Minimum degree and nowhere-zero 3-flows

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ABSTRACT. Let G be a 2-edge-connected simple graph on n vertices, $n \geq 3$. It is known that if G satisfies that $d(x) \geq n/2$ for every vertex $x \in V(G)$, then G has a nowhere-zero 3-flow with several exceptions. In this paper, we prove that with ten exceptions, all graphs with at most two vertices of degree less than $n/2$ have nowhere-zero 3-flows. More precisely, if G is a 2-edge-connected graph on n vertices, $n \geq 3$, in which at most two vertices have degree less than $n/2$, then G has no nowhere-zero 3-flow if and only if G is one of ten completely described graphs.

Key Words: minimum degree, nowhere-zero 3-flow
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1 Introduction

The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For $xy \in E(G)$, we call y a *neighbor* of x, and the set of neighbors of x in G is denoted by $N_G(x)$, or simply $N(x)$. Let H be a subgraph of G and $v \in V(G)$, define that $d_H(v) = |N(v) \cap V(H)|$, the number of the neighbors of v in H. When $H = G$, $d_G(v)$ is called the *degree* of v, and abbreviated to $d(v)$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. For a subgraph A, $d(A)$ denotes the number of edges with exact one end in A.

An edge is *contracted* if it is deleted and its two ends are identified into

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a single vertex. Let H be a connected subgraph of G. G/H denotes the graph obtained from G by contracting all the edges of H and deleting all the resulting loops. For $S \subseteq V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. When $S = \{v\}$, we simplify this notation to $G - v$. The complete graph on n vertices is denoted by K_n. Denote by K_n^- the graph obtained from K_n by deleting an edge.

A *k-circuit* is a circuit of k vertices. A *wheel* W_k is the graph obtained from a k-circuit by adding a new vertex, called the *center* of the wheel, which is joined to every vertex of the k-circuit. W_k is an *odd (even) wheel* if k is odd (even). For simplicity, A 3-circuit (triangle) on vertices $\{x, y, z\}$ is denoted by xyz.

Let G be a graph with an orientation. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail v, and $E^-(v)$ is the set of non-loop edges with head v. Let $\mathbb{Z}_k$ denote an abelian group of k elements with identity 0. Let f be a function from $E(G)$ to $\mathbb{Z}_k$. Set

$$f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e).$$

f is a *$\mathbb{Z}_k$-flow* in G if $f(v) = 0$ for each vertex $v \in V(G)$. For an edge $e \in E(G)$, we call $f(e)$ the *flow value* of e. The *support* of f is defined by $S(f) = \{e \in E(G) : f(e) \neq 0\}$. f is *nowhere-zero* if $S(f) = E(G)$. It is well known that a graph G has a nowhere-zero $\mathbb{Z}_k$-flow if and only if there is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and $f(v) = 0$ for each $v \in V(G)$, which is called a *nowhere-zero k-flow* in G. Therefore, we also call a $\mathbb{Z}_k$-flow a *k-flow*. Tutte [8] conjectured that every 2-edge-connected graph has a nowhere-zero 5-flow. Seymour [7] proved that every 2-edge-connected graph has a nowhere-zero 6-flow. In this paper, we focus on nowhere-zero 3-flow. Since loops play no role with respect to existence of nowhere-zero flows, we only consider loopless graphs. The well-known 3-flow conjecture of Tutte (see unsolved problem 48 of [1]) is that

Conjecture 1.1 *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

A subgraph H of G is *3-flow contractible* if G/H having a nowhere-zero 3-flow implies that G has a nowhere-zero 3-flow. By results in [2, Proposition 2.4 and Observation 1.3], the even wheel W_{2k} is 3-flow contractible for $k \geq 2$. Similarly, by [5, Corollary 3.5] and [2, Observation 1.3], K_n^- and K_n are 3-flow contractible for $n \geq 5$. In summary,

Proposition 1.2 [2, 5] *(i) W_{2k} is 3-flow contractible for $k \geq 2$; (ii) K_n^- and K_n are 3-flow contractible for $n \geq 5$.*

In [3], it is shown that all graphs which satisfies Ore condition have nowhere-zero 3-flows except for six completely described graphs. Precisely,

Proposition 1.3 [3] *Let G be a simple graph on n vertices, $n \geq 3$. If $d(x) + d(y) \geq n$ for each $xy \notin E(G)$, then G has no nowhere-zero 3-flow if and only if G is one of the six graphs $(G_1, G_2, G_7, G_8, G_9, G_{10})$ in Fig. 1.*

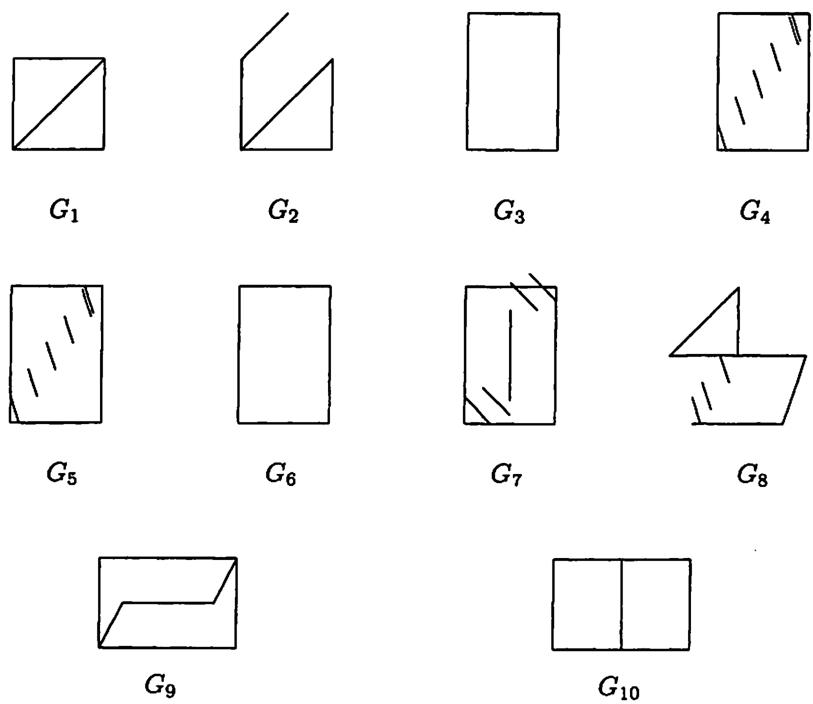

Fig. 1

Proposition 1.4 *None of the ten graphs in Fig. 1 has a nowhere-zero 3-flow.*

Proof. By Proposition 1.3, we need only prove that G_3, G_4, G_5 and G_6 have no nowhere-zero 3-flows. It is not difficult to see that G_3, G_4 and G_5 have no nowhere-zero 3-flows since they are obtained from G_2 by subdividing one edge. Since the subgraph induced by vertices of degree 3 in G_6 contains triangle, it is not bipartite, and thus has no nowhere-zero 3-flows. ■

Let G be a simple graph on n vertices. In [3], it is shown that if G satisfies the Ore-condition [6]: $d(x)+d(y) \geq n$ for every pair of non-adjacent vertices x and y, then G has a nowhere-zero 3-flow with six exceptions. We can directly obtain that if the minimum degree of G is no less than $n/2$, then G also has a nowhere-zero 3-flow with several exceptions. In this paper, we intend to prove the existence of nowhere-zero 3-flow if G has at most two vertices of degree less than $n/2$.

Main Theorem *Let G be a 2-edge-connected simple graph on n vertices, $n \geq 3$. If G has at most two vertices of degree less than $n/2$, then G has no nowhere-zero 3-flow if and only if G is one of the ten graphs in Fig. 1.*

The bound on the number of vertices with degree less than $n/2$ is best possible. For each $n \geq 7$, there exists a simple graph G on n vertices without nowhere-zero 3-flow, in which there are three vertices of degree less than $n/2$. Let G be the graph obtained by joining vertex-disjoint K_3 and K_{n-3} with three independent edges, where $n \geq 7$. Then, $d(x) \geq n/2$ for every $x \in V(G)$ except for three vertices. However, the graph G has no nowhere-zero 3-flow, since the contraction of the K_{n-3} results in a K_4.

As a technique, we introduce the concept of splitting. For a graph G, let v be a vertex of G and e_1, e_2 be two edges incident with v. *Splitting* e_1, e_2 *away from* v means that deleting edges e_1, e_2 and adding an edge e, which joins two ends of e_1, e_2 other than v. We denote the graph obtained from G by splitting e_1, e_2 away from v by $G_{[v;\{e_1,e_2\}]}$. It is proved by Fleischner that property of 2-edge-connectivity can be preserved after splitting. In summary,

Proposition 1.5 (see [4]) *Let G be a 2-edge-connected graph and $v \in V(G)$ with $d(v) \geq 4$. Then there are two edges e_1, e_2 such that $G_{[v;\{e_1,e_2\}]}$ is 2-edge-connected.*

Repeatedly applying Proposition 1.5, if v has degree even, then the graph obtained by splitting v out (splitting all edges incident with v away from v) also remains 2-edge-connected.

2 Lemmas

For a K_4^-, the union of xyz and xyw with xy in common, if it satisfies that $d(z) \geq 4$ or $d(w) \geq 4$, then we say the K_4^- is *desired.*

Lemma 2.1 *Let G be a simple graph on n vertices. If $n \geq 7$ and $\delta(G) \geq n/2$, then either G contains a desired K_4^- or G is bipartite.*

Proof. Clearly, if G contains no triangle, then by Turán's theorem, G is bipartite. Suppose that G contains a triangle xyz. Note that $d(x) + d(y) + d(z) \geq \frac{3}{2}n > n + 3$. Thus, there is a vertex $w \in V(G)$ such that w has at least two neighbors in $\{x, y, z\}$. Without loss of generality, let $x, y \in N(w)$. Then the union of triangles xyz and xyw with xy in common is the desired K_4^-. ∎

Lemma 2.2 *$K_{n,n} - e$ has a nowhere-zero 3-flow, here $e \in E(K_{n,n})$ and $n \geq 4$.*

Proof. Let X and Y be the two parts of $K_{n,n}$ and $X = \{x_1, x_2, \cdots, x_n\}, Y = \{y_1, y_2, \cdots, y_n\}$. Without loss of generality, let $e = x_1y_1$. If n is even , let G' be the graph obtained from G by deleting $n-2$ vertices $x_3, x_4, \cdots, x_n$ and adding $n-2$ copies of edges $y_1y_2, y_3y_4, \cdots, y_{n-1}y_n$. Then by consecutively contracting 2-circuits, which are 3-flow contractible, we get a simple graph G^*. It is easy to see that G^* is K_1. Obviously, K_1 has a nowhere-zero 3-flow, and hence G also has a nowhere-zero 3-flow. If n is odd, let G' be the graph obtained from G by deleting 2 vertices x_1, y_1 and adding $n-1$ edges $x_2x_3, x_4x_5, \cdots, x_{n-1}x_n, y_2y_3, y_4y_5, \cdots, y_{n-1}y_n$. It is not difficult to see that G' can be decomposed into a bipartite cubic graph, which is known to have a nowhere-zero 3-flow, and a set of 2-factors (each has a nowhere-zero 2-flow), and combining these flows yields a nowhere-zero 3-flow in G'. This implies that there is a nowhere-zero 3-flow in G. ∎

Lemma 2.3 *Let G be a 2-edge-connected simple graph on n ($n \leq 7$) vertices. If G has at most two vertices of degree less than $n/2$, then G has a nowhere-zero 3-flow if and only if G is not one of ten graphs in Fig. 1.*

Proof. If G is one of ten graphs in Fig. 1, by Proposition 1.4, G has no nowhere-zero 3-flow. Conversely, suppose that G has no nowhere-zero 3-flow, we are to prove that G is one of graph in Fig. 1. If $n \le 4$, it is not difficult to see that G is K_4, which is G_1 in Fig. 1. From now on, we assume that $n \ge 5$.

(i) $n = 5$. If $\delta(G) = 2$, let $w \in V(G)$ with $d(w) = 2$ and $N(w) = \{u_1, u_2\}$. Let G' be the graph obtained from G by deleting vertex w and adding edge u_1u_2. If G' has a nowhere-zero 3-flow, then G also has a nowhere-zero 3-flow, it is contrary to the hypothesis. Thus, G' has no nowhere-zero 3-flow, which implies that G' is K_4, and hence, G is G_2 in Fig. 1. Suppose that $\delta(G) \ge 3$. Since n is odd, there is a vertex u such that $d(u) = 4$. If each vertex $v \in V(G) \setminus \{u\}$ has degree 3, then G is a W_4 centered at u; If there is a vertex $v \in V(G) \setminus \{u\}$ such that $d(v) = 4$, then G is K_5 or K_5^-. Either in the former case or in the later case, G has a nowhere-zero 3-flow, a contradiction.

(ii) $n = 6$. If $\delta(G) = 2$, let w be a vertex of degree 2 in G and $N(w) = \{u_1, u_2\}$. Let G' be the graph obtained from G by deleting vertex w and adding edge u_1u_2. By the similar method used in case (i), if G' is simple, then G' is G_2 in Fig. 1, and hence, G is G_3, G_4 or G_5 in Fig. 1. If G' is not simple, let G^* denote the graph obtained from G' by contracting the 2-circuit. If G^* is not simple, then G^* has a nowhere-zero 3-flow, which implies that G has a nowhere-zero 3-flow, a contradiction. If G^* is simple, then G^* is K_4, and G is G_6 in Fig. 1. Thus we may assume that $\delta(G) \ge 3$. Then G satisfies Ore condition that $d(x) + d(y) \ge n$ for every pair of non-adjacent vertices x and y, and by Proposition 1.3, G is G_7, G_8, G_9 or G_{10} in Fig. 1.

(iii) $n = 7$. Since n is odd, there is at least one vertex u such that $d(u)$ is even. If $d(u) = 2$, let $N(u) = \{u_1, u_2\}$ and G' be the graph obtained from G by deleting vertex u and adding edge u_1u_2. If G' is simple, then G is one of $G_i (3 \le i \le 10)$ in Fig. 1. But $G_i (3 \le i \le 10)$ has at least four vertices of degree 3, which implies that G has at least four vertices of degree less than $n/2$, a contradiction. Thus, G' is not simple. Denote by G^* the simple graph obtained from G by consecutively contracting 2-circuits, and denote by u^* the new vertex generated by contraction. By the hypothesis, G^* has no nowhere-zero 3-flow, this implies that G^* is G_1 or G_2 in Fig. 1. Note that all vertices, except for u^*, of G^* has the same degree as in G. However, both G_1 and G_2 have four vertices of degree 3, also a contradiction. Thus,

$d(u) \geq 4$. By splitting the vertex u out, we get a 2-edge-connected graph G' by Proposition 1.5. If G' is simple, then G' is one of G_i $(3 \leq i \leq 10)$ in Fig. 1. But the number of vertices with degree 3 gives the contradiction. So, G' is not simple. If G' contains only one 2-circuit, then we get a graph G^* by contracting that 2-circuit. Since G^* has no nowhere-zero 3-flow, G^* is G_1 or G_2 in Fig. 1. It is easy to see that all vertices, except for one vertex, of G^* have the same degree as in G, contrary to the hypothesis that at most two vertices have degree less than $n/2$. If G' has two 2-circuits, let G^* be the graph obtained from G' by contracting these two circuits and let u^*, v^* be the new vertices generated by contraction. Since G^* has no nowhere-zero 3-flow, G^* is K_4 and $V(G^*) \setminus \{u^*, v^*\}$ are vertices of degree less than $n/2$ in the original graph G. Suppose that two 2-circuits in G' are on $\{u_1, u_2\}$ and $\{v_1, v_2\}$, respectively. From $d_{G^*}(u^*) = d_{G^*}(v^*) = 3$, we can get that

$$d_G(u_1) + d_G(u_2) = 3 + 4 = 7, d_G(v_1) + d_G(v_2) = 3 + 4 = 7,$$

which implies that there are at least two vertices of degree 3 in $\{u_1, u_2, v_1, v_2\}$. Together with two vertices in $V(G^*) \setminus \{u^*, v^*\}$ give four vertices of degree less than $n/2$ in G, a contradiction. This completes the proof of Lemma 2.3.
■

Lemma 2.4 *Let G be a 2-edge-connected simple graph on n vertices. If $n \geq 8$ and G has at most 2 vertices of degree less than $n/2$, then G has a nowhere-zero 3-flow or G contains a desired K_4^-.*

Proof. Suppose first that G contains no desired K_4^-. What remains is to show that G has a nowhere-zero 3-flow. By Lemma 2.1, we need only consider the following two cases.

(i) G has two vertices of degree less than $n/2$.

Suppose that x and y are vertices of degree less than $n/2$. Let Z be the subgraph induced by $N(x) \cap N(y)$ and $W = G - (N(x) \cup N(y) \cup \{x, y\})$. Denote the subgraph induced by $N(x) \setminus V(Z)$ and $N(y) \setminus V(Z)$ by X and Y, respectively. Suppose that $Z \neq \emptyset$. If $W = \emptyset$, then, by the hypothesis that $d(x) < n/2$, $d(y) < n/2$, we get $|V(Z)| = 1$ and $d(x) = \frac{n-1}{2}, d(y) = \frac{n-1}{2}$. In this case, n is odd. If there is a path of length 3 (three vertices) in $N(x)$ or $N(y)$, then there is a desired K_4^-, a contradiction. Thus there is no path of length 3 in $N(x)$ and $N(y)$, which means that $d(z) \leq 4$, contrary to that $n \geq 9$. Therefore we may suppose that $W \neq \emptyset$.

Claim 1. There is no edge in W.

Suppose to the contrary that there is an edge $w_1w_2 \in E(W)$. Since $d(w_1) + d(w_2) \geq n$ and $x, y \notin N(w_1) \cap N(w_2)$, we have that $|N(w_1) \cap N(w_2)| \geq 2$. Let $u, v \in N(w_1) \cap N(w_2)$. Then the union of w_1w_2u and w_1w_2v is the desired K_4^-, a contradiction.

Claim 2. There is no edge in Z.

Suppose to the contrary that there is an edge $z_1z_2 \in E(Z)$. If $d(x) \geq 4$ or $d(y) \geq 4$, then the union of z_1z_2x and z_1z_2y is the desired K_4^-, a contradiction. Thus, $d(x) \leq 3$ and $d(y) \leq 3$, which implies that $|X|+|Y|+|Z| \leq 4$. For a vertex $w \in V(W)$, if $wz_1, wz_2 \in E(G)$, then the desired K_4^- (union of z_1z_2x and z_1z_2w) with $d(w) \geq 4$ gives a contradiction. So, $z_1w \notin E(G)$ or $z_2w \notin E(G)$. By Claim 1, we have $d(w) \leq 3$, contrary to the hypothesis that $d(w) \geq n/2 \geq 4$.

Claim 3. There is no edge in X and there is no edge in Y.

If there is an edge, say x_1x_2, in X, then for any $z \in V(Z)$, we have that $zx_1 \notin E(G)$ and $zx_2 \notin E(G)$. Otherwise, let $zx_1 \in E(G)$. Then the union of xx_1x_2 and xx_1z forms a desired K_4^- with $d(z) \geq 4$, a contradiction. Since $d(x_1) + d(x_2) \geq n$ and $Z \cap (N(x_1) \cup N(x_2)) = \emptyset$ we have that $|N(x_1) \cap N(x_2)| \geq n - |V(G) \setminus (V(Z) \cup \{y\})| \geq 2$. Let u be the vertex in $N(x_1) \cap N(x_2)$ other than x, we can get a desired K_4^- (the union of x_1x_2x and x_1x_2u) with $d(u) \geq 4$, a contradiction. Therefore, there is no edge in X. Similarly, there is no edge in Y.

Claim 4. $d_X(z) \leq 1$ and $d_Y(z) \leq 1$ for each $z \in V(Z)$.

Suppose to the contrary that $d_X(z) > 1$ for some $z \in V(Z)$. Let $zx_1, zx_2 \in E(G)$. Then the union of zxx_1 and zxx_2 gives a desired K_4^-, contrary to the hypothesis. Thus, $d_X(z) \leq 1$ for any $z \in V(Z)$. Similarly, we have $d_Y(z) \leq 1$ for any $z \in V(Z)$.

Claim 5. $d_Z(x') \leq 1$ and $d_Z(y') \leq 1$ for each $x' \in V(X)$ and $y' \in V(Y)$, respectively.

The proof is similar to that of Claim 4, we omit the details.

Assume that there is a vertex $z \in V(Z)$ such that $d_X(z) = 1$ and $d_Y(z) = 1$. Let $x_1z, y_1z \in E(G)$. By Claim 4 and Claim 2, $d_W(z) \geq n/2 - 4$. Thus, $|V(W)| \geq n/2 - 4$. Further, by Claim 1, we have that $|V(X)|+|V(Y)|+|V(Z)| \geq n/2$. Suppose first that there is $w \in V(W)$ such that $wz \in E(G)$. If $wx_1 \in E(G)$ or $wy_1 \in E(G)$, then we get a desired K_4^-, a contradiction. So, $wx_1, wy_1 \notin E(G)$, which implies that $|V(X)|+|V(Y)|+|V(Z)| \geq n/2+2$. Thus, $|V(W)| = n/2-4$ and $|V(X)|+|V(Y)|+|V(Z)| = n/2+2$. Since $d(z) \geq n/2$ and z has four neighbors in $V(G) \setminus V(W)$, all vertices in W are neighbors of z, that is, $wz \in E(G)$ for each $w \in V(W)$. Then $N(x_1) \cap W = \emptyset$. Note that $x_1y_1 \notin E(G)$ and $d(x_1) \geq n/2$. Then $|V(Y)| \geq n/2-1$ by Claim 3 and Claim 5. Similarly, $|V(X)| \geq n/2-1$. Combining these two inequations, we have that $|V(X)|+|V(Y)| \geq n-2$. But $|V(X)|+|V(Y)| = n/2-|V(Z)|+2 \leq n/2+1$, a contradiction. Thus, we may suppose that $wz \notin E(G)$ for each $w \in V(W)$. Then $d(z) = 4$, by the hypothesis, we have that $n = 8$. Since $d(x) \leq 3$ and $d(y) \leq 3$, it is derived from $d(w) \geq 4 (w \in V(W))$ that $|V(X)| = 2, |V(Y)| = 2$ and $|V(Z)| = 1$. Then $|V(W)| = 1$ and $N(w) = V(X) \cup V(Y)$, here $w = V(W)$. Suppose that $x_2 \in V(X) \setminus \{x_1\}$. If x_2 has degree at least 2 in $N(w)$, then there is a desired K_4^-, a contradiction. Thus, x_2 has degree at most 1 in $N(w) = V(X) \cup V(Y)$, which implies that $d(x_2) \leq 3 < n/2$, contrary to the hypothesis that $d(x_2) \geq n/2$.

Suppose, then, that there is a vertex $z \in V(Z)$ such that $d_X(z) = 1$ and $d_Y(z) = 0$. Let $zx_1 \in E(G)$. Using the similar method, if there is $w \in V(W)$ such that $zw \in E(G)$, then $|V(W)| = n/2-3$ and $|V(X)|+|V(Y)|+|V(Z)| = n/2+1$. Thus, for each $w \in V(W)$, we have that $x_1w \notin E(G)$. By Claim 3, we have that $n/2 \leq d(x_1) \leq |V(Y)|+2$, which means that $|V(Y)| \geq n/2-2$ and $|V(X)|+|V(Z)| \leq 3$. Similarly, by the hypothesis that $d(y) < n/2$, we can get that $|V(Z)| = 1$, $|V(X)| = 2$ and $|V(Y)| = n/2-2$. Let $x_2 \in V(x) \setminus \{x_1\}$. It is not difficult to see that $d_Y(x_2) \geq 2$ and $d_W(x_2) \geq 1$ because of $d(x_2) \geq n/2$. Let $y_1, y_2 \in V(Y)$ satisfies that $x_2y_1, x_2y_2 \in E(G)$. Then the desired K_4^- (union of wx_2y_1 and wx_2y_2) yields a contradiction. Therefore, $wz \notin E(G)$ for every $w \in V(W)$. By Claim 2 and Claim 4, we have that $d(z) = 3 < n/2$, contrary to the hypothesis. By exchanging X and Y, the case of $d_Y(z) = 1$ and $d_X(z) = 0$ cannot occur.

Thus, we assume that $d_X(z) = 0$ and $d_Y(z) = 0$ for each $z \in V(Z)$. Since $d(z) \geq n/2$, we have that $|V(W)| \geq n/2-2$. Note that $d(w) \geq n/2$ for each $w \in V(W)$. By Claim 1, we have that $|V(X)|+|V(Y)|+|V(Z)| \geq n/2$, which implies that $|V(W)| = n/2-2$ and $|V(X)|+|V(Y)|+|V(Z)| = n/2$. Thus, $N(w) = V(X) \cup V(Y) \cup V(Z)$ for each $w \in V(W)$. If there is $x' \in V(X), y' \in V(Y)$ such that $x'y' \in E(G)$, then by the fact that $W \subset N(x') \cap N(y')$, we get a desired K_4^-, a contradiction. Thus, there is no edge between X and Y. If $V(Y) \neq \emptyset$, then by Claim 3, $d(y') \leq |V(W)| + 1 = n/2-1$ for each $y' \in V(Y)$, impossible. So, $V(Y) = \emptyset$. Similarly, $V(X) = \emptyset$. By Claim 1 and Claim 2, $|V(W)| = n/2 - 2$ and $|V(Z)| = n/2$, contrary to the hypothesis that $d(x) < n/2$.

Therefore, we assume that $V(Z) = \emptyset$. Let $G' = G - \{x, y\}$ and $n' = |V(G')| = n - 2$. Then, $d_{G'}(u) \geq n'/2 = n/2 - 1$ for each $u \in V(G')$. If $n \geq 9$, then, by Lemma 2.1, either G' has a desired K_4^- or G' is bipartite. In the former case, the desired K_4^- in G' is also a desired K_4^- in G, a contradiction. In the later case, if G contains a triangle, then G has a desired K_4^-, a contradiction; if G contains no triangle, then G is $K_{m,m} - xy$, here $m = n/2$. Then G has a nowhere-zero 3-flow by Lemma 2.2. If $n = 8$ and $\delta(G) = 2$, then we get a graph G'' by splitting the degree 2 vertex out. If G'' is simple, by Lemma 2.3, G'', and hence G, has a nowhere-zero 3-flow. If G'' is not simple, Let G^* be the simple graph obtained from G'' by consecutively contracting 2-circuits. If G^* has a nowhere-zero 3-flow, then G'', and also G, has a nowhere-zero 3-flow. Thus, we suppose that G^* has no nowhere-zero 3-flow. By Lemma 2.3, G^* is one of graphs in Fig. 1. Note that all vertices, except for two vertices, in G^* have degree at least 4. But for each graph in Fig. 1, there are at least four vertices of degree less than 4, a contradiction. So, we may assume that $n = 8$ and $\delta(G) \geq 3$. By the hypothesis, both x and y have degree 3 in G. If there is a triangle in G', then it is not difficult to see that there is a desired K_4^- in G, a contradiction. So, there is no triangle in G', then G' is bipartite. If G contains a triangle, then G also contains a desired K_4^-, a contradiction. Thus, there is no triangle in G. In this case, G is $K_{4,4} - xy$. It derives from Lemma 2.2 that G has a nowhere-zero 3-flow.

(ii) G has only one vertex, say x, with degree less than $n/2$.

Let $G' = G - x$ and $|V(G')| = n'$. If n is odd, then $d_{G'}(u) \geq n'/2$ for each $u \in V(G')$. By Lemma 2.1, G' is bipartite or G' contains a desired K_4^-. If G' contains a desired K_4^-, then so does G, contrary to the assumption. Thus G' is bipartite and $d_{G'}(u) = \frac{n'}{2}$ for each $u \in V(G')$. By the hypothesis that all vertices, except for x, have degree at least $\frac{n'}{2} + 1$, we have that $d(x) = n' = n - 1$, contrary to that $d(x) < n/2$. So, n is even. Let $A = \{u \in V(G') : d_{G'}(u) \geq \frac{n'+1}{2}\}$ and $B = \{u \in V(G') : d_{G'}(u) \leq \frac{n'-1}{2}\}$. Since $d(x) < n/2$, we have that $|B| \leq \frac{n'-1}{2}$, which implies that $|A| \geq \frac{n'+1}{2}$. Further, $d_{G'}(b) = \frac{n'-1}{2}$ for each $b \in B$. It is easy to see that there exist two vertices $u, v \in A$ such that $uv \in E(G)$. Note that $d_{G'}(u) + d_{G'}(v) \geq n' + 1$.

Thus, there is a vertex $w \in V(G')$ such that $w \in N(u) \cap N(v)$. Then

$$d_{G'}(u) + d_{G'}(v) + d_{G'}(w) \geq 2 \cdot \frac{n'+1}{2} + \frac{n'-1}{2} = \frac{3}{2}n' + \frac{1}{2}.$$

It is not difficult to see that there exists a vertex $z \in V(G)$, which has at least two neighbors in $\{u, v, w\}$. Then the desired K_4^- contained in the subgraph induced by $\{u, v, w, z\}$ gives a contradiction. We complete the proof of Lemma 2.4. ■

3 Proof of Main Theorem

Proof of Main Theorem: If G is one of the ten graphs in Fig. 1, then, by Proposition 1.4, G has no nowhere-zero 3-flow. Conversely, suppose that G is not any of the ten graphs in Fig. 1. We shall prove that G has a nowhere-zero 3-flow. By the 2-edge-connectivity of G, we have that $\delta(G) \geq 2$.

We use induction on $n = |V(G)|$. If $n \leq 7$, the theorem holds by Lemma 2.3. Suppose then $n \geq 8$ and the theorem holds for any graph $\overline{G}$ with $|V(\overline{G})| < n$. By Lemma 2.4, we may assume that G contains a desired K_4^-. Consider the K_4^- as the union of two triangles xyz and xyw, with edge xy in common. Without loss of generality, we assume that $d(z) \geq 4$. Let G' be the graph obtained from G by deleting zx, zy, and adding xy. We claim that G' is 2-edge-connected or G has a nowhere-zero 3-flow. Otherwise, suppose that G' is disconnected or G' has an edge cut e. We shall to prove that G has a nowhere-zero 3-flow. Let C_1, C_2 be two components of G' or $G' - e$. Without loss of generality, suppose $z \in V(C_1)$ and $x, y, w \in V(C_2)$. If $|V(C_2)| \geq 4$, then, by the hypothesis that there are at most two vertices of degree less than $n/2$, there is a vertex in $V(C_2)$ which has degree at least $n/2$. Then $|V(C_2)| \geq n/2$. If there is a vertex u in $V(C_1) \setminus \{z\}$ such that $d(u) \geq n/2$, then $|V(C_1)| \geq n/2 + 1$, which makes $|V(G)| = |V(C_1)| + |V(C_2)| \geq n + 1$, a contradiction. Thus, all vertices in C_1, except for z, have degree less than $n/2$. It means that $|V(C_1)| \leq 3$. By the 2-edge-connectivity of G, $|V(C_1)| = 3$ and C_1 is a 3-circuit. Let a be the vertex of degree 2 in C_1 and G'' the graph obtained from G by deleting a and adding an edge between the remained vertices of C_1. By consecutively contracting the 2-circuits in G'', we get a simple graph G^*. Then G^* has at most one vertex of degree less than $n/2$. By the induction hypothesis, G^* has a nowhere-zero 3-flow or G^* is one of ten graphs in Fig. 1. In the former case, the nowhere-zero 3-flow on G^* can be extend to a nowhere-zero 3-flow on G; in the later case, since there are at least four vertices of degree less than 4 in each graphs in Fig. 1, contrary to the fact that only one vertex may has degree less than 4 in G^*. If $|V(C_2)| = 3$, then $d(w) = 2$ or $d(w) = 3$. If $d(w) = 3$, then $d(x) = d(y) = 3$, which implies that $n \leq 6$, a contradiction. Thus $d(w) = 2$ and at least one of $\{x, y\}$ has degree 3. Without loss of generality, we may assume that $d(x) = 3$. Then $d(y) = 4$ and $n = 8$. Let G'' be the graph obtained from G by deleting w and adding an edge xy. By consecutively contracting the 2-circuits in G'', we get a simple graph G^* on at most 5 vertices. It is easy to see that G^* has a nowhere-zero 3-flow since all vertices, except for one, of G^* have degree at least 4. So far, we complete the proof of the claim.

In the following argument, we can assume that G' is 2-edge-connected. Note that, in G', we have a 2-circuit on $\{x, y\}$.

Technique 1: Let G^1 be the graph obtained from G' by contracting the 2-circuit on $\{x, y\}$ into a single vertex u^*. Then, G^1 has a 2-circuit on $\{u^*, w\}$. Let G^2 be the graph obtained from G^1 by contracting the 2-circuit on $\{u^*, w\}$, and for convenience, the resulting new vertex is still denoted by u^*. If there is 2-circuit containing u^*, we continue to contract the 2-circuit into u^*, and denote the resulting graph by G^3. Keep going this way until no 2-circuit exists. Let $G^1, G^2, \cdots, G^t$ be a sequence of graphs obtained in the above-described way of contraction 2-circuits. So, for each i, G^i is obtained from G^{i-1} by contracting a 2-circuit into u^*, $1 \leq i \leq t$, where $G^0 = G'$. Note that G^t is a simple graph, in which all vertices, except for u^* and z, have the same degree as in G. Since G^t is obtained from G' by consecutively contracting 2-circuits, if G^t has a nowhere-zero 3-flow, then so does G', and hence G has a nowhere-zero 3-flow. Moreover, there is a subgraph H such that $G^t = G'/H$. Let $|V(G^t)| = n^*$. Since $t \geq 2$, we have that $n^* \leq n - 2$. If $n^* \leq 3$, then G^t has a nowhere-zero 3-flow, which implies that G', and so G, has a nowhere-zero 3-flow. Thus, assume that $n^* \geq 4$ and G^t has no nowhere-zero 3-flow. By the hypothesis, G^t has at most four vertices (z, u^* together with two original vertices of degree less than n/2 in G) of degree less than $n/2$.

For convenience, we consider the following three cases independently.

(i) $n = 8$. If G has a vertex of even degree (including degree of 2), then by splitting this vertex out, we get a 2-edge-connected graph G'' according to Proposition 1.5. If G'' is simple, then $|V(G'')| = 7$ and G'' has at most two vertices of degree less than $n/2$. By the argument in Lemma 2.3 case (iii), G'', and hence G, has a nowhere-zero 3-flow. Thus suppose that G'' is not simple. Let G^* be the graph obtained from G'' by consecutively contracting 2-circuits. Therefore, G^* is one of graphs in Fig. 1. Using the similar method in the proof of Lemma 2.3 case (iii), we have that this case cannot occur. Thus, we assume that all vertices of G have odd degrees.

By applying the Technique 1 described above, we get a graph G^t. Note that $d(z) \geq 5$ and $n^* \leq n-2$. So, $d_{G^t}(z) \geq n^*/2$. If none of two vertices which have degree less than $n/2$ in G is contained in H, then $d_{G^t}(u^*) \geq 5(n-n^*+1)-(n-n^*)(n-n^*+1)-2 = (n^*-3)(9-n^*)-2$. Note that $4 \leq n^* \leq 6$. So $d_{G^t}(u^*) \geq n^*/2$. Suppose that at least one of two vertices which have degree less than $n/2$ in G is contained in H. Then G^t has at most two vertices which have degree less than $n^*/2$. Note that G^t has at most four vertices of degree less than $n/2 = 4$. Thus, by hypothesis, G^t is G_1, G_9 or G_{10} in Fig. 1. If G^t is K_4, then $|V(H)| = 5$. Let H' be the subgraph obtained from H by deleting one edge of 2-circuit on $\{x, y\}$. If H' is K_5, then G/H' is not simple because of $d_{H'}(z) \geq 2$. Moreover, $|V(G/H')| = 4$, then G/H' has a nowhere-zero 3-flow. By Proposition 1.2, H' is 3-flow contractible, hence G has a nowhere-zero 3-flow. Thus, suppose that H' is not K_5. Note that G^t has four vertices of degree 3, in which two vertices have degree less than $n/2$ in the original graph G. Then $d_{G^t}(u^*) = 3$ and $d(a) \geq 5$ for each $a \in V(H)$. However, $d_{G^t}(u^*) = d(H') \geq 5+2-2 = 5$, a contradiction. If G^t is G_9 or G_{10}, then $|V(H)| = 3$. But $d(x) \geq 5, d(y) \geq 5$ and $d(w) \geq 5$, which implies that $d_{G^t}(u^*) \geq 7$, a contradiction.

(ii) $n = 9$. Apply Technique 1 to G. If $d(z) = 4$, then there is a vertex of degree 2 in G^t. By deleting this vertex and adding an edge connecting its two neighbors, we get a graph G^* on n^*-1 vertices. If G^* has a nowhere-zero 3-flow, then G has a nowhere-zero 3-flow. So, G^* has no nowhere-zero 3-flow. Note that there are at most two vertices of degree less than 5 in G^*. Then $n^* = 7$. It is not difficult to see that G^* is K_6^-, which has a nowhere-zero 3-flow, a contradiction. Thus, $d(z) \geq 5$ and $n^* \leq n-2$. Then $d_{G^t}(z) \geq 3$. We intend to prove that G^t has at most two vertices of degree less than $n^*/2$. If $n^* \leq 6$, then $d_{G^t}(z) \geq n^*/2$. If one vertex of degree less than $n/2$ in G is contained in H, then G^t has at most two vertices of degree less than $n^*/2$. Suppose then none of vertices of degree less than $n/2$ is contained in H. Thus, $d_{G^t}(u^*) \geq 5(n-n^*+1)-(n-n^*)(n-n^*+1)-2 =$

$(n^* - 3)(9 - n^*) - 2$. When $n^* \leq 6$, $d_{G^t}(u^*) \geq n^*/2$, and G^t has at most two vertices of degree less than $n^*/2$. Thus, we need only consider the case $n^* = 7$. Then $|V(H)| = 3$. If all vertices of H have degree at least 5, then $d_{G^t}(u^*) = 5 \times 3 - 6 - 2 = 7 > n^* - 1$, this is impossible. If all vertices of G with degree less than $n/2$ are contained in H, then G^t has at most two vertices of degree less than $n^*/2$. Suppose then H contains only one vertex of degree less than $n/2$ in G. In this case, $d_{G^t}(u^*) \geq 5 \times 2 - 4 - 2 = 4 \geq n^*/2$. Also, G^t has at most two vertices of degree less than $n^*/2$. Note that G^t has at most four vertices of degree less than $n/2$. If G^t has a nowhere-zero 3-flow, then G also has a nowhere-zero 3-flow. Suppose then G^t has no nowhere-zero 3-flow. By Lemma 2.3, we have that G^t is G_1 or G_{10}. If G^t is K_4, then $|V(H)| = 6$. Since $d_{G^t}(u^*) = 3$ and degrees of vertices in H are no less than 5, we have that

$$\Sigma_{v \in V(H)} d_H(v) \geq 5 \times 6 - 3 = 27.$$

Note that H has only one 2-circuit on $\{x, y\}$. So, there is a vertex $v \in V(H)$ such that $d_H(v) \geq 5$. By the 2-edge-connectivity of H, we have that $d_H(v') \geq 2$ for each $v' \in V(H)$. Let H' be the subgraph of G obtained from H by deleting one edge of 2-circuit on $\{x, y\}$. If H' contains an even wheel centered at v, then by Proposition 1.2, H' is 3-flow contractible. Then G/H' is a 2-edge-connected graph on 4 vertices. Since $d_{H'}(z) \geq 2$ ($zx, zy \in E(H')$), we have that G/H' is not simple, which means that G/H' has a nowhere-zero 3-flow, and hence G has a nowhere-zero 3-flow. Therefore, assume that the subgraph induced by $N_H(v)$ contains no even circuit, that is, each block of subgraph induced by $N_H(v)$ is an edge or an odd circuit. Since $d_{G^t}(u^*) = 3$, the minimum degree of subgraph induced by $N_H(v)$ is at least 2 and the maximum degree is at least 3. It is not difficult to see that the subgraph induced by $N_H(v)$ is two triangles with one vertex in common. But, $\Sigma_{v \in V(H)} d_H(v) = 5 + 5 + 3 \times 4 + 2 = 24$, this means that $d_{G^t}(u^*) = d(H) \geq 5 \times 6 - 24 = 6$, a contradiction. If $|V(G^t)| = 6$, then $|V(H)| = 4$ and G^t is G_{10}. It is easy to calculate that $d_{G^t}(u^*) = d(H) \geq 5 \times 4 - 3 \times 4 - 2 = 6$. However, G_{10} has four vertices of degree 3, which means that degree of u^* in G^t must be 3, a contradiction.

(iii) $n = 10$. Apply Technique 1 to G. If $d(z) = 4$, then there is a vertex of degree 2 in G^t. By deleting this vertex and adding an edge connecting its two neighbors, we get a graph G^* on $n^* - 1$ vertices. If G^* has a nowhere-zero 3-flow, then G has a nowhere-zero 3-flow. So, assume that G^* has no nowhere-zero 3-flow. Note that there are at most two vertices of degree less than 5 in G^*. If G^* is simple, then by case (ii), G^* has a nowhere-zero 3-flow, and so does G. If G^* is not simple, then by consecutively contracting 2-circuits in G^*, we get a simple graph G'. Note that $|V(G')| \leq 6$. Since G' has at most two vertices of degree less 5, G' has a nowhere-zero 3-flow, a contradiction. Suppose then $d(z) \geq 5$. We intend to prove that G^t has at least two vertices of degree less than $n^*/2$. Since $d(z) \geq 5$, $d_{G^t}(z) \geq 3$. If $d_{G^t}(z) < \frac{n^*}{2}$, then $7 \leq n^* \leq 8$ and $d_{G^t}(z) = 3$, which means that $|V(H)| \leq 4$. Note that G contains at most two vertices, say a, b, of degree less than $n/2$. If $a \in H$ or $b \in H$, then $d_{G^t}(u^*) \geq 4 > n^*/2$. In these cases, G^t has at most two vertices of degree less than $n^*/2$. Then we consider the case that $a, b \notin V(H)$. If $d_{G^t}(a) \geq 4$ or $d_{G^t}(b) \geq 4$, then G^t has at most two vertices of degree less than $n^*/2$. Thus, we may assume that $d_{G^t}(a) \leq 3$ and $d_{G^t}(b) \leq 3$. Let $X = \{v : d(v) \geq 5, v \in V(G^t) \setminus \{u^*\}\}$. Thus, $3 \leq |X| \leq 4$. If $|X| = 3$, then $n^* = 7$ and $d_{G^t}(u^*) \geq 5\times4-3\times4-2 = 6$. If the subgraph induced by X is a circuit, then one vertex of $\{z, a, b\}$ has at least two neighbor in X. Thus G^t contains an even wheel centered at u^*. By contracting this even wheel, we have that G^t, and then G, has a nowhere-zero 3-flow. Suppose then the subgraph induced by X is not a circuit. Thus $d_{G^t}(X) \geq 4+4+3 = 11$. So, $e(X, \{z, a, b\}) \geq 8$. Since $d_{G^t}(u^*) = 6$ and $d_{G^t}(z) = 3$, there is one vertex in $\{a, b\}$ which has degree more than 3 in G^t. So, G^t has at most two vertices of degree less than $n^*/2$. If $|X| = 4$, then $n^* = 8$ and $d_{G^t}(u^*) = 7$. Similarly, suppose that the subgraph induced by X contains no even circuit. Then $d_{G^t}(X) \geq 3+3+2+4 = 12$. So, $e(X, \{z, a, b\}) \geq 8$. Since $d_{G^t}(u^*) = 7$ and $d_{G^t}(z) = 3$, at least one of $\{a, b\}$ has degree more than 3 in G^t. So, G^t has at most two vertices of degree less than $n^*/2$. If $d_{G^t}(z) \geq n^*/2$, then we need only to prove that $d_{G^t}(u^*) \geq n^*/2$. If $|V(H)| \leq 5$, then it is not difficult to see that $d_{G^t}(u^*) \geq n^*/2$. What remains is to prove the cases $|V(H)| = 6$ and $|V(H)| = 7$. If $|V(H)| = 6$, then there is a vertex $a \in V(G) \setminus V(H)$ such that $d(a) \geq 5$, which means that a has at least two neighbors in $V(H)$, contrary to the simplicity of G^t. If $|V(H)| = 7$, then G^t is K_4. Let H' be subgraph of G obtained from H by deleting one edge of 2-circuit on $\{x, y\}$. Then H' is simple and 2-edge-connected. If H' is 3-flow contractible, then G/H' is a 2-edge-connected graph on 4 vertices and not simple (zx and zy form a 2-circuit in G/H'). It is clearly that G/H' has a nowhere-zero 3-flow, and also, G has a nowhere-zero 3-flow. Thus, we assume that H' is not 3-flow contractible. Since $d_{G^t}(u^*) = 3$, there are at least two vertices, say u, v, of H' such that $d_{H'}(u) \geq 5$ and $d_{H'}(v) \geq 5$, respectively. Suppose first that $uv \in E(G)$. Let S be the subgraph induced by $N_{H'}(u) \cap N_{H'}(v)$. Then $|V(S)| \geq 3$. If there is $s \in V(S)$ such that $d_S(s) \geq 2$, then there is an even wheel centered at u, which implies that H' is 3-flow contractible, a contradiction. So, $d_S(s) \leq 1$ for each $s \in V(S)$. If $|V(S)| = 3$, let $u_1 \in N_{H'}(u) \setminus V(S), v_1 \in N_{H'}(v) \setminus V(S)$. If $d_S(u_1) \geq 2$, then subgraph induced by $N(u) \cup \{u\}$ contains an even wheel, also gives a contradiction. Thus $d_S(u_1) \leq 1$. By the 2-edge-connectivity of H', we have that $d_S(u_1) = 1$. Similarly, $d_S(v_1) = 1$. Then $d_{H'}(u_1) \leq 2, d_{H'}(v_1) \leq 2$ and $d_H(s) \leq 4$ for each $s \in V(S)$. In this case, $d_{G^t}(u^*) \geq 2 + 2 + 3 - 2 = 5$. But in G^t, u^* has degree 3, a contradiction. Suppose that $|V(S)| = 4$. If $d_{H'}(u) = 6$, then by the similar method used above, we have a contradiction. Thus, we assume that $|V(S)| = 4$ and $d_{H'}(u) = d_{H'}(v) = 5$. Let $u_1 \in V(H') \setminus (V(S) \cup \{u, v\})$. Since $d_S(s) \leq 1$ for each $s \in V(S)$ and $d_{G^t}(u^*) = 3$, we have that $u_1 s \in E(H')$ and $d_S(s) = 1$ for each $s \in V(S)$. Let G'' be the graph obtained from G by deleting edges us_1, us_2 and adding edge $s_1 s_2$, here $s_1, s_2 \in V(S)$ and $s_1 s_2 \in E(G)$. Then by consecutively contracting 2-circuits, we get a simple graph G^*. Since z has at least two neighbors in $V(H)$. we have that $|V(G^*)| \leq 3$, which means that G has a nowhere-zero 3-flow. If $|V(S)| = 5$, then $d_{H'}(s) \leq 3$, and so $d_{G^t}(u^*) \geq 2 \times 5 - 2 = 8$, impossible. Therefore, assume that $uv \notin E(G)$. Then $N_H(u) = N_H(v)$. Let S denote the subgraph induced by $N_H(u)$. Then $|V(S)| = 5$. If there is $s \in V(S)$ such that $d_S(s) \geq 3$, then it is the case $d(u) \geq 5, d(s) \geq 5$ and $us \in E(G)$, which we have discussed above. Thus, suppose that $d_S(s) \leq 2$ for each $s \in V(S)$. Note that there is no even wheel in H'. Then S is a 5-circuit. Let $s_1, s_2 \in V(S)$ and $s_1 s_2 \in E(G)$. Let G'' be the graph obtained from G by deleting edges us_1, us_2 and adding edge $s_1 s_2$. By consecutively contracting 2-circuits, we get a simple graph G^*. It is easy to see that $|V(G^*)| \leq 3$. Thus, G^*, and also G, has a nowhere-zero 3-flow. Therefore, we suppose that $|V(H)| \leq 6$. By the hypothesis and G^t has at most four vertices of degree less than 5, we have that G^t is G_{10} in Fig. 1. Let H' be the subgraph of G obtained from H by deleting one edge of 2-circuit on $\{x, y\}$. Note that $d_{G^t}(u^*) = 3$. Then H' is K_5, which is 3-flow contractible. Thus G/H' is not simple (z has two neighbors in $V(H')$). It is not difficult to see that G has a nowhere-zero 3-flow by consecutively contracting the 2-circuits in G/H'.

From now on, we may suppose that $n \geq 11$. After applying Technique 1, we have a simple graph G^t.

Claim 1. G^t has at most two vertices of degree less than $n^*/2$ or G^t has a nowhere-zero 3-flow.

Suppose that there are two vertices, say a and b, in $V(G)$ with degree less than $n/2$. If $a, b \in V(H)$, then G^t has at most two vertices u^* and z of degree less than $n^*/2$. If $a, b \notin V(H)$, then consider the following two cases.

(i) $V(G) \setminus (V(H) \cup \{a, b, z\}) \neq \emptyset$. Let $v \in V(G) \setminus (V(H) \cup \{a, b, z\})$ and $|V(H)| = h$. Since $d_H(v) \leq 1$ and $d(v) \geq n/2$, we have that $h \leq n/2$. If $h \geq 5$, then $d_{G^t}(z) = d_G(z) - 2 \geq n^*/2$. Further, $d_{G^t}(u^*) \geq h \cdot (n/2 - h + 1) - 2 \geq n^*/2$. So, the vertices of G^t, which possibly have degree less than $n^*/2$, are in $\{a, b\}$, and Claim 1 holds for this case. Thus, we assume that $h \leq 4$. If $h = 3$, then $d(x) + d(y) + d(w) \geq \frac{3}{2}n$, which implies that $d_{G^t}(u^*) \geq \frac{3}{2}n - 8$. On the other hand, $d_{G^t}(u^*) \leq n^* - 1 = n - 3$. Thus, we have that $n - 3 \geq d_{G^t}(u^*) \geq \frac{3}{2}n - 8$, that is, $n \leq 10$, contrary to the hypothesis that $n \geq 11$. If $h = 4$, then by the similar calculation, we have that $d_{G^t}(u^*) \geq 2n - 14$. But $d_{G^t}(u^*) \leq n^* - 1 = n - 4$. Combining these two inequations, we get $n \leq 10$, also a contradiction.

(ii) $V(G) \setminus (V(H) \cup \{a, b, z\}) = \emptyset$. Then $n^* = 4$ and G^t is K_4. Notice that $d_{G^t}(z) = 3$. Thus $d_G(z) = 3 + 2 = 5$, which implies that $n \leq 10$, a contradiction.

Suppose then only one of a, b is contained in $V(H)$. Without loss of generality, let $a \in V(H)$. If $|V(H)| \geq 5$, then $d_{G^t}(z) = d_G(z) - 2 \geq n^*/2$, and hence the vertices in G^t, which possibly have degree less than $n^*/2$, are in $\{u^*, b\}$. In this case, Claim 1 holds. Thus, we need only consider two cases $|V(H) = 3|$ and $|V(H)| = 4$. If $|V(H)| = 3$, then $d_{G^t}(u^*) \geq d(x) + d(y) + d(a) - 8 \geq n - 6$. Note that $n^* = n - 2$ and $n > 10$. Then $d_{G^t}(u^*) \geq n^*/2$. If $|V(H)| = 4$, then $d_{G^t}(u^*) \geq \frac{3}{2}n - 11$. Note that $n^* = n - 3$ and $n > 10$. Then $d_{G^t}(u^*) \geq n^*/2$. In either case, the vertices in G^t, which possibly have degree less than $n^*/2$, are in $\{z, b\}$, and we are done.

If G has one vertex of degree less than $n/2$, then we need only consider two cases $|V(H)| = 3$ and $|V(H)| = 4$. By the similar analysis, we can easily prove that Claim 1 holds for these two cases.

If G has no vertex of degree less than $n/2$, then the vertices in G^t, which possibly have degree less than $n^*/2$, are in $\{u^*, z\}$. We complete the proof of Claim 1.

By Claim 1 and induction hypothesis, either G^t has a nowhere-zero 3-flow or G^t is one of the ten graphs in Fig. 1. In the former case, since G^t is obtained from G' by consecutively contracting 2-circuits, which is 3-flow contractible, we see that G', and so G, has a nowhere-zero 3-flow, and we are done. In the later case, G^t contains at least 4 vertices of degree 3, at most two of which have degree 3 in the original graph G, then $d_{G^t}(z) = 3$. But $d_G(z) = d_{G^t}(z) + 2 = 5 \geq n/2$, which implies that $n \leq 10$, contrary to the hypothesis that $n > 10$. This completes the proof of the Main Theorem. ∎

References

[1] J.A. Bondy and U.S.R Murty, "Graph Theory with Applications", North-Holland, New York, 1976.

[2] M. DeVos, R. Xu and G. Yu, Nowhere-zero $\mathbb{Z}_3$-flows through $\mathbb{Z}_3$-connectivity, *Discrete Mathematics*, **306**(2006), 26-30.

[3] G.H. Fan and C.X. Zhou, Ore condition and nowhere-zero 3-flows, *SIAM Discrete Mathematics*, **22**(2008), 288-294.

[4] H. Fleischner, Eulerian graphs and related topics, Part 1, Vol. 2 *Ann. Discrete Mathematics*, Vol. 50, North-Holland, (1990).

[5] H.-J. Lai, Group connectivity of 3-edge-connected chordal graphs, *Graphs and Combinatorics*, **16**(2000), 165-176.

[6] O. Ore, Note on Hamilton circuits, *Amer. math. Monthly*, **27**(1960), 55.

[7] P. D. Seymour, Nowhere-zero 6-flows, *J. Combin. Theory Ser. B*, **30**(1981), 130-155.

[8] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.*, **6**(1954), 80-91.