ON RIGHT DERIVATIONS OF WEAK BCC-ALGEBRAS

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Abstract. In this paper,the notion of right derivation of a weak BCC-algebra is introduced and some related properties are investigated. Also, we consider regular right derivation and d-invariant on weak BCC-ideals in weak BCC-algebras.

1. Introduction

In the theory of rings and near rings, the properties of derivations are important .Several authors [16], [15], [11], [10], [5], [4] have studied BCI-algebras, BCK-algebras, BCC-algebras and weak BCC-algebras. In [12], Jun and Xin applied the notion of derivations in rings and near-rings theory to BCI-algebras and also introduced a regular derivation in BCI-algebras and in [8], [18], [21]. In [17], Prabpayak and Leerawat applied the notion of a regular derivation in BCI-algebras to BCC-algebras and also investigated some of its related properties. In [20], Thomys describes derivations of weak BCC-algebras which the condition $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch. In [8], [18], [21], Szasz, Ferrari and Xin, Li and Lu applied the notion of derivations to lattices. They investigated some of its properties, defined a d-invariant ideal and gave conditions for an ideal to be d-invaryant. In [5], Dudek and Zhang introduced the notion of f-derivations of BCI-algebras and they gave a characterizations of a p-semisimple BCI-algebra using regular f-derivations. In [9], [19], Firat and Thomys applied the notion of f-derivation in BCI-algebras to BCC-algebras and weak BCC-algebras and also investigated some of its related properties.
In [13], Lee and Kim introduced the notion of derivation in lattice implication algebra, and considered the properties of derivations in lattice implication algebras. In [1], Abujabal and Al-Shehri introduced the notion of left derivation and gave a condition for left derivation to be regular,and they gave a characterization of a left derivation of a semisimple BCI-algebra.

In this paper , we introduced the notion of right derivation of a weak BCC-algebra, and investigated some related properties.

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2. Preliminaries

The BCC-operation will be denoted by juxtaposition. Dots will be only used to avoid repetitions of brackets. For example , the formula $((xy)(zy))(xz) = 0$ will be written in the abbreviated form as $(xy.zy).xz = 0$.

Definition 2.1. A *weak BCC-algebra* is a system $(G;.,0)$ of type (2,0) satisfying the following axioms :

(i) $(xy.zy).xz = 0$,

(ii) $xx = 0$,

(iii) $x0 = x$,

(iv) $xy = yx = 0 \Longrightarrow x = y$.

Weak BCC-algebras are called *BZ-algebras* by many mathematicians, especially from China and Korea, (cf.[6], [22], [23]), but we save the first name because it coincides with the general concept of names presented in the book [14] for algebras of logic.

A weak BCC- algebra satisfying the identity

(v) $0x = 0$

is called a *BCC- algebra.* A BCC-algebra with the condition

(vi) $(x.xy)y = 0$

is called a *BCK-algebra.*

One can prove (see [2] or [3]) that a BCC-algebra is BCK-algebra if and only if it satisfies the identity

(vii) $xy.z = xz.y$.

In any weak BCC-algebra we can define a natural partial order $\leq$ by putting

$$x \leq y \Longleftrightarrow xy = 0.$$

Directly from the axioms of weak BCC-algebras we can see that the following two implications

$(viii)$ $x \leq y \Longrightarrow xz \leq yz$,

(ix) $x \leq y \Longrightarrow zy \leq zx$

are valid for all $x, y, z \in G$.

The set of all minimal (with respect to $\leq$) elements of G is denoted by $I(G)$. Elements belonging to $I(G)$ are called *initial.*

In the investigation of algebras connected with various types of logics an important role plays the so-called *Dudek's map* φ defined as $\varphi(x) = 0x$. The main properties of this map in the case of weak BCC-algebras are collected in the following theorem proved in [6].

Theorem 2.2. Let G be a weak BCC-algebra. Then

$(1)\varphi^2(x) \leq x$,

$(2)x \leq y \Longrightarrow \varphi(x) = \varphi(y)$,

$(3)\varphi^3(x) = \varphi(x)$,

$(4)\varphi^2(xy) = \varphi^2(x)\varphi^2(y)$,

for all $x, y \in G$.

Theorem 2.3. $I(G) = \{a \in G : \varphi^2(a) = a\}$.

Comparing this result with Theorem 2.2 (4) we obtain

Corollary 2.4. $I(G)$ is a subalgebra of G.

Corollary 2.5. $I(G) = \varphi(G)$ for any weak BCC-algebra G.

The set $B(a) = \{x \in G : a \leq x\}$, where $a \in I(G)$,is called a *branch* of G

initiated by a. The branch initiated by 0 is the greatest BCC-algebra contained in G.

Definition 2.6. For any weak BCC-algebra G we consider three subsets

$Ker\varphi = \{x \in G : \varphi(x) = 0\}$,

$G(G) = \{x \in G : \varphi^2(x) = x\}$,

$T(G) = \{x \in G : \varphi(x) \leq x\}$,

which are called the *BCC-part* ([3]), the *group-like* part and *T-part* of G, respectively.

Theorem 2.7. [6]In a weak BCC-algebra G the following conditions are equivalent:

(a)*a is an atom,*

(b)$a = \varphi(x)$,

(c)$\varphi^2(a) = a$,

(d)$\varphi(xa) = ax$,

(e)$\varphi^2(ax) = ax$,

(f)*ax is an atom,*

(g)*$y \leq z$ implies $ay = az$,*

where x, y, z are arbitrary elements of G.

Definition 2.8. [6] A non-empty subset I of a weak BCC-algebra(or BCK/BCI-algebra) G is called an *ideal* of G if

(i) $0 \in I$

(ii) $xy \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in G$

Definition 2.9. [6] A non-empty subset I of a weak BCC-algebra G is called an *a weak BCC-ideal* of G if

(i) $0 \in I$

(ii) $xy.z \in I$ and $y \in I$ imply $xz \in I$ for all $x, y, z \in G$.

Putting $z = 0$ in (ii) we see that a weak BCC-ideal is an ideal and the following lemma is true.

Lemma 2.10. If I is a weak BCC-ideal of a weak BCC-algebra G ,then $xy \in I$ and $y \in I$ imply $x \in I$. In particular, $x \leq y$, $y \in I$ imply $x \in I$.

Using definition(2.1)$(viii)$ and (ix), it is not difficult to see that $B(0) = Ker\varphi$ is a weak BCC-ideal of each weak BCC-algebra. The relation $\sim$ defined by

$$x \sim y \iff xy, yx \in B(0)$$

is congruence on G. Its equivalence classes coincide with branches of G ,i.e, $B(a)$=C_a for any $a \in I(G)$ (cf. [7]). So, $B(a)B(b) = B(ab)$ and $xy \in B(ab)$ for $x \in B(a)$, $y \in B(b)$.

In the following part,we will need two propositions proved in [7].

Proposition 2.11. Elements $x, y \in G$ are in the same branch if and only if $xy \in B(0)$.

Proposition 2.12. If $x, y \in B(a)$, then also $x.xy$ and $y.yx$ are in $B(a)$.

One of important classes of weak BCC-algebras is the class of group-like weak BCC-algebras called also *anti-grouped BZ-algebras* [23] ,i.e, weak BCC-algebras containing only one-element branches.

The conditions under which a weak BCC-algebra is group-like are found in [7] and [23]. Below we present some of these conditions.

Theorem 2.13.A weak BCC-algebra G is group-like if and only if at least one of the following conditions is satisfied:

(1)$\varphi^2(x) = x$ for all $x \in G$,

(2)$\varphi(xy) = yx$ for all $x, y \in G$,

(3)$Ker\varphi = \{0\}$.

Definition 2.14.[20] Let G be a weak BCC-algebra. A map $d : G \longrightarrow G$ is called a *left-right derivation (briefly, (l,r)-derivation)* of G , if it satisfies the identity $d(xy) = d(x)y \wedge xd(y)$,where $x \wedge y$ means $y.yx$.

If d satisfies the identity $d(xy) = xd(y) \wedge d(x)y$, then it is called a *right-left derivation (briefly, (r,l)-derivation)* of G. A map d which is both a (l,r) and a (r,l)-derivation is called a *derivation.* Any d with the property $d(0) = 0$ is called *regular.*

3. On Right Derivations of Weak BCC-algebras

Definition 3.1. Let G be a weak BCC-algebra and a map $d : G \longrightarrow G$ is said to be right derivation of G , if it satisfies the identity $d(xy) = d(x)y \wedge d(y)x$ for all $x, y \in G$.

Example 3.2. Let G be a proper weak BCC-algebra with table as follows.

.	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

Table 3.1

Define a map $d_1 : G \longrightarrow G$ for all in G by

$$d_1(x) = \begin{cases} 0, & x = 0,1 \\ 2, & x = 2,3 \end{cases}$$

Then d_1 is a right derivation of G.

Define a map $d_2 : G \longrightarrow G$ for all in G by

$$d_2(x) = \begin{cases} 2, & x = 0,1 \\ 0, & x = 2,3 \end{cases}$$

Then d_2 is a right derivation of G.

Define a map $d_3 : G \longrightarrow G$ for all in G by $d_3(x) = 0$

Then d_3 is not a right derivation of G since $d(32) = 0 \neq 2 = d(3)2 \wedge d(2)3$

Definition 3.3. Let G be a weak BCC-algebra and a map $d : G \longrightarrow G$ is a right derivation of G. If $d(0) = 0$,then d is called *regular.*

Example 3.4. Let G and d_1,d_2 be as in Example(3.2). Then obviously d_1 is regular, but d_2 is not regular.

Proposition 3.5. A right derivation d of a weak BCC-algebra G is regular if and only if for every x in G , $d(x) \leq x$.

Proof. Let d be a regular right derivation of a weak BCC-algebra G. Then from definition(2.1) (ii) and (iii) we get

$0 = d(0) = d(xx) = d(x)x \wedge d(x)x$

$= d(x)x.(d(x)x.d(x)x)$

$= d(x)x.0$
$= d(x)x.$

Thus we obtain $d(x) \leq x$ for all $x \in G$.

Conversely , let be $d(x) \leq x$ for all $x \in G$, then in particular $d(0) \leq 0$. Thus $0 = d(0).0 = d(0)$,i.e.,d is regular.

Corollary 3.6. Let G be a group-like weak BCC-algebra and d be a right derivation of G.If d is regular, then $d(x) = x$ for every $x \in G$.

Proof. Let d be a right derivation of G. We have $d(x) \leq x$ for all x in G by Proposition(3.5). By Definition(2.1)(ii) and (ix), we obtain $0 = xx \wedge xd(x)$. From this we get $0 \leq xd(x)$ and $0.xd(x) = 0 = \varphi(xd(x))$. Thus we find $xd(x) \in Ker\varphi$ and so $xd(x) = 0$ since $Ker\varphi = 0$ by Theorem (2.13).From this $x \leq d(x)$. Thus we obtain $d(x) = x$ by Definition(2.1)(iv) and the proof is completed.

Theorem 3.7. A right derivation d of a weak BCC-algebra G is regular if and only if for every $x \in G$ elements x and $d(x)$ belongs to the same branch.

Proof. Let d be a regular right derivation of a weak BCC-algebra G. Since d is regular, $d(x) \leq x$. From Definition(2.1) (ix) and (ii) , $0 = xx \leq xd(x)$ and thus we find $xd(x) \in B(0)$. This, according to Proposition (2.11) ,shows that x and $d(x)$ are in the same branch.

Conversely, let elements x and $d(x)$ be in the same branch for every $x \in G$. Then $d(x)x \in B(0)$ and from this $0 \leq d(x)x$ and $0.d(x)x = 0$.

Hence we get

$$\begin{aligned} d(0) &= d(0.d(x)x) = d(0).d(x)x \wedge d(d(x)x).0 \\ &= d(0).d(x)x \wedge d(d(x)x) \\ &= d(0).d(x)x \wedge (d(d(x))x \wedge d(x)d(x)) \\ &= d(0).d(x)x \wedge (d(d(x))x \wedge 0) \\ &= d(0).d(x)x \wedge (0.0(d(d(x))x)) \\ &= d(0).d(x)x \wedge \varphi(0.(d(d(x))x)). \end{aligned}$$

Since by hypothesis x and $d(x)$ are in the same branch and $d(x)$ and $d(d(x))$ are in the same branch , this gives that x and $d(d(x))$ are in the same branch. According to Proposition(2.11) $d(d(x))x \in B(0)$. Thus we obtain $0 \leq d(d(x))x$ and $0.d(d(x)x) = 0$.

By Theorem (2.2) we can find that

$$\begin{aligned} d(0) &= d(0).d(x)x \wedge \varphi(0.(d(d(x))x)) \\ &= d(0).d(x)x \wedge \varphi(0) \\ &= d(0).d(x)x \wedge 0 \\ &= 0(0(d(0).d(x)x)) \\ &= \varphi^2(d(0).d(x)x) \\ &= \varphi^2(d(0)).\varphi^2(d(x)x) \\ &= \varphi^2(d(0)).\varphi(0.d(x)x) \\ &= \varphi^2(d(0)).\varphi(0) \\ &= \varphi^2(d(0)).0 \\ &= \varphi^2(d(0)). \end{aligned}$$

By Theorem (2.3) we obtain $d(0) \in I(G)$. Since by hypothesis 0 and $d(0)$ are in the same branch , it gives that $0d(0) \in B(0)$,i.e, $0 \leq 0d(0)$. From this we find that $0 = 0(0d(0)) = \varphi^2(d(0)) = d(0)$. Hence we get $d(0) = 0$,i.e., d is regular.

Corollary 3.8. A right derivation d of a weak BCC-algebra G is regular if and only if for every x in G, x and $d(x)$ belongs to the same branch.

Corollary 3.9. A right derivation d of a weak BCC-algebra G is regular if and only if for every a in $I(G)$, $d(a) = a$.

Proof. Let be d a regular right derivation of G.By Proposition (3.5), $d(a) \leq a$ for every $a \in G$ and by Theorem(3.7) a and $d(a)$ are in the same branch. Since $a \in B(a)$, we obtain that $d(a) \in B(a)$. Then we have $a \leq d(a)$.
Thus we can find that $a = d(a)$ by Definition(2.1)(iv).
Conversely , we suppose that $a = d(a)$ for every $a \in I(G)$. Now, we'll prove that d is regular. Firstly we will show that $a \in I(G)$. Let be $x \leq 0$ for every $x \in G$. Hence $x.0 = 0$ and $x = 0$ by definition(2.1)(iii). Thus we obtain $0 \in I(G)$. Since $0 \in I(G)$, this gives that $d(0) = 0$. Hence d is regular.

Corollary 3.10. Let G be a weak BCC-algebra and d be any right derivation of G. Then $d(B(a)) \subset B(a)$.

Proof. We suppose that $x \in B(a)$. Then $d(x) \in d(B(a))$. Since d is regular , according to Theorem(3.7), x and $d(x)$ are in the same branch. So, this gives that $d(x) \in B(a)$ and we obtain that $d(B(a)) \subset B(a)$.

Corollary 3.11. Let G be a weak BCC-algebra and d be any regular right derivation of G. Then $d(x)y \leq xd(y)$ for every $x, y \in G$.

Proof. Since d is regular, $d(x) \leq x$ and $d(y) \leq y$ for every $x, y \in G$ by Proposition (3.5.). Then We obtain $d(x)y \leq xy$ and $xy \leq xd(y)$ by definition(2.1)$(viii)$ and (ix). Thus it gives that $d(x)y \leq xy \leq xd(y)$,i.e., $d(x)y \leq xd(y)$.

Theorem 3.12. If a weak BCC-algebra G is group-like , then $Kerd = \{0\}$ for any regular right derivation d of G.

Proof. Let G be a group-like. Then firstly we will prove that $B(0) = 0$. We suppose that $x \in B(0)$. Then $0 \leq x$ and from this $0.0x = 0.0 = 0$,i.e, $\varphi^2(x) = 0$.Since G is group-like ,$\varphi^2(x) = x = 0$,i.e., $B(0) = \{0\}$.
Suppose that $x \in Kerd$. According to Proposition(3.5) , since d is regular $d(x) = 0 \leq x$.So , $x \in B(0) = \{0\}$. Hence $x = 0$. Then $Kerd = \{0\}$ and the proof is completed.

Proposition 3.13. Let d be a right derivation of a weak BCC-algebra G.
Then the following holds:

(1)$d(0) \in I(G)$

(2)$d(I(G)) \subset I(G)$

Proof.(1) Suppose that $x \leq d(0)$ for every $x \in G$. Then $xd(0) = 0$. Thus we get $d(x)0 \leq xd(0) = 0$ by Corollary(3.11). So, $d(x).0 = d(x) \leq 0$. From this $d(x)0 = 0 = d(x)$. Thus we can find that $d(0)x \leq 0d(x) = 0.0 = 0$ by Corollary (3.11). So, $d(0)x.0 = 0 = d(0)x$. Hence we obtain that $d(0) \leq x$. Then we can find $d(0) = x$ by definition(2.1)(iv) . This gives that $d(0) \in I(G)$.
(2) Suppose that $d(a) \in d(I(G))$ for every $a \in I(G)$. Since d is a right derivation of G , we get

$$\begin{aligned} d(0) &= d(a.a) = d(a)a \wedge d(a)a \\ &= d(a)a.(d(a)a.d(a)a) \\ &= d(a)a.0 \end{aligned}$$

$= d(a)a.$

Then $d(0) = d(a)a \in I(G)$ by (1). By Theorem (2.7.$(f) \Longrightarrow (a)$) , we can find $d(a) \in I(G)$ and the proof is completed.

Definition 3.14. Let G be a weak BCC-algebra and A be a weak BCC-ideal of G and d be a right derivation of G. If $d(A) \subset A$, A is called *d-invariant.*

Theorem 3.15. A right derivation d of a weak BCC-algebra G is regular if and only if all weak BCC-ideals of G are d-invariant.

Proof. Let d be a right derivation of a weak BCC-algebra G and let be $x \in A$ for all weak BCC-ideals A of G. Since d is regular $d(x) \leq x$ according to Proposition(3.5). Thus $d(x)x = 0 \in A$. Since any weak BCC-ideal A is ideal by lemma(2.10), we obtain that $d(x) \in A$. So, $d(A) \subset A$,i.e., according to definition(3.14), all weak BCC-ideals of G are d-invariant.

Conversely , let all weak BCC-ideals A of G be d-invariant . Then $d(A) \subset A$. In particular for $A = \{0\}$, $d(\{0\}) \subset \{0\}$. It is not difficult to see that $d(0) = 0$. So , d is regular.

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