number of odd components of graph G.

The following preliminary results are important to our work.

Lemma 2.1 (Hall's Theorem) [6] *Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for any $S \subseteq X$.*

Lemma 2.2 (Tutte's Theorem) [6] *G has a perfect matching if and only if $o(G - S) \leq |S|$ for any $S \subset V(G)$.*

A matching M is called a *forbidden matching* if it is a bipartite matching and $V(M)$ is a vertex cut such that $G-V(M)$ has an odd component. We state some necessary conditions of BM-extendable graphs as follows.

Lemma 2.3 *If G is BM-extendable, then:*

(a) there is no forbidden matching in G;

(b) G is 2-connected;

(c) if $\{u,v\}$ is a vertex cut of G and $uv \notin E(G)$, then $G-\{u,v\}$ has exactly two components and both of them are odd;

(d) for a bipartite matching M of G and an independent set X in $G-V(M)$, $|N_{G-V(M)}(X)| \geq |X|$.

Proof (a) If M is a forbidden matching, then $G-V(M)$ has no perfect matchings, contradicting the BM-extendability of G.

(b) If G has a cut vertex x, then $G-x$ has at least two components G_1 and G_2, and at most one is odd by Tutte's Theorem. So we may assume G_1 is an even component. Take a vertex y in G_1 adjacent to x. Then $M=\{xy\}$ is a forbidden matching in G, contradicting (a).

(c) Let G_1 and G_2 be two components of $G-\{u,v\}$. Take a vertex x in G_1 adjacent to u and a vertex y in G_2 adjacent to v. Then $M=\{ux,vy\}$ is a bipartite matching. If either G_1 or G_2 is even, then M is forbidden, contradicting (a). Therefore all components of $G-\{u,v\}$ are odd. If $G-\{u,v\}$ has at least three components, let G_3 be one other than G_1 and G_2. Then M is also forbidden, leading to a contradiction.

(d) Suppose to the contrary that $|N_{G-V(M)}(X)| < |X|$. Let $S = N_{G-V(M)}(X)$. Then $o(G-V(M)-S) \geq |X| > |S|$. By Tutte's Theorem, $G-V(M)$ has no perfect matchings, contradicting the BM-extendability of G. The proof is completed. ■

3 Local structure analysis

In the following, let G be a 4-regular graph. In this section, we will show that if G is BM-extendable, then the subgraph induced by the neighbor set of every vertex of G is isomorphic to either $\overline{K}_4$ or $K_1 \cup K_3$. For convenience, at a vertex u of G, we set $N_u = N(u)$, $e(N_u)=|E(N_u)|$, $N_u^2 = N(N_u) \setminus \{u\}$. By the 4-regularity of G, we have $0 \leq e(N_u) \leq 6$ and $|N_u^2| \leq 12$. Let $N_u = \{x_1,x_2,x_3,x_4\}$ and let Y_i be the set of neighbors of x_i in N_u^2, i.e., $Y_i = N_{x_i} \cap N_u^2$.

Lemma 3.1 *If G is BM-extendable, then $e(N_u) \leq 3$ for every vertex u of G.*

Proof Suppose that G is a BM-extendable graph, but the assertion fails. Then there is a vertex $u \in V(G)$ such that $4 \leq e(N_u) \leq 6$. If $e(N_u)=6$, then $G[N_u] \cong K_4$. By the 4-regularity of G, we have $G \cong K_5$, and so G has no perfect matchings, a contradiction.

If $e(N_u) = 5$, then $G[N_u] \cong K_4 - e$, where e is an edge of K_4. Suppose that $x_1, x_2 \in N_u$ are two vertices such that $x_1x_2 \notin E(N_u)$. The 4-regularity of G implies $1 \leq |N_u^2| \leq 2$. If $N_u^2 = \{y_1\}$, then y_1 is a cut vertex of G, contradicting Lemma 2.3 (b). If $N_u^2 = \{y_1, y_2\}$ and $x_iy_i \in E(G)$ for $i = 1, 2$, then $\{x_1, y_2\}$ is a vertex cut such that $x_1y_2 \notin E(G)$ and $G - \{x_1, y_2\}$ has an even component composed of u, x_2, x_3 and x_4, contradicting Lemma 2.3 (c).

If $e(N_u) = 4$, then $G[N_u]$ is isomorphic to either C_4 or $K_{1,3} + e$, where e is an edge joining two pendent vertices of $K_{1,3}$. In the former case, suppose $G[N_u] = x_1x_2x_3x_4x_1$. Then $M = \{x_1x_2, x_3x_4\}$ is a bipartite matching of G such that u is an isolated vertex in $G - V(M)$, contradicting Lemma 2.3 (a). In the latter case, let $x_1x_2x_3$ be the triangle in $G[N_u]$ and $x_3x_4 \in E(G)$. The 4-regularity of G implies that each of x_1 and x_2 has one neighbor in N_u^2 and x_4 has two neighbors in N_u^2. Suppose $Y_4 = \{y_1, y_2\}$. If $y_1y_2 \in E(G)$, then $M = \{ux_3, y_1y_2\}$ is a forbidden matching with x_4 being an isolated vertex in $G - V(M)$, a contradiction. In the following, we suppose $y_1y_2 \notin E(G)$, and consider two cases as follows.

Case 1 $(Y_1 \cup Y_2) \cap Y_4 \neq \emptyset$. If $Y_1 \cup Y_2 \subseteq Y_4$, say either $x_1y_1, x_2y_2 \in E(G)$ or $x_1y_1, x_2y_1 \in E(G)$, then in both cases $M = \{x_1y_1, x_4y_2\}$ is a forbidden matching with ux_2x_3 being an odd component of $G - V(M)$. If $Y_1 \cup Y_2 \nsubseteq Y_4$, suppose that $x_1y_1, x_2y_3 \in E(G)$ but $y_3 \notin Y_4$. Then $M = \{x_2y_3, x_4y_1\}$ is a forbidden matching with ux_1x_3 being an odd component of $G - V(M)$.

Case 2 $(Y_1 \cup Y_2) \cap Y_4 = \emptyset$. If there are two vertices $z_1 \in N(y_1) \setminus \{x_4\}$ and $z_2 \in N(y_2) \setminus \{x_4\}$ such that $z_1z_2 \notin E(G)$, then $M = \{ux_3, y_1z_1, y_2z_2\}$ is a forbidden matching with x_4 being an isolated vertex in $G - V(M)$. Otherwise, each vertex in $N(y_1) \setminus \{x_4\}$ is adjacent to each vertex in $N(y_2) \setminus \{x_4\}$. Then the 4-regularity implies that $N(Y_4) \setminus \{x_4\}$ induces a K_3 or $K_{3,3}$. It follows that x_4 is a cut vertex of G such that y_1, y_2 and their neighbors are separated from u, contradicting Lemma 2.3 (b).

To summarize, the contradiction to BM-extendability proves the lemma. ■

Lemma 3.2 *If there is a vertex u of G such that $e(N_u) = 3$ and $G[N_u]$ is isomorphic to either P_4 or $K_{1,3}$, then G is not BM-extendable.*

Proof Let u be the vertex of G such that $e(N_u) = 3$. To prove the result, we will find a forbidden matching M of G. Then by Lemma 2.3 (a), G is not BM-extendable.

If $G[N_u] \cong P_4$, let $P_4 = x_1x_2x_3x_4$, then $M = \{x_1x_2, x_3x_4\}$ is the desired matching with u being an isolated vertex in $G - V(M)$.

If $G[N_u] \cong K_{1,3}$, let $\{x_1, x_2, x_3\}$ be the independent set and x_4 the center of $K_{1,3}$. Then each of x_1, x_2, x_3 has two neighbors in N_u^2 and x_4 has none. If the two neighbors y_1, y_2 of x_1 are adjacent, i.e., $y_1y_2 \in E(G)$, then $M = \{ux_4, y_1y_2\}$ is a forbidden matching with x_1 being an isolated vertex in $G - V(M)$. The same result holds for x_2 and x_3. So we may assume that $Y_i, 1 \le i \le 3$, are independent. We assert that among Y_1, Y_2 and Y_3 there must be two, say Y_1 and Y_2, such that $y_1'y_2' \notin E(G)$ for some $y_1' \in Y_1$ and $y_2' \in Y_2$. Indeed, suppose to the contrary that, for any i, j, $1 \le i, j \le 3$ and $i \ne j$, each vertex in Y_i is adjacent to each vertex in Y_j. If any pair of Y_i and Y_j do not intersect, then $d(y) \ge 5$ for any $y \in Y_k$ $(1 \le k \le 3)$, a contradiction to the 4-regularity of G; if there are two Y_i's which intersect, say $Y_1 \cap Y_2 \ne \emptyset$, then both Y_1 and Y_2 are not independent sets, a contradiction to the above assumption. Therefore, $M = \{x_1y_1', x_2y_2', ux_3\}$ is a forbidden matching with x_4 being an isolated vertex in $G - V(M)$. The proof is completed. ■

We see from this lemma that if G is BM-extendable and $e(N_u) = 3$ for a vertex u of G, then the only possibility is $G[N_u] \cong K_3 \cup K_1$. We will deal with this case later.

Lemma 3.3 *If there is a vertex u of G such that $e(N_u) = 2$, then G is not BM-extendable.*

Proof Let u be the vertex such that $e(N_u) = 2$. We can see that $G[N_u]$ is isomorphic to either $2K_2$ or $P_3 \cup K_1$. If $G[N_u] \cong 2K_2$, then $M = E(N_u)$ is a forbidden matching of G with u being an isolated vertex in $G - V(M)$.

If $G[N_u] \cong P_3 \cup K_1$, suppose $P_3 = x_2x_1x_3$ and $K_1 = x_4$. Then, in N_u^2, x_1 has one neighbor, say y_1, each of x_2 and x_3 has two neighbors, and x_4 has three neighbors. If $y_1x_2, y_1x_3 \in E(G)$, then $e(N_{x_1}) = 4$, and so by Lemma 3.1 G is not BM-extendable. If y_1 is adjacent to one of x_2 and x_3, then $e(N_{x_1}) = 3$ and $G[N_{x_1}] \cong P_4$. By Lemma 3.2, G is not BM-extendable. So, we suppose that y_1 is adjacent to neither x_2 nor x_3, and proceed to find a forbidden matching M of G in what follows.

Let $Y_2 = \{y_2, y_3\}$. If $y_2y_3 \in E(G)$, then $G[N_{x_2}] \cong 2K_2$ which is excluded as before. Otherwise, Y_2 is independent, and so is Y_3. If $Y_4 \cap \{y_2, y_3\} \ne \emptyset$, say $x_4y_2 \in E(G)$, then, noting that $\{x_1, y_2, y_3\}$ and $\{x_2, x_3, x_4\}$ are independent sets, we see that $M = \{x_1x_3, x_2y_3, x_4y_2\}$ is a forbidden matching with u being an isolated vertex in $G - V(M)$. We now suppose that $Y_4 \cap (Y_2 \cup Y_3) = \emptyset$. Let $y_4 \in Y_4$. Then, $\{x_2, x_3, y_4\}$ and $\{x_1, x_4, y_3\}$ are independent sets, and so $M = \{x_1x_3, x_2y_3, x_4y_4\}$ is a forbidden matching of G with u being an isolated vertex in $G - V(M)$. This completes the proof. ■

Lemma 3.4 *If there is a vertex u of G such that $e(N_u) = 1$, then G is not BM-extendable.*

Proof Let u be the vertex of G such that $e(N_u) = 1$, $x_1x_2 \in E(N_u)$ and x_3, x_4 the two isolated vertices in $G[N_u]$. Then $|Y_1| = |Y_2| = 2, |Y_3| = |Y_4| = 3$. If $Y_1 \cap Y_2 \ne \emptyset$ or the two vertices in Y_1 are adjacent, then $e(N_{x_1}) \ge 2$ and $G[N_{x_1}] \not\cong K_1 \cup K_3$. Thus G is not BM-extendable by Lemma 3.1 – 3.3. So we may suppose that Y_1 and Y_2 are independent and $Y_1 \cap Y_2 = \emptyset$. There are two cases to consider, depending on whether $Y_3 = Y_4$ or not.

Case 1 If $Y_3 \ne Y_4$, we can choose $y_1 \in Y_3 \setminus Y_4, y_2 \in Y_4 \setminus Y_3$. Then

$x_3y_1, x_4y_2 \in E(G)$ and $x_3y_2, x_4y_1 \notin E(G)$. Since x_1 and x_2 cannot be both adjacent to any one of y_1 and y_2, we may assume that $x_1y_1, x_2y_2 \notin E(G)$. So $\{x_1, y_1, x_4\}$ and $\{x_2, x_3, y_2\}$ are independent. Thus $M = \{x_1x_2, x_3y_1, x_4y_2\}$ is a forbidden matching with u being an isolated vertex in $G - V(M)$.

Case 2 If $Y_3 = Y_4$, there must be two nonadjacent vertices y_1 and y_2 in it. For, otherwise, $Y_3 = Y_4$ will be a 3-clique, and so u is a cut vertex, contradicting Lemma 2.3 (b). Therefore $\{x_3y_1, x_4y_2\}$ is a bipartite matching of G. If furthermore $M = \{x_1x_2, x_3y_1, x_4y_2\}$ is a bipartite matching, then the proof is completed as the previous case. Otherwise, x_1 and x_2 are adjacent to y_1 and y_2 respectively, say $x_1y_1, x_2y_2 \in E(G)$ (an odd cycle $x_1x_2y_2x_3y_1x_1$ occurs in $G[V(M)]$). We will change this matching M to a forbidden matching M' with x_1 being an isolated vertex in $G - V(M')$ in the sequel. Let $Y_3 = Y_4 = \{y_1, y_2, y_3\}$. Since $Y_1 \cap Y_2 = \emptyset$, y_3 is adjacent to at most one of x_1 and x_2. So, we can suppose $x_1y_3 \notin E(G)$. Then $Y_1 \cap \{y_1, y_2, y_3\} = \{y_1\}$. Let $Y_1 = \{y_1, y_4\}$. Observing that $y_4 \notin Y_2 \cup Y_3 \cup Y_4$ and $y_1 \in Y_1 \cap Y_3 \cap Y_4$, by the 4-regularity of G, we can choose a vertex $z \notin N_u$ such that $y_4z \in E(G)$ and $y_1z \notin E(G)$. Then, $\{u, y_1, z\}$ and $\{x_2, x_3, y_4\}$ are independent sets, and so $M' = \{ux_2, y_1x_3, y_4z\}$ is a forbidden matching as required. The proof is completed. ■

So far, all we have left to consider are two cases when $G[N_u] \cong \overline{K}_4$ ($e(N_u) = 0$) and when $G[N_u] \cong \overline{K}_1 \cup K_3$. This leads to a classification of local structures: for a vertex u in G, the neighbor set N_u is said to be of *type 1* if it is an independent set, and of *type 2* if it induces a subgraph $K_1 \cup K_3$. From Lemmas 3.1 through 3.4, we see that for each vertex u in BM-extendable graph G, N_u is either of type 1 or of type 2.

4 Two types of neighborhoods

In this section we concentrate our attention on a vertex u with neighbor set N_u of type 1, i.e., $G[N_u] \cong \overline{K}_4$. Recall that $N_u = \{x_1, x_2, x_3, x_4\}$ and Y_i is the set of neighbors of x_i in N_u^2 ($1 \leq i \leq 4$). Then $|Y_i| = 3$ and $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 = N_u^2$. We apply the above classification to the neighbor sets Y_i: when N_{x_i} is of type 1 (type 2 resp.), we say that Y_i is of type 1 (type 2 resp.). So Y_i is of type 1 if and only if it is an independent set ($G[Y_i] \cong \overline{K}_3$); Y_i is of type 2 if and only if it is a clique ($G[Y_i] \cong K_3$).

Lemma 4.1 *Let G be a BM-extendable graph and N_u of type 1 for some vertex u of G.*

(a) If Y_i and Y_j are of type 2, then $Y_i \cap Y_j = \emptyset$.

(b) If Y_i is of type 2, then there is at most one Y_j of type 1 such that $Y_i \cap Y_j \neq \emptyset$ (thus $|Y_i \cap Y_j| = 1$).

(c) If Y_i is of type 1 and the other three Y_j are of type 2, then there is at least one Y_j ($j \neq i$) such that $Y_i \cap Y_j = \emptyset$.

Proof (a) Suppose $Y_i \cap Y_j \neq \emptyset$. Then the 4-regularity implies that $Y_i = Y_j$. It follows that u is a cut vertex, contradicting Lemma 2.3 (b).

(b) Let $Y_1 = \{y_1, y_2, y_3\}$ be the one of type 2. Suppose to the contrary that Y_2, Y_3 are the two of type 1 such that $y_2 \in Y_2$ and $y_3 \in Y_3$. Then $x_2y_2, x_3y_3 \in E(G)$, and both Y_2 and Y_3 are independent. Let z be the only element of $N_{y_1} \setminus \{x_1, y_2, y_3\}$. If $z \in Y_2$ (similarly for $z \in Y_3$), i.e., $x_2z \in E(G)$, then take y_4 as the only element of $Y_2 \setminus \{y_2, z\}$, else take y_4 arbitrarily from $Y_2 \setminus \{y_2\}$. In the former case, $\{u, z, y_4\}$ and $\{x_2, x_3, y_1\}$ are independent; in the latter case, $\{u, y_1, y_4\}$ and $\{x_2, x_3, z\}$ are independent. Hence $M = \{ux_3, x_2y_4, y_1z\}$ is a forbidden matching with $x_1y_2y_3$ being a triangle component in $G - V(M)$, a contradiction to Lemma 2.3 (a).

(c) Let Y_1, Y_2, Y_3 be of type 2 and Y_4 of type 1. Suppose to the contrary that $Y_i \cap Y_4 = \{y_i\}, i = 1, 2, 3$, and denote by Q_i the 4-clique $\{x_i\} \cup Y_i$. In addition to x_i and y_i, let a_i and b_i be the other two vertices of Q_i. To concentrate ourself, we consider Q_1 at the moment. Let x_1' be the only element of $N_{a_1} \setminus \{x_1, y_1, b_1\}$ and y_1' the only element of $N_{b_1} \setminus \{x_1, y_1, a_1\}$. Noting that N_{a_1} is not of type 1, and then is of type 2, we have $x_1' \neq y_1'$. We further assert that $x_1' \neq y_i$ for each $i = 2, 3$. Otherwise, say $x_1' = y_2$, since Y_2 is of type 2, y_2 has five neighbors, a_1, a_2, b_2, x_2 and x_4, a contradiction to the 4-regularity of G.

If there is a neighbor z of x_1' such that y_1' is adjacent to at most one of z and x_1', then $M = \{ux_4, x_1'z, b_1y_1'\}$ is a forbidden matching with $x_1y_1a_1$ being a triangle component of $G - V(M)$, contradicting Lemma 2.3 (a). Otherwise, for any neighbor z of x_1', y_1' is adjacent to both z and x_1'. Then $N_{x_1'}$ is of type 2, and x_1' and y_1' are in a common 4-clique, denoted by Q_1' (not excluding $Q_1' = Q_2$ or Q_3). The same argument can be applied on Q_1' and a new 4-clique Q_1'' is obtained, and so on. Furthermore, the same argument can be applied on Q_2 and Q_3. In this way, three chains of 4-cliques starting at Q_1, Q_2, Q_3, respectively, are generated. Due to the finiteness and the 4-regularity of the graph, two of chains will eventually join together; but the extending process of the third chain can never be stopped, and this contradicts the finiteness of G. The proof is completed. ■

Let $X = N_u = \{x_1, x_2, x_3, x_4\}$, and let X^0 be the set of x_i with N_{x_i} being of type 2 (i.e., Y_i is of type 2). We denote by E_0 the set of edges between different Y_i's in G.

Lemma 4.2 *Let G be a BM-extendable graph. If there is a vertex u of G with N_u of type 1, then N_x is of type 1 for every vertex $x \in N_u$.*

Proof Suppose to the contrary that there is a vertex $x \in N_u$ such that N_x is of type 2. Since Y_i is either of type 1 or of type 2, we have $1 \leq |X^0| \leq 4$. We perform the following transformation: if Y_i is of type 2, then it is contracted to a single vertex (still denoted by Y_i). Here, the resulting parallel edges should be removed. Let G' denote the resulting graph, and Y the set of N_u^2

in G'. Since X is independent, we see that $H = G'[X \cup Y] - E_0$ is bipartite. Then, by Lemma 4.1 (a), $M^0 = \{x_i Y_i : x_i \in X^0\}$ is an induced matching of H. If we can extend M^0 to a matching M' of H saturating every vertex of X, then, coming back to graph G, by changing M^0 in H to a matching in G (still denoted by M^0) so that each $x_i \in X^0$ matches an arbitrary vertex y_i in Y_i, we will get a matching M of G from M'. Furthermore, if M is a bipartite matching, then it is a forbidden matching with u being an odd component of $G - V(M)$. This is a contradiction to Lemma 2.3(a), completing the proof.

We first give the following key observation, which will help us to insure that the M we find is forbidden: the matching M^0 is also an induced matching in $G - E_0$; each of those y_i in $V(M^0)$ is incident to at most one edge of E_0 (due to the 4-regularity), implying that these y_i cannot be contained in any cycle invoking edges of E_0.

We now consider four cases, depending on the cardinality of X^0.

Case 1 $|X^0| = 4$, i.e., $X^0 = X$. We change M^0 to $M = \{x_i y_i : 1 \leq i \leq 4\}$ ($y_i \in Y_i$). Then M is an induced matching in $G - E_0$. From the above observation, $G[V(M)]$ has no cycles and so is bipartite. Thus M is a bipartite matching in G, as required.

Case 2 $|X^0| = 3$. Let $X^0 = \{x_1, x_2, x_3\}$. By Lemma 4.1 (c), x_4 can be matched to a vertex $y_4 \in Y_4 \setminus (Y_1 \cup Y_2 \cup Y_3)$. Thus H has a matching $M' = M^0 \cup \{x_4 y_4\}$ saturating every vertex of X. Let $M = \{x_i y_i : 1 \leq i \leq 4\}$, where $y_i \in Y_i$ $(1 \leq i \leq 3)$. Also by the observation, we can see that $G[V(M)]$ is bipartite, implying that M is bipartite in G, as required.

Case 3 $|X^0| = 2$. Let $X^0 = \{x_1, x_2\}$ and $H^0 = H - V(M^0)$, then by Lemma 4.1 (b), either $d_{H^0}(x_3) \geq 2$ and $d_{H^0}(x_4) \geq 2$ or $d_{H^0}(x_3) \geq 1$ and $d_{H^0}(x_4) = 3$. By Hall's theorem, H^0 has a matching $M^1 = \{x_i y_i : i = 3, 4\}$, where $y_i \in Y_i \setminus (Y_1 \cup Y_2)$. Thus H has a matching $M' = M^0 \cup M^1$ saturating every vertex of X. Let $M = \{x_i y_i : 1 \leq i \leq 4\}$, where $y_i \in Y_i$ $(i = 1, 2)$. For $3 \leq i \leq 4$, the independence of Y_i implies that if $x_i y_j \in E(G)$ $(j \neq i)$ then $y_i y_j \notin E_0$, and as mentioned before, no cycles using edges of E_0 could pass through y_1 or y_2. So, there are no odd cycles in $G[V(M)]$ and M is bipartite.

Case 4 $|X^0| = 1$. Let $X^0 = \{x_1\}$. We proceed to find a bipartite matching $M = \{x_i y_i : 1 \leq i \leq 4\}$. First, let x_1 match a vertex y_1 in Y_1 and we ignore edge $x_1 y_1$ below. By Lemma 4.1 (b), $Y_1 \cap Y_i \neq \emptyset$ for at most one Y_i $(i \neq 1)$. Without loss of generality, suppose that $Y_1 \cap Y_2 \neq \emptyset$, that is, x_2 is adjacent to a vertex in Y_1. Let $Y_2' = Y_2 \setminus Y_1$. If $Y_1 \cap Y_i = \emptyset$ for all $i \neq 1$, let $Y_2' = Y_2 \setminus \{v\}$, where v is an arbitrary vertex of Y_2. Then $|Y_2'| = 2$. Let H' be the bipartite subgraph consisting of the edges between $\{x_2, x_3, x_4\}$ and $Y_2' \cup Y_3 \cup Y_4$. If $\{x_2, x_3, x_4\}$ can be matched into Y_3 or Y_4, say $Y_3 = \{y_2, y_3, y_4\}$ and $y_2 \in Y_2', y_3 \in Y_3, y_4 \in Y_4$, then $M' = \{x_i y_i : 2 \leq i \leq 4\}$ is a bipartite matching, as Y_3 is independent in G.

Otherwise, neither Y_3 nor Y_4 is an SDR (system of distinct representatives) of the family $\{Y_2', Y_3, Y_4\}$. Then there will be three possibilities as follows: (i) one of Y_2', Y_3, Y_4 is disjoint from the other two; (ii) any two of Y_2', Y_3, Y_4 intersect exactly at a common vertex z, i.e., $Y_2' \cap Y_3 = Y_2' \cap Y_4 = Y_3 \cap Y_4 = \{z\}$; (iii) $Y_3 \cap Y_4 = \emptyset$ and $Y_2' \cap Y_3 = \{z'\}, Y_2' \cap Y_4 = \{z\}$ $(z \neq z')$. Three possibilities are illustrated in Figure 1(a), (b), (c). For any possibility, we claim that there are $y_2 \in Y_2', y_3 \in Y_3, y_4 \in Y_4$ satisfying: (1) $y_2, y_3, y_4 \neq z$; (2) $\{y_2, y_3, y_4\}$ doesnot form a triangle by using edges of E_0. In fact, (1) is easy to do. As to (2), suppose that $\{y_2, y_3, y_4\}$ forms a triangle by using edges of E_0. Then for the other vertex $y_4' \in Y_4 \setminus \{z\}$, we have $y_2y_4' \notin E_0$. Otherwise, by the two type classification, we see that $\{x_2, y_3, y_4, y_4'\}$, the neighbor set of y_2, is of type 2. Thus $\{y_3, y_4, y_4'\}$ induces a K_3, contradicting that Y_4 is independent. So we can change y_4 to y_4', and $\{y_2, y_3, y_4'\}$ satisfies condition (2).

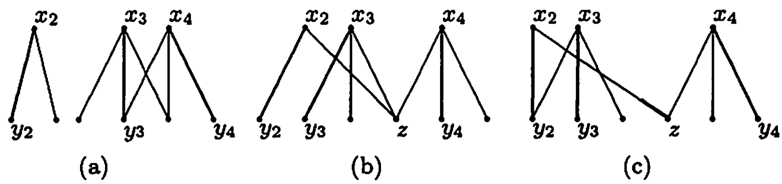


Figure 1. Subgraph H

Using these chosen y_2, y_3, y_4, we construct a matching $M' = \{x_iy_i : 2 \leq i \leq 4\}$. If there is no z, then the induced subgraph of $V(M')$ in the bipartite graph H', denoted by $H'[V(M')]$, is disconnected. Otherwise, z is a cut vertex of H', and $z \notin V(M')$. Then $H'[V(M')]$ is also disconnected. As a result, when edges of E_0 are put into this subgraph, there are no odd cycles occurring in $G[V(M')]$. In fact, it is known by (2) that $G[\{y_2, y_3, y_4\}]$ is not a 3-cycle in $G[V(M')]$. Further, the absence of other 3-cycles is due to the independence of Y_i $(2 \leq i \leq 4)$; the absence of 5-cycles is due to the disconnection of $H'[V(M')]$. Therefore M' is bipartite. Since y_1 is adjacent to at most one vertex in $V(M')$, $M = \{x_1y_1\} \cup M'$ is bipartite, and so is forbidden.

To summarize, we get the contradiction that G is not BM-extendable. The proof is completed. ■

5 Triangle-free graphs

Lemma 5.1 *Let G be a triangle-free (i.e., all N_u are of type 1) BM-extendable graph. Then $G \cong K_{4,4}$.*

Proof Let $u \in V(G)$ be given, $X = N_u = \{x_i : 1 \leq i \leq 4\}$, and $Y = N_u^2$. We can see that $H = G[X \cup Y] - E_0$ is bipartite. We first suppose that

$|Y| \geq 4$. The main burden of the proof is to show that $|Y| \geq 4$ is impossible.

By the fact that $d_H(x_i) = 3$ $(1 \leq i \leq 4)$, for any subset $S \subseteq X$, if $|S| \leq 3$, then $|N_H(S)| \geq d_H(x_i) = 3 \geq |S|$ $(x_i \in S)$; if $S = X$, then $|N_H(S)| = |Y| \geq 4 = |S|$. Thus, by Hall's theorem, there exists a matching M of H saturating every vertex of X. If $E_0 = \emptyset$, M is bipartite in G, and so is forbidden with u being an isolated vertex in $G - V(M)$, contradicting Lemma 2.3(a).

If $E_0 \neq \emptyset$ and $|Y| = 4$, let $Y = \{y_i : 1 \leq i \leq 4\}$ and $y_3y_4 \in E_0$. Since G is triangle-free, $d_H(y_3) + d_H(y_4) \leq |X| = 4$. This implies that $d_H(y_1) = d_H(y_2) = 4$. Therefore, $M = \{y_3y_4\}$ is a forbidden matching with $G[X \cup \{u, y_1, y_2\}]$ being an odd component in $G - V(M)$, a contradiction.

We next consider $5 \leq |Y| \leq 6$. First, if $G[E_0]$ has two independent edges, then we may take them as a matching M. Note that $N_{G-V(M)}(X) = (Y \setminus V(M)) \cup \{u\}$. We have $|N_{G-V(M)}(X)| \leq 3 < 4 = |X|$, contradicting Lemma 2.3 (d). Second, if $G[E_0]$ has no independent edges, then $G[E_0]$ is a star $K_{1,r}, 1 \leq r \leq 3$ (for G is triangle-free). Let y be the center of this star. Instead of H, we may consider the bipartite graph $H' = H - y$ with no edges of E_0 between its vertices. By a similar discussion as the case $E_0 = \emptyset$ above, G is not BM-extendable.

For $|Y| \geq 7$, we consider three cases below, according to the value of $\max\{d_H(y) : y \in Y\}$. Here, the following observations are needed.

Observation 1 *If each* Y_i $(1 \leq i \leq 4)$ *contains a pendent vertex of* H, *then* G *is not BM-extendable.*

In fact, let $y_i \in Y_i$ be the pendent vertices of H and $M = \{x_iy_i : 1 \leq i \leq 4\}$. Clearly, M is an induced matching in H. Since there is no triangle in G, $G[V(M)]$ is bipartite, implying that M is forbidden. Thus G is not BM-extendable.

Observation 2 *If* Y_4 *contains pendent vertices of* H, $Y_i \cap Y_4 = \emptyset$ *for some* $i \in \{1,2,3\}$, *and* Y_i *contains at most one pendent vertex of* H, *then* G *has a forbidden matching which saturates every vertex in* X *(and so* G *is not BM-extendable).*

To show this, suppose that y' is a pendent vertex of H in Y_4, and suppose, without loss of generality, that $i = 1$. Then $Y_1 \cap Y_4 = \emptyset$, and Y_1 contains at most one pendent vertex of H. Let $Y_1 = \{y_1, y_2, y_3\}$ (if Y_1 has a pendent vertex, assume that it is y_3). If $|N_H(Y_1)| = 3$, then $N_H(Y_1) = \{x_1, x_2, x_3\}$. By Hall's theorem, the bipartite graph $G[\{x_1, x_2, x_3\} \cup Y_1]$ has a perfect matching M' with $|M'| = 3$. Set $M = M' \cup \{x_4y'\}$. Since $\{x_1, x_2, x_3, y'\}$ and $\{y_1, y_2, y_3, x_4\}$ are independent in G, M is bipartite. If $|N_H(Y_1)| = 2$, say $N_H(Y_1) = \{x_1, x_2\}$. Then $Y_1 \cap Y_3 = Y_2 \cap Y_3 = \emptyset$. Since Y_4 contains pendent vertices, Y_3 has a vertex y'' which does not belong to $Y_1 \cup Y_2 \cup Y_4$. then $\{x_1, x_2, y', y''\}$ and $\{y_1, y_2, x_3, x_4\}$ are independent, and so $M = \{x_1y_1, x_2y_2, x_3y'', x_4y'\}$ is bipartite. In both cases M is a forbidden matching of G with u being an isolated vertex in $G - V(M)$, completing the proof of Observation 2.

From Observation 1, there is a Y_i which contains no pendent vertices of H, and so $\max\{d_H(y) : y \in Y\} \geq 2$. By symmetry, we may suppose that $Y_1 = \{y_1, y_2, y_3\}$ and $d_H(y_1) \geq d_H(y_2) \geq d_H(y_3) \geq 2$.

Case 1 $\max\{d_H(y) : y \in Y\} = 2$. By Observation 1, we have $|Y| \leq 9$. We will show that there is a bipartite matching M in H saturating X, and thus G is not BM-extendable. Here, three subcases arise:

(2.1) $|Y| = 7$. Suppose that the degree sequence of Y in H is $(d_H(y_1), d_H(y_2), d_H(y_3), d_H(y_4), d_H(y_5), d_H(y_6), d_H(y_7)) = (2,2,2,2,2,1,1)$. If the two pendent vertices y_6, y_7 are contained in a single Y_i, say Y_4, since Y_4 has only one non-pendent vertex, Y_4 intersects only one of Y_1, Y_2 and Y_3. If y_6 and y_7 are contained in two distinct Y_i's, say $y_6 \in Y_3, y_7 \in Y_4$, since Y_4 has two non-pendent vertices, Y_4 intersects at most two of Y_1, Y_2 and Y_3. In both cases there is a Y_l, $1 \leq l \leq 3$, such that $Y_4 \cap Y_l = \emptyset$. Since Y_l has at most one pendent vertex, by Observation 2, we are done.

(2.2) $|Y| = 8$. Let $(2,2,2,2,1,1,1,1)$ be the degree sequence of Y in H, for which Y has four pendent vertices y_5, y_6, y_7, y_8 in H. Among them there must be two contained in distinct Y_i's, say $y_7 \in Y_3, y_8 \in Y_4$. We make a transformation on H by contracting y_7 and y_8 into a single vertex, and denote the resulting graph by H'. This reduces to the case of $|Y| = 7$. By Observation 2, H' has a bipartite matching M saturating X. Clearly, M is also bipartite with respect to H.

(2.3) When $|Y| = 9$, the same transformation can be made and the proof of Case 2 is completed.

Case 2 $\max\{d_H(y) : y \in Y\} = 3$. By Observation 1, we have $|Y| \leq 8$. There are three subcases to consider.

(3.1) $|Y| = 7$ and (3,2,2,2,1,1,1) is the degree sequence of Y in H. If two (or three) of the pendent vertices are contained in some Y_i, say Y_4, then Y_4 is disjointed to at least one of Y_1, Y_2, Y_3. Otherwise, each of the three pendent vertices is contained in a distinct Y_i, $2 \leq i \leq 4$. Then Y_4 intersects at most two of Y_1, Y_2, Y_3, i.e., $Y_l \cap Y_4 = \emptyset$ for some l $(1 \leq l \leq 3)$. Note that Y_l has at most one pendent vertex. In both cases the result follows from Observation 2.

(3.2) $|Y| = 7$ and (3,3,2,1,1,1,1) is the degree sequence of Y in H. Let y_4, y_5, y_6, y_7 be the pendent vertices of H. Without loss of generality, there are three cases for the distribution of these pendent vertices as follows: (a) $y_4 \in Y_3, Y_4 = \{y_5, y_6, y_7\}$; (b) $y_4 \in Y_2, y_5 \in Y_3, \{y_6, y_7\} \subseteq Y_4$; (c) $\{y_4, y_5\} \subseteq Y_3, \{y_6, y_7\} \subseteq Y_4$. For (a) or (b), the proof is the same as that of (3.1) by using Observation 2. We need only consider (c) below.

Note that $Y_2 = Y_1$. Suppose, without loss of generality, that $x_3y_1, x_4y_2 \in E(H)$ (see Figure 2). Then there are no edges from y_1 to $\{y_4, y_5\}$ and no edges form y_2 to $\{y_6, y_7\}$. Since y_1 has only one additional neighbor, one of y_6 and y_7 is not adjacent to y_1, say y_6. Similarly, suppose that $y_2y_4 \notin E(G)$.

If $y_4y_6 \notin E_0$, let $M = \{x_1y_1, x_2y_2, x_3y_4, x_4y_6\}$. Since $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_4, y_6\}$ are independent, M is bipartite as required. We now suppose that $y_4y_6 \in E_0$.

If both y_1y_7 and y_2y_5 are edges of G (see Figure 2(a)), we take $M = \{y_1y_7, y_2y_5, y_4y_6\}$. Since $Y_3 = \{y_1, y_4, y_5\}$ and $Y_4 = \{y_2, y_6, y_7\}$ are independent, M is bipartite. By the fact that $|N_{G-V(M)}(X)| = 2 < 4 = |X|$ occurs in Lemma 2.3(d), G is not BM-extendable. Therefore at most one of y_1y_7 and y_2y_5 is an edge of G. If $y_5y_7 \in E(G)$, let $M = \{x_1y_1, x_2y_2, y_4y_6, y_5y_7\}$. Since G is triangle free and there is at most one edge between $G[\{x_1, y_1, x_2, y_2\}]$ and $G[\{y_4, y_5, y_6, y_7\}]$, M is a bipartite matching of G such that

$$|N_{G-V(M)}(\{x_3, x_4\})| = 1 < |\{x_3, x_4\}|,$$

which implies that G is not BM-extendable. So $y_5y_7 \notin E(G)$. If $y_1y_7, y_2y_5 \notin E(G)$, then $M = \{x_1y_1, x_2y_2, x_3y_5, x_4y_7\}$ is a bipartite matching as required. Otherwise, by symmetry, suppose $y_1y_7 \in E(G)$ (see Figure 2(b)). Then $M = \{x_1u, x_2y_2, x_3y_5, x_4y_7\}$ is a forbidden matching of G with y_1 being a component of $G - V(M)$.

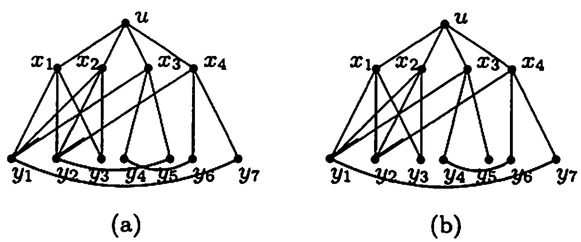


Figure 2. Avoiding odd cycles

(3.3) $|Y| = 8$ and (3,2,2,1,1,1,1,1) is the degree sequence of Y in H. We make a transformation on H by contracting two pendent vertices lying in distinct Y_i's into a vertex of degree two and reduce it to Case (3.1).

Case 3 $\max\{d_H(y) : y \in Y\} = 4$. By Observation 1, we have $|Y| = 7$ and the unique degree sequence of Y in H is (4,2,2,1,1,1,1). So $d_H(y_i) = 1$ for $4 \le i \le 7$. Let $Y_i' = Y_i \setminus \{y_1\}$. then $|Y_i'| = 2$ $(1 \le i \le 4)$. There are two cases for the distribution of pendent vertices: either $Y_3' = \{y_4, y_5\}, Y_4' = \{y_6, y_7\}$ or $y_4 \in Y_2', y_5 \in Y_3', Y_4' = \{y_6, y_7\}$. Consequently, we distinguish these two cases:

(4.1) $Y_3' = \{y_4, y_5\}$ and $Y_4' = \{y_6, y_7\}$ (see Figure 3(a)). Let $M = \{x_1y_2, x_2y_3, x_3y_4, x_4y_6\}$. If $y_2y_6, y_3y_4, y_4y_6 \in E_0$, then $x_1y_3y_4y_6y_2x_1$ is a 5-cycle in $G[V(M)]$. If no such cycle exists, then, by the fact that x_3, x_4 are pendent vertices in $G[V(M)]$, M is a bipartite matching of G saturating X. Otherwise, we may change x_3y_4 to x_3y_5 and/or change x_4y_6 to x_4y_7 so as to avoid the 5-cycles. If this is unavoidable, then the graph G is the one shown in Figure 3(a). For this, we may take $M = \{y_2y_7, y_3y_4, y_5y_6\}$, which is a bipartite matching with $\{y_2, y_4, y_5\}$ and $\{y_3, y_6, y_7\}$ being independent in $G[V(M)]$. Noting that $|N_{G-V(M)}(X)| = 2 < 4 = |X|$, we see that G is not BM-extendable by Lemma 2.3(d).

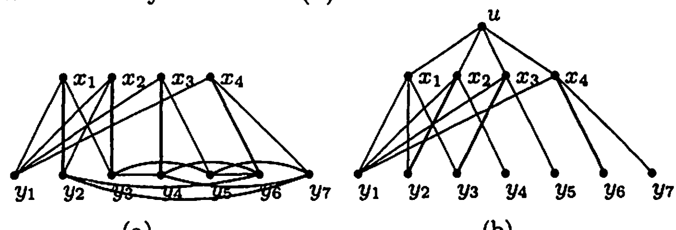


Figure 3.

(4.2) $y_4 \in Y_2', y_5 \in Y_3', Y_4' = \{y_6, y_7\}$ (see Figure 3(b)). Let $M = \{x_2y_2, x_3y_3, x_4y_6\}$. Observing that $\{x_2, x_3, y_6\}$ and $\{y_2, y_3, x_4\}$ are independent and $\{u, x_1, y_1\}$ comprises an odd component in $G - V(M)$, we know that M is forbidden, and so G is not BM-extendable.

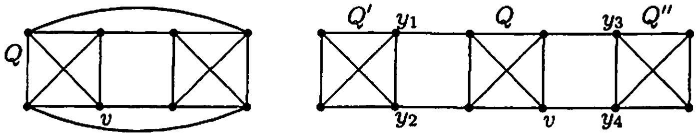
To summarize, we have $|Y| = 3$. Then, by the 4-regularity of G, $G \cong K_{4,4}$. The proof is completed. ■

6 Proof of the Theorem

We first show that $K_{4,4}$ *and* T_{4n} *are BM-extendable.* In fact, every matching of $K_{4,4}$ extends to a perfect matching, $K_{4,4}$ is BM-extendable. For T_{4n}, suppose that M is a bipartite matching and $Q_i = \{u_i, v_i, x_i, y_i\}$ $(1 \leq i \leq n)$ are cliques of T_{4n}. Then $|V(M) \cap Q_i| \leq 2$. For any $e \in M$, if $e \in E(Q_i)$, then let $\varphi(e)$ be the edge in $E(Q_i)$ independent to e; if $e = x_iu_{i+1}$ then let $\varphi(e) = y_iv_{i+1}$ or conversely. Clearly, $\cup_{e \in M}\{e, \varphi(e)\}$ is a matching of T_{4n} containing M. The remaining vertices (if any) can be matched by pairing x_i and y_i or u_i and v_i. In this way, we obtain a perfect matching of T_{4n} containing M. Thus T_{4n} is BM-extendable.

To complete the proof of the theorem, invoking Lemma 5.1, we need only prove that *for a BM-extendable graph G, if there is a vertex v of G such that N_v is of type 2, then $G \cong T_{4n}$.* We start with the following claim.
Claim *Let G be a BM-extendable graph, v a vertex of G with N_v of type 2, and Q the 4-clique containing v. If $|V(G)| = 8$, then $G \cong T_8$; otherwise, there are two vertices in $N(Q)$ contained in a 4-clique Q' and the other two vertices contained in another 4-clique Q'' $(Q' \neq Q'')$ (see Figure 4).*

To show this, let $v = x_1$ and $Q = \{x_1, x_2, x_3, x_4\}$. Clearly, all N_{x_i} are of type 2. If there are two vertices in Q, say x_1, x_2, having a neighbor $y \notin Q$ in common, then G is K_5, a contradiction. Thus Q has four distinct neighbors, i.e., $|N(Q)| = 4$. Let $N(Q) = \{y_i : 1 \leq i \leq 4\}$ and $x_iy_i \in E(G)$. If there is a vertex y_i being of type 1, then N_{x_i} is of type 1 by Lemma 4.2, a contradiction. Thus all N_{y_i} are of type 2. Let Q_i $(i = 1, 2, 3, 4)$ denote the 4-clique of G containing y_i. We assert that if $Q_i \neq Q_j$, then $Q_i \cap Q_j = \emptyset$ $(i \neq j)$. Otherwise, the 4-regularity of G implies that $|Q_i \cap Q_j| = 3$ and $M = \{x_i y_i, x_j y_j\}$ is a forbidden matching with $G[Q_i \setminus \{y_i\}]$ being a triangle component of $G - V(M)$, contradicting Lemma 2.3. Let q denote the number of 4-cliques containing some y_i $(i = 1, 2, 3, 4)$. Clearly, $1 \leq q \leq 4$.



(a) (b)

Figure 4. Recursive structure

If $q = 4$, i.e., Q_i are pairwise disjoint, let $z_i \in Q_i$ $(z_i \neq y_i)$ and $M = \{y_1 z_1, y_2 z_2, y_3 z_3, x_4 y_4\}$; if $q = 3$, let $Q_1 = Q_2$, $z_3 \in Q_3$ $(z_3 \neq y_3)$ and $M = \{y_1 y_2, y_3 z_3, x_4 y_4\}$. We can see that in both cases M is a forbidden matching with $x_1 x_2 x_3$ being a triangle component of $G - V(M)$, a contradiction. If $q = 1$, i.e., $Q_1 = Q_2 = Q_3 = Q_4$, then the 4-regularity of G gives $G \cong T_8$ (see Figure 4(a)). If $q = 2$, then, without loss of generality, we get that either $Q_1 = Q_2$ and $Q_3 = Q_4$ or $Q_1 = Q_2 = Q_3$. For the former case, y_1 and y_2 are in a 4-clique Q', and y_3 and y_4 in another 4-clique Q'' (as in Figure 4(b)). For the latter case, let $z \neq y_i$ $(1 \leq i \leq 3)$ be in Q_3, then $M = \{y_3 z, x_4 y_4\}$ is a forbidden matching with $G[\{x_1, x_2, x_3, y_1, y_2\}]$ being an odd component of $G - V(M)$, a contradiction. Thus the claim is proved.

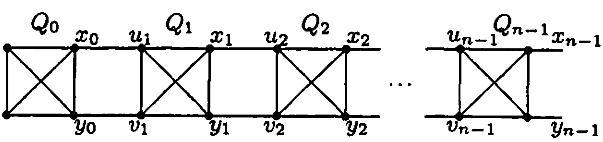


Figure 5. A chain of 4-cliques

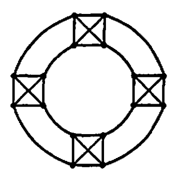


Figure 6. T_{4n} $(n = 4)$

As a result, we need only consider $|V(G)| > 8$. Let x_1 be a vertex of G with N_{x_1} being of type 2. From the claim, we can suppose that $Q_1 = \{x_1, y_1, u_1, v_1\}$ is the 4-clique of G containing x_1, $Q_2 = \{x_2, y_2, u_2, v_2\}$ the 4-clique containing two neighbors u_2, v_2 of Q_1, say $x_1 u_2, y_1 v_2 \in E(G)$, and Q_0 the 4-clique containing two neighbors x_0, y_0 of Q_1, say $x_0 u_1, y_0 v_1 \in E(G)$ (see Figure 5). For the vertex x_2, by the claim again, we have a 4-clique $Q_3 = \{x_3, y_3, u_3, v_3\}$ such that $x_2 u_3, y_2 v_3 \in E(G)$. Continuing in this way, since G is finite, let n be the maximum integer such that there are n 4-cliques $Q_i = \{x_i, y_i, u_i, v_i\}$, $0 \leq i \leq n-1$, with $x_i u_{i+1}, y_i v_{i+1} \in E(G)$. For x_{n-1}, by the claim we have a 4-clique $Q_n = \{x_n, y_n, u_n, v_n\}$ containing two neighbors u_n, v_n of Q_{n-1}, say $x_{n-1} u_n, y_{n-1} v_n \in E(G)$. By the 4-regularity of G and the maximality of n, we can see that $Q_n = Q_0$. Thus, $G \cong T_{4n}$, as shown in Figure 6. The theorem is proved. ■

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