# The yin–yang structure of the affine plane of order four

Stefano Innamorati(1) and Daniela Tondini(2)

(1)Dipartimento di Ingegneria Elettrica e dell'Informazione
Università de L'Aquila
Via G. Gronchi, 18
I-67100 L'Aquila
e-mail: stefano.innamorati@univaq.it

(2)Dipartimento di Scienze della Comunicazione
Università di Teramo
Coste Sant'Agostino
I-64100 Teramo
e-mail: dtondini@unite.it

**Abstract.** In this paper we show the yin–yang structure of the affine plane of order four by characterizing the unique blocking set as the Möbius–Kantor configuration $8_3$.

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## 1. Introduction

Configuration is one of the oldest combinatorial structure, since it appeared for the first time in 1876 in the second edition of Theodor Reye's book *Geometrie der Lage* [10]. A $(v_r,b_k)$ *configuration* $C=(X,B)$ is a pair where $X$ is set of $v$ points and $B$ is a family of $b$ subsets called *lines*, with $k$ points on each line and $r$ lines through each point. Two different lines intersect each other at most once and two different points are connected by a line at most once. By definition it easily follows that the parameters of a configuration $(v_r,b_k)$ must satisfy $vr=bk$ and $v\geq r(k-1)+1$, see [2], [4], [5]. If $v=b$ and hence $r=k$, then the configuration is *symmetric* and denoted by $v_k$. Symmetric configurations $v_2$ are regular graphs with equally many vertices and edges and so $v$–cycles. Configurations $v_3$ exist if and only if $v\geq 7$. The Fano configuration, i.e. the projective plane of order two, is the unique configuration $7_3$. In 1881 S. Kantor, proved the uniqueness of the configuration $8_3$, see [7] and [11].

A *blocking b–set* of $C$ is a set $B$ of $b$ points such that any line contains a point of $B$ and a point outside $B$.
An *affine plane of order q* is a $((q^2)_{q+1},(q^2+q)_q)$ configuration such that:

1. Any two distinct points belong to exactly one line.
2. If the point $P$ is not on the line $l$, then there is a unique line on $P$ missing $l$.

The aim of this paper is to look at the affine plane of order 4 from a new point of view, see [8]. We shall show that AG(2,4) contains, up to isomorphism, exactly one blocking set which is the unique Möbius–Kantor configuration $8_3$. The blocking set with its complementary set leads to a yin–yang structure of the affine plane of order four.
Let $K$ denote a $k$–set in AG(2,$q$). We recall that the *characters* of $K$ are the numbers $t_i=t_i(K)$ of lines meeting $K$ in exactly $i$ points, $0\leq i\leq q$. The set $K$ is called *of class* $[m_1,m_2,\ldots,m_h]$ if $t_i\neq 0 \Rightarrow i\in\{m_1,m_2,\ldots,m_h\}$. We prove the following

Theorem.- AG(2,4) *contains, up to isomorphism, exactly one blocking set which is the unique Möbius–Kantor configuration* $8_3$.

## 2. The proof of the Theorem

Let $B$ denote a blocking $b$–set of AG(2,4). The set $B$ is of class [1,2,3]. By [1] and [6], $b\geq 7$.
By counting in double way the number of lines, the number of pairs $(P,l)$, where $P\in B$ and $l$ is a line through $P$, and the number of pairs $(\{P,Q\},l)$, where $\{P,Q\}\subset B$ and $l$ is a line through $P$ and $Q$, we get the following equations on the integers $t_i$,

$$\begin{cases} t_1+t_2+t_3=20 \\ t_1+2t_2+3t_3=5b \\ 2t_2+6t_3=b(b-1) \end{cases}.$$

If $b$=7, we get

$$\begin{cases} t_1+t_2+t_3=20 \\ t_1+2t_2+3t_3=35 \\ 2t_2+6t_3=42 \end{cases}$$

which gives $t_1$=11, $t_2$=3, $t_3$=6. Consider the configuration formed by the six 3–secant lines.
Two 3–secant lines intersect in a point of $B$. Indeed, if two 3–secant lines meet in a point outside $B$, say P, each of the other three lines through P must contain another point of $B$, so $b\geq 9$, a contradiction.
Four 3–secant lines cannot contain a same point of $B$, otherwise $b\geq 9$.

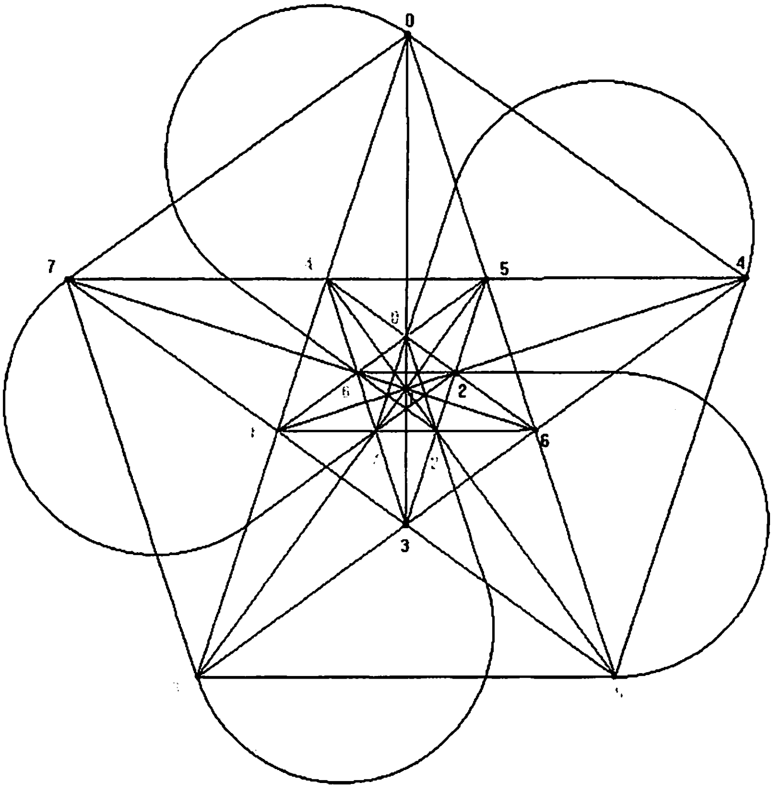
We call 2–points and 3–points the points of $B$ on exactly two and three 3–secant lines, respectively.
Any 3–secant line contains exactly one 2–point and exactly two 3–points.
Let us denote by $x$ and $y$ the number 2–points and 3–points of $B$, respectively. Counting in two different ways the number of points of $B$, the number of pairs $(P,r)$, where $P\in B$ and $r$ is a 3–secant line through $P$, we get

$$\begin{cases} x+y=7 \\ 2x+3y=18 \end{cases},$$

which gives $x=3$ and $y=4$.
Therefore the four 3–points form a quadrangle and the three 2–points are its diagonal points. The contradiction follows because in even characteristic the diagonal points of a quadrangle are collinear.
Since a complementary set of a blocking set is a blocking set too, $b\neq 7$ implies that $b\neq 9$.
Therefore $b=8$. We get

$$\begin{cases} t_1+t_2+t_3=20 \\ t_1+2t_2+3t_3=40 \\ 2t_2+6t_3=56 \end{cases}$$

which gives $t_1=8$, $t_2=4$, $t_3=8$. Consider the configuration formed by the eight 3–secant lines.
Two non parallel 3–secant lines intersect in a point of $B$. Indeed, if two 3–secant lines meet in a point outside $B$, say P, each of the other three lines through P must contain another point of $B$, so $b\geq 9$, a contradiction.
Four 3–secant lines cannot contain a same point of $B$, otherwise $b\geq 9$. A 3–secant line meets exactly other six 3–secant lines and has one 3–secant parallel line.
Therefore $B$ with its eight 3–secant lines is a Möbius–Kantor configuration $8_3$.
Now consider the complementary set $B^c$ of $B$ with the eight tangent lines of $B$. Since $B^c$ is also a blocking 8–set of AG(2,4), then $B^c$ is a Möbius–Kantor configuration $8_3$.
The following picture shows the affine plane of order 4, equally partitioned by the two Möbius–Kantor configurations $8_3$, see [9]. We have used the cyclic representation of the Möbius–Kantor configuration $8_3$, in which $X=\{0,1,2,3,4,5,6,7\}$ and the lines are constructed starting from the line $l=\{0,1,3\}$, by adding 1 modulo 8 to each element of $l$, see [3].




The yin-yang structure of AG(2,4).

## 3. Conclusion

The concept of yin–yang is used to describe how polar opposites or seemingly contrary forces are interconnected and interdependent and how they give rise to each other in turn. In this paper we show the yin–yang structure of the affine plane of order four by characterizing the unique blocking set as the Möbius–Kantor configuration $8_3$.

## References

[1] A. E. Brower and A. Schrijver, The blocking number of an affine space, *J. Combin. Theory Ser. A* 24 (1978), 251–253.

[2] H. S. M. Coxeter, Self–dual configurations and regular graphs, *Bull.*

*Amer. Math. Soc.* 56 (1950), 413–455.

[3] H. Gropp, On the existence and non-existence of configuration $n_k$, *Journal of Combinatorics, Information & System Sciences*, 15 (1990), 34-48.

[4] H. Gropp, *Configurations*, The CRC Handbook of Combinatorial designs, C.J.Colbourn J.H.Dinitz Eds, Boca Raton 1996, 253-255.

[5] J. W. P. Hirschfeld, *Projective geometries over finite fields*, Clarendon Press, Oxford, 2nd edition, 1998.

[6] R. Jamison, Covering finite fields with cosets of subspaces, *J. Combin. Theory Ser. A* 22 (1977), 253–266.

[7] S. Kantor, *Die Configurationen* $(3,3)_{10}$ *mit den Indices* 8,9 *etc.*, Sitzb. Der Wiener Akad. 1881.

[8] C. F. Laywine A derivation of an affine plane of order 4 from a triangle-free 3-colored $K_{16}$, *Discrete Math.* 235 (2001), 165-171.

[9] B. Polster, *A geometrical picture book*, Springer-Verlag New York, Inc., 1998.

[10] T. Reye, *Geometrie der Lage I*, C. Rumpler, Hannover, 2 Aufl., 1876.

[11] E. W. Weisstein, Möbius-Kantor Configuration. From *MathWorld*, A Wolfram Web Resource. http://mathworld.wolfram.com/Moebius-KantorConfiguration.html