Sums and products of the Jacobsthal and Jacobsthal-Lucas numbers and their determinantal representation *

Zhaolin Jiang Fuliang Lu†
School of Sciences, Linyi University, Linyi, Shandong 276005, China.

Abstract

Denote by $\{J_n\}$ and $\{j_n\}$ the Jacobsthal number and the Jacobsthal-Lucas number respectively. Let $\mathcal{J}_n = J_n \times j_n$ and $\mathfrak{J}_n = J_n + j_n$. In this paper, we give some determinantal and permanental representations of $\mathcal{J}_n$ and $\mathfrak{J}_n$. Also, complex factorization formulas for the numbers are presented.

Keywords: Jacobsthal number; Jacobsthal-Lucas number; matrix

1. Introduction

The Jacobsthal and Jacobsthal-Lucas sequences are defined by the following recurrence relations, respectively:

$$J_{n+2} = J_{n+1} + 2J_n \quad \text{where } J_0 = 0, J_1 = 1;$$

$$j_{n+2} = j_{n+1} + 2j_n \quad \text{where } j_0 = 2,\ j_1 = 1.$$

The properties of Jacobsthal and Jacobsthal-Lucas numbers have been considered by many mathematicians. Horadam [6, 7] gave Cassini-like formulas of Jacobsthal and Jacobsthal-Lucas numbers. Čerin [2] considered sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products. Djordjevic and Srivastava [4] presented a systematic

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†Corresponding author. *E-mail addresses:* jzh1208@sina. com, flianglu@163. com.

investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal-Lucas numbers. Yılmaz and Bozkurt [15] investigated a family of matrices such that the permanents of the matrices are Jacobsthal and Jacobsthal-Lucas numbers. They also gave complex factorization formulas for the Jacobsthal sequence.

Let $\mathcal{J}_n = J_n \times j_n$ and $\mathfrak{J}_n = J_n + j_n$. Then we can get the following recurrence relations:

$$\mathcal{J}_{n+2} = 5\mathcal{J}_{n+1} - 4\mathcal{J}_n \quad \text{where } \mathcal{J}_0 = 0, \mathcal{J}_1 = 1;$$

$$\mathfrak{J}_{n+2} = \mathfrak{J}_{n+1} + 2\mathfrak{J}_n \quad \text{where } \mathfrak{J}_0 = 2, \ \mathfrak{J}_1 = 2.$$

Let $A = [a_{i,j}]$ be an $n \times n$ matrix with row vectors $r_1, r_2, \cdots, r_m$. The permanent of A is defined by

$$\mathrm{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n [11]. We call A is contractible on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. If A is a nonnegative matrix and B is a contraction of A, then $\mathrm{per}A = \mathrm{per}B$ [9].

Matrix method is used frequently in the proof of the property of some numbers. By the determinant of tridiagonal matrix, an identity of Fibonacci number is proved [5]. They also constructed a type of determinants to give new proof of the Fibonacci identities. Following them, Yaşar and Bozkurt [14] gave another proof of Pell identities by using the determinant of tridiagonal matrix. By defining some new matrices, Dasdemir [3] presented some elementary identities between modified Pell and Pell-Lucas numbers. Yılmaz and Bozkurt [16] considered relationships between Hessenberg matrices and the Pell and Perrin numbers.

A directed pseudo graph $G = (V, E)$, with set of vertices $V(G) = \{1, 2, \cdots, n\}$ and set of edges $E(G) = \{e_1, e_2, \cdots, e_m\}$, is a graph in which loops and multiple edges are allowed. A directed graph represented with arrows on its edges, each arrow pointing towards the head of the corresponding arc. The adjacency matrix $A(G) = [a_{i,j}]$ is $n \times n$ matrix, defined by the rows and the columns of $A(G)$ are indexed by $V(G)$, in which $a_{i,j}$ is the number of edges joinning v_i and v_j [8].

There are many relations between well-known number sequences and the properties of graphs. Bogdanowicz [1] derived an explicit formula which corresponds to the Fibonacci numbers for the number of spanning trees for a type of graph. Startek et al. [12] described the number of independent sets in graphs with two elementary cycles using Fibonacci and Lucas numbers. Lee [10] considered k-Lucas and k-Fibonacci sequences and investigated the relationships between these sequences and 1-factors of a bipartite graph. Tesler [13] got the number of perfect matchings for a type of lattice on a Möbius strip involving q-analogue of the Fibonacci numbers.

In this paper, we investigate relationships between adjacency matrices of graphs and the $\mathcal{J}_n$ and the $\mathfrak{J}_n$ sequences. We also give complex factorization formulas for the numbers.

2. Determinantal representations of $\mathcal{J}_n$ and $\mathfrak{J}_n$

In this section, we consider a class of pseudo graph given in Figure 1 and Figure 2, respectively. Then we investigate relationships between permanents of the adjacency matrices of the graphs and $\mathcal{J}_n$ and $\mathfrak{J}_n$.

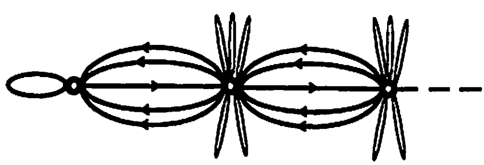

Figure 1.

Let $A_n = [a_{ij}]_{n\times n}$ be the adjacency matrix of the graph given by Figure 1, in which $a_{11} = a_{t,t+1} = 1, a_{ss} = 5, a_{l,l-1} = -4$ for $t = 1, 2, \cdots, n-1$,

$s = 2, 3, \cdots, n$ and $l = 3, 4, \cdots, n$ and otherwise 0. That is

$$A_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 5 & 1 & 0 & \cdots & 0 \\ 0 & 4 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 4 & 5 & 1 \\ 0 & 0 & \cdots & 0 & 4 & 5 \end{bmatrix}.$$

Let S be a $(1, -1)$-matrix of order n, defined as

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 \end{bmatrix}.$$

Denote the matrices $A_n \circ S$ by H_n, where $A_n \circ S$ denotes Hadamard product of A_n and S. That is

$$H_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 5 & 1 & 0 & \cdots & 0 \\ 0 & -4 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -4 & 5 & 1 \\ 0 & 0 & \cdots & 0 & -4 & 5 \end{bmatrix}. \tag{1}$$

Theorem 1. *Let H_n be the matrix as in* (1). *Then* $\text{per}H_n = \text{per}H_n^{(n-2)} = \mathcal{J}_n$.

Proof. By definition of the matrix H_n, it can be contracted on column 1. Let $H_n^{(r)}$ be the rth contraction of H_n. If $r = 1$, then

$$H_n^{(1)} = \begin{bmatrix} 5 & 1 & 0 & \cdots & 0 & 0 \\ -4 & 5 & 1 & 0 & \cdots & 0 \\ 0 & -4 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -4 & 5 & 1 \\ 0 & 0 & \cdots & 0 & -4 & 5 \end{bmatrix}.$$

Note that $H_n^{(1)}$ also can be contracted by the first column, then

$$H_n^{(2)} = \begin{bmatrix} 21 & 5 & 0 & \cdots & 0 & 0 \\ -4 & 5 & 1 & 0 & \cdots & 0 \\ 0 & -4 & 5 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -4 & 5 & 1 \\ 0 & 0 & \cdots & 0 & -4 & 5 \end{bmatrix}.$$

Going with this process, we have $H_n^{(n-2)} = \begin{bmatrix} \mathcal{J}_{n-1} & \mathcal{J}_{n-2} \\ -4 & 5 \end{bmatrix}$, so

$$\text{per}H_n = \text{per}H_n^{(n-2)} = \mathcal{J}_n. \quad \blacksquare$$

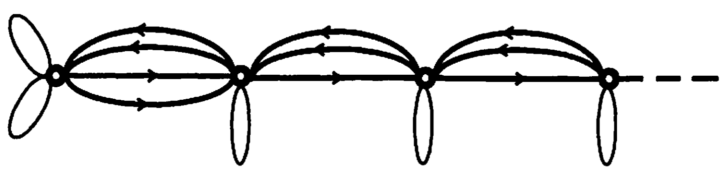

Figure 2.

Let $K_n = [k_{ij}]_{n\times n}$ be the adjacency matrix of the pseudo graph given in Figure 2, with $k_{11} = k_{12} = k_{t,t-1} = 2, k_{ss} = 1, k_{l,l+1} = 1$ for $t = 2, \cdots, n$, $s = 2, 3, \cdots, n$ and $l = 2, 3, \cdots, n$ and otherwise 0. That is

$$K_n = \begin{bmatrix} 2 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 2 & 1 \end{bmatrix}. \tag{2}$$

Theorem 2. *Let K_n be the matrix as in* (2). *Then* $\text{per}K_n = \text{per}K_n^{(n-2)} = \mathfrak{J}_n$.

Proof. By definition of the matrix K_n, it can be contracted on column 1. Let $K_n^{(r)}$ be the rth contraction of K_n. If $r = 1$, then

$$K_n^{(1)} = \begin{bmatrix} 6 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 2 & 1 \end{bmatrix}.$$

Note that $K_n^{(1)}$ also can be contracted by the first column, then

$$K_n^{(2)} = \begin{bmatrix} 10 & 6 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 2 & 1 \end{bmatrix}.$$

Going with this process, we have $K_n^{(n-2)} = \begin{bmatrix} \mathfrak{J}_{n-1} & \mathfrak{J}_{n-2} \\ 2 & 1 \end{bmatrix}$, so

$$\text{per}K_n = \text{per}K_n^{(n-2)} = \mathfrak{J}_n. \quad \blacksquare$$

Denote the matrices $K_n \circ S$ by B_n. That is

$$B_n = \begin{bmatrix} 2 & 2 & 0 & \cdots & 0 & 0 \\ -2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -2 & 1 \end{bmatrix}.$$

Then we have $\det A_n = \text{per}H_n = \mathcal{J}_n$ and $\det B_n = \text{per}K_n = \mathfrak{J}_n$.

Let C_{n+1} be an $(n+1) \times (n+1)$ matrix defined as

$$C_{n+1} = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -2 & 1 \end{bmatrix}.$$

Similar to Theorem 2, it can be proved that $\text{per}(C_{n+1} \circ S_{n+1}) = \det C_{n+1} = \mathfrak{J}_n$.

3. Complex factorization formulas

In this section, we give complex factorization formulas for $\mathcal{J}_n$ and $\mathfrak{J}_n$.

Theorem 3. $\mathcal{J}_n = \prod_{k=1}^{n-1}(5 + 2\cos\frac{k\pi}{n})$.

Proof. The characteristic equation of A_n is

$$0 = \det(A_n - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 5-\lambda & 1 & 0 & \cdots & 0 \\ 0 & 4 & 5-\lambda & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 4 & 5-\lambda & 1 \\ 0 & 0 & \cdots & 0 & 4 & 5-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 5-\lambda & 1 & & & \\ 4 & 5-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 4 & 5-\lambda & 1 \\ & & & 4 & 5-\lambda \end{vmatrix},$$

here I is the identity matrix. By [17], the eigenvalues of the matrix

$$\begin{bmatrix} 5-\lambda & 1 & & & \\ 4 & 5-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 4 & 5-\lambda & 1 \\ & & & 4 & 5-\lambda \end{bmatrix}$$

are $5 + 2\cos\frac{k\pi}{n} (k = 1, 2, \cdots, n-1)$. So the result follows. ■

Theorem 4. $\mathfrak{J}_n = 2\prod_{k=1}^{n}(1 + \sqrt{2}i\cos\frac{k\pi}{n+1})$.

Proof. The characteristic equation of C_{n+1} is

$$0 = \det(C_{n+1} - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 & \cdots & 0 \\ 0 & -2 & 1-\lambda & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2 & 1-\lambda & 1 \\ 0 & 0 & \cdots & 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 & & & \\ -2 & 1-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 1-\lambda & 1 \\ & & & -2 & 1-\lambda \end{vmatrix}.$$

By [17], the eigenvalues of the matrix

$$\begin{bmatrix} 1-\lambda & 1 & & & \\ -2 & 1-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 1-\lambda & 1 \\ & & & -2 & 1-\lambda \end{bmatrix}$$

are $1+\sqrt{2}i\cos\frac{k\pi}{n+1}(k=1,2,\cdots,n)$. So the result follows. ■

References

[1] Z. R. Bogdanowicz, Formulas for the number of spanning trees in a fan, Applied Mathematical Sciences 2 (2008), 781–786.

[2] Z. Čerin, Sums of squares and products of Jacobsthal numbers, Journal of Integer Sequences 10 (2007), 07. 2. 5.

[3] A. Dasdemir, On the Pell, Pell-Lucas and modified Pell numbers by matrix method, Applied Mathematical Sciences 5 (2011), 3173–3181.

[4] G. B. Djordjevic and H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, Mathematical and Computer Modelling 42 (2005), 1049–1056.

[5] J. Feng, Fibonacci identities via the determinant of tridiagonal matrix, Appl. Math. Comput. 217 (2011), 5978–5981.

[6] A. F. Horadam, Jacobsthal representation numbers, Fibinacci Quarterly 34 (1996), 40–54.

[7] A. F. Horadam, Jacobsthal and Pell curves, Fibonacci Quarterly 26 (1988), 79–83.

[8] F. Koken and D. Bozkurt, On the Jacobsthal numbers by matrix method, Int. J. Contemp. Math. Sciences 3 (2008), 605–614.

[9] E. Kılıç and Dursun Taşçı, On families of bipartite graphs associated with sums of Fibonacci and Lucas numbers, Ars Combinatoria 89 (2008), 31–40.

[10] G. Y. Lee, k-Lucas numbers and associated bipartite graphs, Linear Algebra Appl. 320 (2000), 51–61.

[11] H. Minc, Encyclopedia of mathematics and its applications, permanents, Vol. 6, Addison-Wesley Publishing Company, London, 1978.

[12] M. Startek, A. Wloch, I. Wloch, Fibonacci numbers and Lucas numbers in graphs, Discrete Appl. Math., 157 (2009), 864–868.

[13] G. Tesler, Matchings in graphs on non-orientable surfaces, J. Combin. Theory Ser. B 78 (2) (2000), 198–231.

[14] M. Yaşar and D. Bozkurt, Another proof of Pell identities by using the determinant of tridiagonal matrix, Appl. Math. Comput. (2011), doi:10.1016/j.amc.

[15] F. Yılmaz and D. Bozkurt, The adjacency matrix of one type of directed graph and the Jacobsthal Numbers and their determinantal representation, Journal of Applied Mathematics, (2012), doi:10.1155/2012/423163.

[16] F. Yılmaz and D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers, Journal of Number Theory, 131 (2011), 1390–1396.

[17] W. C. Yueh, Eigenvalues of several tridiagonal matrices, Applied Mathematics E-Notes 5 (2005), 66–74.