DUAL HELLY-TYPE THEOREMS FOR UNIONS OF SETS STARSHAPED VIA STAIRCASE PATHS

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ABSTRACT. Let d be a fixed integer, $0 \leq d \leq 2$, and let $\mathcal{K}$ be a family of sets in the plane having simply connected union. For every countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}$, assume that $\cup\{K_n : n \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cup\{K : K \text{ in } \mathcal{K}\}$ has these properties as well.

In the finite case, define function g on $\{0, 1, 2\}$ by $g(0) = 2, g(1) = g(2) = 4$. Let $\mathcal{K}$ be a finite family of nonempty compact sets in the plane such that $\cup\{K : K \text{ in } \mathcal{K}\}$ has a connected complement. For d fixed, $d \in \{0, 1, 2\}$, and for every $g(d)$ members of $\mathcal{K}$, assume that the corresponding union is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cup\{K : K \text{ in } \mathcal{K}\}$ has these properties, also.

Most of these results are dual versions of theorems that hold for intersections of sets starshaped via staircase paths. The exception is the finite case above when $d = 2$. Surprisingly, although the result for $d = 2$ holds for unions of sets, no analogue for intersections of sets is possible.

1. INTRODUCTION

We begin with some definitions from [3]. Let S be a nonempty set in the plane. Set S is called an *orthogonal polygon* if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Set S is *horizontally convex* if and only if for each pair x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Set S is *orthogonally convex* if and only if S is both horizontally and vertically convex.

Let λ be a simple polygonal path in the plane whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Path λ is a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction.

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Edge $[v_{i-1}, v_i]$ will be called north, south, east, or west according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points. For $n \geq 1$, if the staircase path λ is a union of at most n edges, then λ is called a *staircase n-path*. For convenience, here we will assume that a staircase n-path is a union of exactly n edges.

Let $S \subseteq \mathbb{R}^2$. For points x and y in set S, we say x sees y (x is *visible* from y) via staircase paths if and only if there is a staircase path in S that contains both x and y. Set S is called *convex via staircase paths* (*staircase convex*) if and only if for every x, y in S, x sees y via staircase paths. Similarly, set S is *starshaped via staircase paths (staircase starshaped)* if and only if for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, denoted $Ker S$. Observe that a staircase starshaped set cannot be empty.

Many theorems in convexity that involve the usual concept of visibility via straight line segments have analogues that employ the notion of visibility via staircase paths. For example, the following staircase results for intersections of sets appear in [3]. The first theorem is a staircase analogue of [1, Theorem 1]. The second is a staircase variation of [6, Theorem 2], with a different result when $d = 2$.

Theorem A ([3, Theorem 1]). Let d be a fixed integer, $0 \leq d \leq 2$, and let $\mathcal{K}$ be a family of simply connected sets in the plane. For every countable subfamily $\{K_n : n \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_n : n \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. The $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties as well.

Theorem B ([3, Theorem 2]). Define function f on $\{0, 1\}$ by $f(0) = 3, f(1) = 4$. Let $\mathcal{K} = \{K_i : 1 \leq i \leq n\}$ be a finite family of compact sets in the plane, each having connected complement. For d fixed, $d \in \{0, 1\}$, and for every $f(d)$ members of $\mathcal{K}$, assume that the corresponding intersection is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cap\{K_i : 1 \leq i \leq n\}$ has these properties, also. There is no analogous Helly number for the case in which $d = 2$.

Each of the numbers $f(d)$ above is best possible.

Because theorems which hold for intersections of sets sometimes have dual versions for unions of sets (see [9] or [1, Theorem 2], for example), it is reasonable to ask if such duals hold here. Indeed, with one exception, dual versions do exist. However, an interesting aberration occurs in the finite case when $d = 2$. Although the result fails for intersections of sets and, in fact, no analogue is possible by [3, Example 4], the corresponding result does hold for unions of sets.

Throughout the paper, we will use the following terminology and notation. We say that a planar set S is simply connected if and only if for every simple closed curve $\delta \subseteq S$, the bounded region determined by δ lies in S. For points x and y, dist (x, y) will be the distance from x to y. If λ is a simple path containing points x and y, then $\lambda(x, y)$ will denote the subpath of λ from x to y (ordered from x to y). When x and y are distinct, $L(x, y)$ will represent their corresponding line. For any line L in $\mathbb{R}^2$, we let L_1 and L_2 denote the associated open halfplanes, $cl\, L_1$ and $cl\, L_2$ the corresponding closed halfplanes. Readers may refer to Valentine [11], to Lay [10], to Danzer, Grünbaum, Klee [7], and to Eckhoff [8] for discussions concerning Helly-type theorems, visibility via straight line segments, and starshaped sets.

2. THE RESULTS.

We begin with an analogue of [1, Theorem 2] for sets starshaped via staircase paths. Note that this result provides a dual version of Theorem A, with intersections of sets replaced by unions of sets.

Theorem 1. Let d be a fixed integer, $0 \le d \le 2$, and let $\mathcal{K}$ be a family of sets in the plane having simply connected union. For every countable subfamily $\{K_n : n \ge 1\}$ of $\mathcal{K}$, assume that $\cup\{K_n : n \ge 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cup\{K : K \text{ in } \mathcal{K}\}$ also is starshaped via staircase paths, and its staircase kernel contains a convex set of dimension d.

Proof. As in the proof of [1, Theorem 2], let $T = \cup\{K : K \text{ in } \mathcal{K}\}$ and, for each K_α in $\mathcal{K}$, define $N_\alpha = \{x : x \text{ in } T, x \text{ sees each point of } K_\alpha \text{ via staircase paths in } T\}$. Clearly $Ker\, K_\alpha \subseteq N_\alpha$ so $N_\alpha \neq \phi$. Using [4, Lemma 2] and an argument like one in the proof of [3, Theorem 1, Part 2], it is easy to show that each set N_α is simply connected.

Let $\mathcal{N}$ denote the family of all the N_α sets. We will follow the proof of [1, Theorem 2] to show that $\mathcal{N}$ satisfies the hypotheses of Theorem A. That is, we will show that for every countable subfamily $\{N_m : m \ge 1\}$ of $\mathcal{N}$, $\cap\{N_m : m \ge 1\}$ is starshaped via staircase paths and its staircase kernel contains a convex set of dimension at least d. It suffices to show that, for the associated subfamily $\{K_m : m \ge 1\}$ of $\mathcal{K}$, $Ker \cup \{K_m : m \ge 1\} \subseteq Ker \cap \{N_m : m \ge 1\}$. Choose $x \in Ker \cup \{K_m : m \ge 1\}$ and $y \in \cap \{N_m : m \ge 1\}$ to show that x sees y via staircase paths in $\cap\{N_m : m \ge 1\}$. Certainly $x \in \cup \{K_m : m \ge 1\}$ and x sees each point of $\cup\{K_m : m \ge 1\}$ via staircase paths in $\cup\{K_m : m \ge 1\}$. Also, y sees each point of $\cup\{K_m : m \ge 1\}$ via staircase paths in T, so y sees x via a staircase $\lambda(y, x) \subseteq T$. We assert that $\lambda(y, x) \subseteq \cap\{N_m : m \ge 1\}$. For $p \in \cup \{K_m : m \ge 1\}, x$ sees p via a staircase path $\delta(x, p)$ in $\cup\{K_m : m \ge 1\} \subseteq T$. Similarly, y sees p via a staircase path $\mu(y, p) \subseteq T$. By [4, Lemma 2] the region A bounded by $\lambda \cup \delta \cup \mu$

is an orthogonally convex (and staircase convex) polygon, and, since T is simply connected, $A \subseteq T$. Thus p sees each point of λ via staircase paths in T. Since this is true for every $p \in \cup\{K_m : m \geq 1\}, \lambda \subseteq \cap\{N_m : m \geq 1\}$, establishing the assertion. It follows that x sees y via staircase paths in $\cap\{N_m : m \geq 1\}$ and hence $x \in Ker \cap \{N_m : m \geq 1\}$, the desired result.

We conclude that $\mathcal{N}$ is a family of sets satisfying the hypotheses of Theorem A. By that theorem, $\cap\{N_\alpha : N_\alpha \text{ in } \mathcal{N}\}$ is starshaped via staircase paths and its staircase kernel contains a convex set of dimension at least d. It is easy to see that $\cap\{N_\alpha : N_\alpha \text{ in } \mathcal{N}\} = Ker\,T$, so $Ker T$ contains a d-dimensional convex set, finishing the proof of Theorem 1.

Example 1 demonstrates that we cannot replace *countable* with *finite* in the hypothesis of Theorem 1. (See Example 2 and Example 3 as well.)

Example 1. For every integer $n \geq 1$, let $A_{2n} = \{(x, y) : 0 \leq x \leq 2n+1, -1 \leq y \leq 0\}$ and let λ_{2n} denote the staircase 2-path from $(2n, 0)$ north to $(2n, 1)$, then west to $(2n-1, 1)$. Define $K_{2n} = A_{2n} \cup \lambda_{2n}$. (Figure 1 illustrates $K_2 \cup K_6$.) Every finite subfamily of $\{K_{2n} : n \geq 1\}$ has a union that is starshaped via staircase paths, and the corresponding staircase kernel contains a convex set of dimension two. However, the closed set $\cup\{K_{2n} : n \geq 0\}$ is not staircase starshaped.

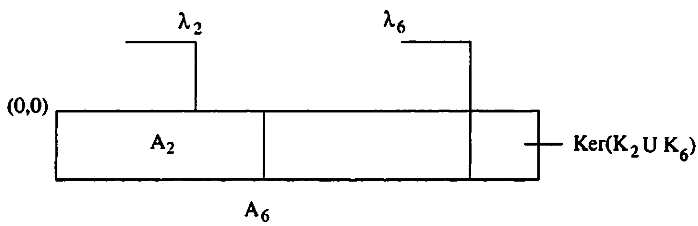

Figure 1

To obtain a dual version of Theorem B, we use a different approach.

The following preliminary lemma will be useful.

Technical Lemma. Let δ be a fixed positive number. Let S be a compact subset of $\mathbb{R}^2$, with $\mathbb{R}^2 \backslash S$ connected. Assume that S is starshaped via staircase paths and that $Ker\, S$ contains distinct points a and b, with $[a, b]$ horizontal and a west of b. Let $L = L(a, b)$, with L_1 the corresponding open halfplane north of L. Assume that $Ker\, S$ contains no point strictly west or strictly north of a and contains no point strictly east or strictly north of b. If every two points of S see via staircase paths a common δ

by δ square region in $cl\,L_1$, then $Ker\,S$ contains a (nondegenerate) square region in $cl\,L_1$.

Proof. By [5, Theorem 1], $Ker\,S$ is orthogonally convex (and staircase convex). Hence $[a,b] \subseteq Ker\,S$. There are two cases to consider.

Case 1. First consider the case in which $S \cap cl\,L_1$ contains a nondegenerate square region (say $\frac{\delta}{2}$ by $\frac{\delta}{2}$ or smaller) north of $[a,b]$. That is, each point of the square region is north of some point of $[a,b]$. Then such a square C exists on $[a,b]$. We will show that each point of S sees via staircase paths each point of C. Since $[a,b] \subseteq Ker\,S$, it is easy to show that each point of S in $L \cup L_2$ sees via staircase paths each point of C. Select x in $S \cap L_1$ to show that x sees via staircase paths each point of C. Let u be any point on the north edge of C. By hypothesis, x and u see via staircase paths a common δ by δ square region D in $cl\,L_1$. Certainly there are either points of D northeast of u or points of D northwest of u (or both). Without loss of generality, assume that there are points of D northeast of u, and hence there is a northeast $u - D$ staircase in S. Of course, each point of $[a,b]$ sees via staircase paths each point of D, and clearly C, D, and $[a,b]$ lie in a common staircase convex subset T of S. (See Figure 2.)

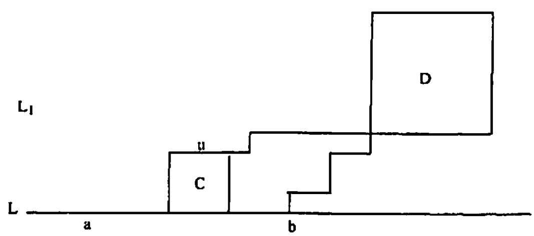

Figure 2

Considering possible locations for x in L_1 and using the fact that $[a,b] \subseteq Ker\,S$, it is not hard to show that T may be extended to include x, and hence x sees C via staircase paths. We conclude that $C \subseteq Ker\,S$, finishing Case 1.

Case 2. Now consider the case in which $S \cap cl\,L_1$ contains no square region north of $[a,b]$. By hypothesis, there must be a δ by δ square region in $cl\,L_1$, so such a region must be either northwest of a or northeast of b. Notice that we cannot have both situations, since points from the north edges of the two squares could not see via staircase paths a common δ by δ square region in $cl\,L_1$ without introducing a square region north of $[a,b]$, contradicting our hypothesis for Case 2. Without loss of generality, assume that $S \cap cl\,L_1$ contains a δ by δ square region northeast of b but no such region northwest of a. Since points of this region see b via staircase paths, there is at b either a north segment or an east segment $[b, c], b \neq c$. By hypothesis, no point of $(b, c]$ can lie in $Ker\, S$. Thus there is on (b, c) a sequence $\{b_n\}$ converging to b and a corresponding sequence $\{w_n\}$ in S such that w_n cannot see b_n via staircase paths. It follows by an easy argument that w_n sees via staircase paths no point of $[b_n, c]$. Without loss of generality, assume that $dist\,(b_n, b) < \frac{\delta}{2}$ for each n.

By hypothesis, for each n, w_n and c see (via staircase paths) a common δ by δ square region D_n in $cl\, L_1$, and by comments above such a region must be northeast of b. We assert that this forces w_n to see via staircase paths points of (b_n, c) near b_n:

First consider the case in which c is east of b. If w_n is west of (or on) the vertical line at b_n, it is clear that w_n sees via staircase paths each point of $[b_n, c]$. If w_n is east of this vertical line and if there is a point p of $(b_n, c]$ strictly south of D_n, then using the facts that w_n and c see via staircase paths each point of D_n and $[a, b] \subseteq Ker\, S$, it is easy to show that w_n sees p and hence each point of $[b_n, p]$ via staircase paths. (See Figure 3a.) If w_n is east of the vertical line at b_n and there are no points of $(b_n, c]$ strictly south of D_n, a similar argument shows that w_n sees each point of $[b_n, c]$ via staircase paths.

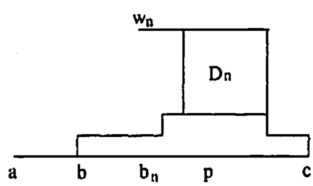

Figure 3a

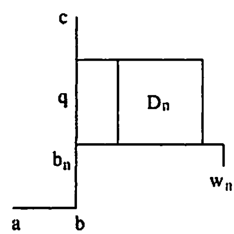

Figure 3b

Figure 3

Now consider the case in which c is north of b. If w_n is south of (or on) the horizontal line at b_n, then w_n sees via staircase paths each point of $[b_n, c]$. If w_n is north of this horizontal line and if there is a point q of $(b_n, c]$ strictly west of D_n, then w_n sees each point of $[b_n, q]$ via staircase paths. (See Figure 3b.) If w_n is north of this horizontal line and there are no points of $(b_n, c]$ strictly west of D_n, again w_n sees each point of $[b_n, c]$ via staircase paths.

In each case, w_n sees via staircase paths points of (b_n, c) near b_n. However, this contradicts our choice of w_n and b_n. Our hypothesis for Case 2 must be false, and this situation cannot occur. Case 1 is the only possibility, finishing the proof of the lemma.

We will establish the following "dual" to Theorem B. Note that the associated Helly numbers $f(d)$ and $g(d)$ agree only when $d = 1$.

Theorem 2. Define function g on $\{0,1,2\}$ by $g(0) = 2, g(1) = g(2) = 4$. Let $\mathcal{K}$ be a finite family of compact sets in the plane such that $\cup\{K : K \text{ in } \mathcal{K}\}$ has a connected complement. For d fixed, $d \in \{0,1,2\}$, and for every $g(d)$ members of $\mathcal{K}$, assume that the corresponding union is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least d. Then $\cup\{K : K \text{ in } \mathcal{K}\}$ has these properties, also.

Proof. Let $\mathcal{K} = \{K_i : 1 \leq i \leq n\}$, and define $S = \cup\{K_i : 1 \leq i \leq n\}$. We begin with some preliminary observations. First notice that set S is a nonempty compact set in the plane, with $\mathbb{R}^2 \backslash S$ connected. Moreover, for each d value, every two points of S are visible via staircase paths from a common point of S. Hence by [5, Theorem 1], S is starshaped via staircase paths and $Ker\, S$ is orthogonally convex (and staircase convex). Further, by [5, Proposition 1], for every point s in S, the associated visibility set $V_s \equiv \{y : s \text{ sees } y \text{ via staircase paths in } S\}$ is closed. Thus $Ker\, S \equiv \cap\{V_s : s \text{ in } S\}$ is closed as well. Finally, for s in S and x, y in V_s, if S contains an $x - y$ staircase path λ, then by the proof of [2, Lemma 1] or the proof of [5, Theorem 1], $\lambda \subseteq V_s$. Of course, an analogous result holds when $x, y \in Ker\, S$.

Choose $p \in Ker\, S$. If $d = 0$, there is nothing more to prove. If $d = 1$, we use an argument like one in Theorem 1: For each set K_i in $\mathcal{K}$, define $N_i = \{x : x \text{ in } S, x \text{ sees each point of } K_i \text{ via staircase paths in } S\}$. We will show that the family $\{N_i : 1 \leq i \leq n\}$ satisfies the hypothesis of Theorem B: Fix $i, 1 \leq i \leq n$. For each s in K_i, the corresponding visibility set V_s is closed, so $N_i \equiv \cap\{V_s : s \text{ in } K_i\}$ is closed as well, hence compact.

To see that set N_i has a connected complement, observe that $\mathbb{R}^2 \backslash N_i$ has exactly one unbounded component, and this component contains the connected set $\mathbb{R}^2 \backslash S$. Hence if $\mathbb{R}^2 \backslash N_i$ is not connected, it must have a bounded component $B \subseteq S \backslash N_i$. Let $b \in B$, with L the vertical line at b. Since N_i is compact, there is a first point a of N_i north of b on L. Similarly, there is a first point c of N_i south of b on L. For each point p in $K_i, a, c \in V_p$. Since $[a, c]$ is (trivially) a staircase path in S, it follows from our preliminary observations that $[a, c] \subseteq V_p$. Thus b sees p via staircase paths in S. Since this is true for all p in $K_i, b \in N_i$, impossible. We conclude that no such B exists, and $\mathbb{R}^2 \backslash N_i$ is connected, the desired result.

We have shown that $\{N_i : 1 \leq i \leq n\}$ is a finite family of compact sets in the plane, each having connected complement. Moreover, using our hypothesis, for every $g(1) = 4$ of these sets, the corresponding intersection is starshaped via staircase paths, and the associated staircase kernel contains a convex set of dimension at least one. We may apply Theorem B to conclude that $\cap\{N_i : 1 \le i \le n\} \equiv Ker\, S$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least one. This finished the proof when $d = 1$.

Since Theorem B has no analogue when $d = 2$, the argument for $d = 2$ will require a different approach. For every four of the K_i sets, say K_1, K_2, K_3, K_4, the associated union is starshaped via staircase paths, and its kernel contains a full two-dimensional convex set, hence a square region $U(1,2,3,4)$ in S. Selecting one square region for every four of the K_i sets, we obtain a finite family of square regions. Assume that the smallest of these is δ by $\delta, \delta > 0$. Then certainly every four points of S see via staircase paths a common δ by δ square in S.

By the earlier argument for $d = 1, Ker\, S$ contains at least a nondegenerate segment. Suppose that $Ker\, S$ contains no two-dimensional convex set, to reach a contradiction. There are several cases to consider.

Case 1. Suppose that $Ker\, S$ contains three or more noncollinear segments. Recall that $Ker\, S$ is staircase convex and hence any two of its points are joined by a staircase path in $Ker\, S$. For such a staircase with edges $[v_{i-1}, v_i], 1 \le i \le n, 3 \le n$, and for any horizontal edge $e = [v_{i-1}, v_i], i \ne 1, i \ne n, S$ contains no point strictly north or strictly south of e. (Using our preliminary observations, it is easy to show that such a point would introduce new staircase 2-paths in $Ker\, S$ to produce a rectangular region in $Ker\, S$.) An analogous statement holds for a vertical edge and points strictly east or strictly west of it.

Similarly, if $[v_1, v_2] \cup [v_2, v_3]$ is any east-north staircase in $Ker\, S, S$ can contain no point strictly northeast of v_2, for such a point also would introduce a rectangular region in $Ker\, S$. Analogous statements hold for east-south, west-north, and west-south staircase 2-paths in $Ker\, S$. It is not hard to see that $Ker\, S$ contains a staircase 3-path and that, for an appropriate pair of opposing directions, say southwest and northeast, all staircase 3-paths in $Ker\, S$ rise from southwest to northeast (fall from northeast to southwest).

From the points of $Ker\, S$ which lie as far south as possible, select the point a_0' as far west as possible. Similarly, from the points of $Ker\, S$ as far west as possible, select a_0 as far south as possible. By earlier comments, either $a_0 = a_0'$ or a_0 sees a_0' via a staircase 2-path in $Ker\, S$. Similarly, from points of $Ker\, S$ as far north as possible, select b' as far east as possible, and from points of $Ker\, S$ as far east as possible, select b as far north as possible. Either $b = b'$ or b sees b' via a staircase 2-path in $Ker\, S$ (See Figure 4a.) It is not hard to see that either $a_0 = a_0'$ or $b = b'$ (or both): Otherwise, any interior point of S would introduce interior points in $Ker\, S$, violating our hypothesis. Thus without loss of generality we assume that $b = b'$. (See

Figure 4b.)

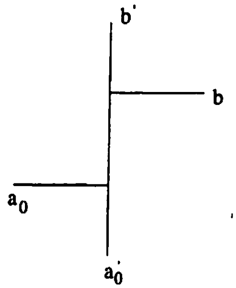

Figure 4a

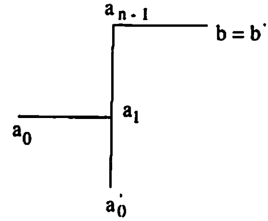

Figure 4b

Figure 4

Clearly each point a_0, a_0' sees b via a staircase path in $Ker\, S$, and for at least one of a_0, a_0', say a_0, a_0 sees b via a staircase n-path λ in $Ker\, S$ with $n \geq 3$. Moreover, λ is the only $a_0 - b$ staircase path in S.

Let $\lambda_i = [a_{i-1}, a_i], 1 \leq i \leq n$, denote the edges of λ, where $a_n = b$ (and $n \geq 3$). Without loss of generality, assume that $\overrightarrow{a_{n-1}a_n}$ is east. Observe that any point in $S \backslash \lambda$ lies either southwest of a_1 or northeast of a_{n-1}. Thus any square region in S has its location either southwest of a_1 or northeast of a_{n-1}. Moreover, if $a_0 \neq a_0'$, such a square region must be northeast of a_{n-1}. Since every four points of S see a common δ by δ square region, for one of these two locations, and we may assume that it is northeast of a_{n-1}, every two points of S see via staircase paths a δ by δ square region here. We have exactly the situation in the technical lemma, letting a_{n-1} play the role of a. Thus $Ker\, S$ contains a square region. Of course, this violates our assumption that $Ker\, S$ contain no two-dimensional convex set, and the situation in Case 1 cannot occur.

Case 2. Suppose that $Ker\, S$ has exactly two noncollinear segments, say s and t. For convenience, assume that s meets t at the origin θ. If θ were relatively interior to both s and t, then there could be no interior points of S without introducing a two-dimensional set in $Ker\, S$, impossible. If θ were relatively interior to t but not s, assume that $t = [a, b]$ lies on the x axis, with s south of θ. There can be no interior points of S below the x axis (without introducing interior points in $Ker\, S$), so any δ by δ square in S necessarily lies north of the x axis. Again we have the situation in the technical lemma, forcing $Ker\, S$ to contain a square region, impossible.

The only remaining possibility is the one in which θ is not relatively interior to s or to t. Assume that t is west of θ, s south of θ. Let L and M denote the x and y axes, respectively, with corresponding open halfplanes labeled so that $s \cap L_1 = t \cap M_1 = \phi$. Clearly $L_2 \cap M_2 \cap S = \phi$, since any point of S in $L_2 \cap M_2$ would introduce a two-dimensional set in $Ker\, S$. If some two points of S could see no $\frac{\delta}{2}$ by $\frac{\delta}{2}$ square region in L_1 and some two points of S could see no $\frac{\delta}{2}$ by $\frac{\delta}{2}$ square region in M_1, then these (not necessarily distinct) four points of S would see a $\frac{\delta}{2}$ by $\frac{\delta}{2}$ square region only in $L_2 \cap M_2$, impossible. Thus for at least one of L_1 or M_1, say L_1, every two points of S see via staircase paths a $\frac{\delta}{2}$ by $\frac{\delta}{2}$ square region in L_1. Using $\frac{\delta}{2}$ in place of δ, again we have the situation in the technical lemma, and $Ker\, S$ contains a square region, impossible. Thus the situation in Case 2 cannot occur either.

Case 3. Assume that $Ker\, S$ is just a segment $[a, b]$, with a west of b. Let L be the associated line, with L_1 and L_2 the corresponding open halfplanes. Using our hypothesis, for at least one of L_1 or L_2, say L_1, every two points of S see a $\frac{\delta}{2}$ by $\frac{\delta}{2}$ square region in $cl\, L_1$. Again using $\frac{\delta}{2}$ in place of δ, the technical lemma yields interior points in $Ker\, S$, so Case 3 cannot occur.

We conclude that our supposition must be false, and $Ker\, S$ does contain a full two-dimensional convex set. This finishes the argument when $d = 2$ and completes the proof of Theorem 2.

It is interesting to observe that no finite analogues of the Helly numbers $g(d)$ exist if we replace finite by infinite in Theorem 2, as Example 2 illustrates.

Example 2. Let T denote the triangular region having vertices $(0,0), (1,0), (1, -\frac{1}{2})$. For $n \geq 1$, let S_n denote the northeast 2-staircase having vertices $(\frac{1}{2^n}, 0), (\frac{1}{2^n}, \frac{1}{2^n}), (\frac{3}{2^{n+1}}, \frac{1}{2^n})$. For $j \geq 1$, let $K_j = \cup\{T \cup S_n : 1 \leq n \leq j\}$. (See Figure 5.) Every four and indeed every finite family of sets K_j have a union whose staircase kernel contains a full two-dimensional set. However, the compact set $\cup\{K_j : j \geq 1\}$ has kernel $\{(0,0)\}$.

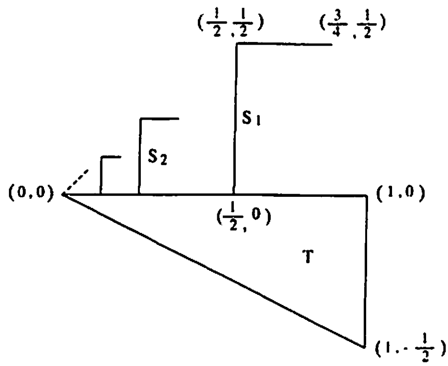

Figure 5

By adapting the example, allowing $\cup\{K_j : j \geq 1\}$ to be bounded but not closed, the union need not even be starshaped.

Example 3. For $n \geq 1$, let $T_n = \{(x, y) : \frac{3}{2^{n+2}} \leq x \leq 1, -\frac{1}{2} \leq y \leq 0\}$, and let S_n denote the northeast 2-staircase with vertices $(\frac{1}{2^n}, 0), (\frac{1}{2^n}, \frac{1}{2})$, $(\frac{3}{2^{n+1}}, \frac{1}{2})$. Define $K_j = \cup\{T_n \cup S_n : 1 \leq n \leq j\}$. (See Figure 6 for K_3.) Then every finite family of sets K_j has a union whose kernel contains a convex set of dimension two, yet $\cup\{K_j : j \geq 1\}$ is not starshaped via staircase paths.

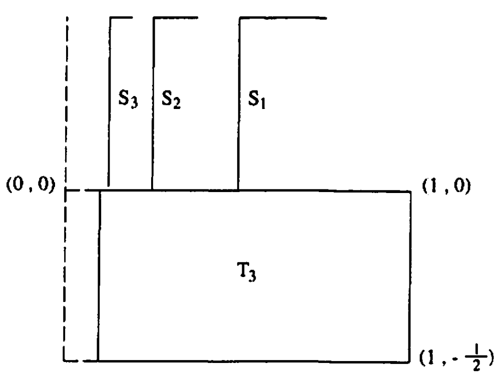

Figure 6

Of course, Example 1 provides a similar example, with $\cup\{K_{2n} : n \geq 0\}$ closed, unbounded, and not starshaped via staircase paths.

To see that $g(1) = 4$ is best, consider the following easy example.

Example 4. Let S_1 denote the set in Figure 7a, and let S_2, S_3, S_4 represent rotations of S_1 clockwise about point p through $\frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$, respectively. (See Figure 7b.) Every three (not necessarily distinct) S_i sets have a union whose staircase kernel contains a one-dimensional convex set, yet $Ker\,(\cup\{S_i : 1 \leq i \leq 4\}) = \{p\}$.

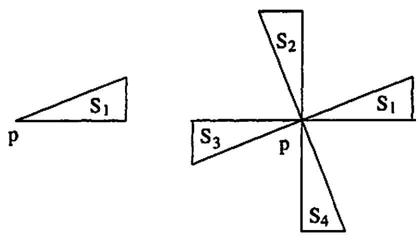

Figure 7a

Figure 7b

Figure 7

The following example concerns the situation for $g(2)$.

Example 5. Let A_1, A_2, A_3, A_4 denote the four rectangular regions along segment $[x, y]$ in Figure 8. Let $B_i = A_i \cup [x, y], 1 \leq i \leq 4$, and let $C = B_3 \cup B_4$. Every two (not necessarily distinct) sets from $\{B_1, B_2, C\}$ have a union whose staircase kernel contains a two-dimensional convex set, yet $Ker\,(B_1 \cup B_2 \cup C) = [x, y]$. Thus the associated Helly number for $d = 2$ is at least three.

Similarly, every three (although not every two) distinct B_i sets have a union whose staircase kernel contains a two-dimensional convex set. Again, $Ker\,(\cup\{B_i : 1 \leq i \leq 4\}) = [x, y]$. Because the kernels of $B_1 \cup B_3$ and $B_2 \cup B_4$ have no interior points, this example does not prove that the associated Helly number is four. However, perhaps it would be sufficient that every four distinct sets in the finite family satisfy the hypothesis in Theorem 2. If so, then this example would show that $g(2) = 4$ is best.

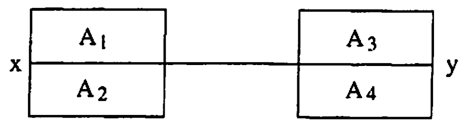

Figure 8

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