The $2-(v, 13, 1)$ designs with block transitive automorphism *

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Abstract

This article is a contribution to the study of the automorphism groups of $2-(v, k, 1)$ designs. Let $\mathcal{D}$ be $2-(v, 13, 1)$ design and suppose that G is a group of automorphisms of $\mathcal{D}$ which is block transitive and point primitive. Then $Soc(G)$, the socle of G, is not isomorphic to ${}^2G_2(q)$ or to ${}^2F_4(q^2)$ for any prime power q.

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1 Introduction

A $2-(v, k, 1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of k−subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v, k, 1)$ design $\mathcal{D}$. Then G is said to be block transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag transitive on $\mathcal{D}$ if G is transitive on the set of flags.

The classification of block transitive $2-(v, 3, 1)$ designs was completed about thirty years ago (see [2]). In [3], Camina and Siemons classified $2-(v, 4, 1)$ designs with a block transitive, solvable group of automorphisms.

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Li classified $2-(v,4,1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [7]). Tong and Li [12] classified $2-(v,5,1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [4] classified $2-(v,5,1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [9] classified $2-(v,k,1)$(where $k=6,7,8,9,10$) designs with a block transitive, solvable group of automorphisms. In [5], Han and Ma classified $2-(v,11,1)$ designs with a block transitive classical simple groups of automorphisms.

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be $2-(v,13,1)$ design, $G \leq Aut(D)$ be block transitive and point primitive. We prove that following theorem.

Main Theorem Let $\mathcal{D}$ be $2-(v,13,1)$ design and suppose that G is a group of automorphisms of $\mathcal{D}$ which is block transitive and point primitive. Then $Soc(G)$, the socle of G, is not isomorphic to ${}^2G_2(q)$ or to ${}^2F_4(q^2)$ for any prime power q.

2 Preliminary Results

Let $\mathcal{D}$ be a $2-(v,k,1)$ design defined on the point set $\mathcal{P}$ and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v,k,1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. Also, G^B denotes the permutation group induced by the action of G_B on the points of B, and so $G^B \cong G_B/G_{(B)}$.

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [11] as subgroups of $GL(7,q)$. Let $GF(q)$ be finite finite field of q elements, where $q=3^{2n+1}$ for some positive integer $n \geq 1$. Set $t=3^{n+1}$ so that $t^2=3q$. We give the following information about subgroups of ${}^2G_2(q)$. For each l dividing $2n+1$, ${}^2G_2(3^l)$ denotes the subgroup of ${}^2G_2(q)$ consisting of all matrices in ${}^2G_2(q)$ with entries in subfield of 3^l. We use the symbols Q and K to note a Sylow 3-subgroup and a cyclic subgroup of order $q-1$ of ${}^2G_2(q)$, respectively.

Lemma 2.1 *([6]) Let $T \leq {}^2G_2(q)$ and T be maximal in ${}^2G_2(q)$. Then either T is conjugate to $P_6(l) = {}^2G_2(3^l)$ for some divisor l of $2n+1$, or T*

is conjugate to one of the subgroups P_i in Table 1.

Table 1

Group	Structure	Remarks
P_1	$Q : K$	*The normaliser of Q in ${}^2G_(q)$*
P_2	$(Z_2^2 \times D_{(q+1)/2}) : Z_3$	*The normaliser of a fours-group*
P_3	$Z_2 \times PSL(2,q)$	*An involution centraliser*
P_4	$Z_{q+t+1} : Z_6$	*The normaliser of Z_{q+t+1}*
P_5	$Z_{q-t+1} : Z_6$	*The normaliser of Z_{q-t+1}*

Lemma 2.2 *([10]) Let $G = {}^2F_4(q^2)$, $q^2 = 2^{2n+1}$, $n \geq 1$. Then every maximal subgroup of G is conjugate to one of the following:*

(1) $P_1 = [q^{22}] : (PSL(2,q^2) \times (q^2-1))$;
(2) $P_2 = [q^{20}] : ({}^2B_2(q^2) \times (q^2-1))$;
(3) $SU(3,q^2) : Z_2$;
(4) $(Z_{q^2+1} \times Z_{q^2+1}) : GL(2,3)$;
(5) $(Z_{q^2-\sqrt{2}q+1} \times Z_{q^2-\sqrt{2}q+1}) : [96]$, if $q^2 > 8$;
(6) $(Z_{q^2+\sqrt{2}q+1} \times Z_{q^2+\sqrt{2}q+1}) : [96]$;
(7) $(Z_{q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1}) : Z_{12}$;
(8) $(Z_{q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1}) : Z_{12}$;
(9) $PGU(3,q^2) : Z_2$;
(10) ${}^2B_2(q^2) : Z_2$;
(11) $B_2(q^2) : Z_2$;
(12) ${}^2F_4(q_0^2)$, where $q_0^2 = 2^{2m+1}$ and $\frac{2n+1}{2m+1}$ be prime.

Lemma 2.3 *([8]) Let $T = T(q)$ be an exceptional simple group of Lie type over $GF(q)$, and let G be a group with $T \trianglelefteq G \leq Aut(T)$. Suppose that M is a maximal subgroup of G not containing T. Then one of the following holds:*

(1) $|M| < q^k|G:T|$, where q^k is defined in Table 2;
(2) $T \cap M$ is a parabolic subgroup of T.
(3) $T \cap M$ is as Table 2.

Table 2

T	q^k	$T \cap M$	condition
${}^2G_2(q)$	q^3	*none*	$q = 3^{2m+1} \geq 27$
${}^2F_4(q^2)$	q^{24}	$L(3,3) : Z_2$ $L(2,25)$	$q = 2$

Lemma 2.4 *([5]) Let G and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq Aut(\mathcal{D})$ be block transitive, point-primitive but not flag-transitive. Let $Soc(G) = T$. Then*

$$|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G:T|,$$

where $\alpha \in \mathcal{P}$, λ is the length of the longest suborbit of G on $\mathcal{P}$.

3 Proof of the Main Theorem

Proposition 3.1 *Let $\mathcal{D}$ be $2-(v,13,1)$ design, G be block transitive, point primitive but not flag transitive. Then $v = 156b_2 + 1$.*

Proof. Let $b_1 = (b,v)$, $b_2 = (b,v-1)$, $k_1 = (k,v)$, $k_2 = (k,v-1)$. Obviously,

$$k = k_1k_2,\ b = b_1b_2,\ r = b_2k_2,\ \ v = b_1k_1.$$

Since $k = 13$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [8], G is flag transitive, a contradiction. Thus $v = k(k-1)b_2 + 1 = 156b_2 + 1$.

Proposition 3.2 *Let $\mathcal{D}$ be $2-(v,13,1)$ design, G be block transitive, point primitive but not flag transitive and $|T|$ be even. Then $|T| \leq 79|T_\alpha|^2|G:T|$.*

Proof. Let $B = \{1,2,\cdots,13\} \in \mathcal{B}$. Then the structure of G^B, the rank and subdegree of G do not occur:

Type of G^B	Rank of G	Subdegree of G
$\langle 1 \rangle$	157	$1,\overbrace{b_2,\cdots,b_2}^{156}$

Otherwise, $|G^B| = 1$ is odd and $|B| = 13$. We have $|G_B|$ and b_2 are odd. Since $v = 156b_2 + 1 = b_1$ and $b = b_1b_2$, then b is odd and $|G| = b|G_B|$ is also odd, a contradiction with $|T|$ be even. Thus $\lambda \geq 2b_2$. By Lemma 2.4 and Proposition 3.1,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda}\cdot|G:T| \leq \frac{v}{2b_2}\cdot|G:T| \leq \frac{156b_2+1}{2b_2}\cdot|G:T| \leq 79|G:T|.$$

Now we may prove our main theorem.

Suppose that $Soc(G) = {}^2G_2(q) = T$. Then ${}^2G_2(q) \trianglelefteq G \leq Aut({}^2G_2(q))$. We have $G = T:\langle x\rangle$, where $x \in Out(T)$, the outer automorphisms group of T which may be generated by an automorphism of field. We may assume that x is an automorphism of field. Set $o(x) = m$, then $m \mid (2n+1)$. Obviously, $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$. By [1] and $k = 13$, G is not flag transitive. Since G is point primitive, G_α $(\alpha \in \mathcal{P})$ is the maximal subgroup of G, T is block transitive in $\mathcal{D}$. Hence $M = G_\alpha$ satisfies one of the two cases in Lemma 2.3. We will rule out these cases one by one.

Case (1) $|M| < q^3|G:T|$.

By Proposition 3.2, we have an upper bound of $|T|$,

$$|T| < 79|T_\alpha|^2|G:T| < 79q^6|G:T| = 79q^6m.$$

We get

$$q-1 < 79(2n+1).$$

Let $2n+1 = s \geq 3$, then $3^s < 80s$. Thus $s = 3, 5$.

If $s = 3$, then $|^2G_2(3^3)| = 3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37$. Since $v = 156b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.1, $T_\alpha \cong {}^2G_2(3)$, $(Z_2^2 \times D_{(q+1)/2}) : Z_3$ or $Z_2 \times PSL(2,q)$, where $q = 3^3$.

(i) $T_\alpha \cong {}^2G_2(3)$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7} - 1 = 6662330.$$

By Proposition 3.1, $156b_2 = 6662330$, a contradiction.

(ii) $T_\alpha \cong (Z_2^2 \times D_{(q+1)/2}) : Z_3$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3 \cdot 2^3 \cdot 7} - 1 = 59960978.$$

By Proposition 3.1, $156b_2 = 59960978$, a contradiction.

(iii) $T_\alpha \cong Z_2 \times PSL(2,q)$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7 \cdot 13} - 1 = 512486.$$

By Proposition 3.1, $156b_2 = 512486$, a contradiction.

If $s = 5$, then $|^2G_2(3^5)| = 3^{15} \cdot (3^{15}+1)) \cdot (3^5 - 1)$. Since $v = 156b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.1, $T_\alpha \cong {}^2G_2(3)$, $(Z_2^2 \times D_{(q+1)/2}) : Z_3$ or $Z_2 \times PSL(2,q)$, where $q = 3^5$. It is not difficult to exclude them by Proposition 3.1.

Case (2) $T \cap M$ is a parabolic subgroup of T.

By Lemma 2.1, the parabolic subgroup of $^2G_2(q)$ is conjugate to QK. Then the order of parabolic subgroup is $q^3(q-1)$ and $v = q^3 + 1$. By Proposition 3.1, we have $q^3 = v - 1 = 156b_2$ and so $156 \mid q^3$, a contradiction.

Suppose that $Soc(G) = {}^2F_4(q^2) = T$. Then $T \trianglelefteq G \leq Aut(T)$. We have $G = T : \langle x \rangle$, where $x \in Out(T)$, the outer automorphisms group of T which may be generated by an automorphism of field. We may assume that x is an automorphism of field. Set $o(x) = f$, then $f \mid (2n+1)$. Obviously, $|^2F_4(q^2)| = q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)$. By [1] and $k = 13$, G is not flag transitive. Since G is point primitive, G_α ($\alpha \in \mathcal{P}$) is the maximal subgroup of G, T is block transitive in $\mathcal{D}$. Hence $M = G_\alpha$ satisfies one of the three cases in Lemma 2.3. We will rule out these cases one by one.

Case (1) $|M| < q^{24}|G : T|$.

By Proposition 3.2, we have an upper bound of $|T|$,

$$|T| < 79|T_\alpha|^2|G : T| < 79q^{48}f.$$

We get

$$q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1) < 79q^{48}f < 79q^{48}q^2,$$

that is

$$(q-1)^2 < 79.$$

Thus $q^2 = 2^5$ or 2^3 and $T = {}^2F_4(2^5)$ or ${}^2F_4(2^3)$. Since $v = 156b_2+1$ is odd, then $|T_\alpha|$ contains a Sylow 2-subgroup of T. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.2, $T_\alpha \cong T_i$, where $T_i \leq P_i (i = 1, 2)$. Then $v \mid (2^5-1)(2^{15}+1)(2^{20}-1)(2^{30}+1)$ or $v \mid (2^3-1)(2^9+1)(2^{12}-1)(2^{18}+1)$. By Proposition 3.1 and rather long and repetitive numerical calculations, we get a contradiction.

Case (2) $T \cap M = L(3,3) : Z_2$, or $L(2,25)$, if $q^2 = 2$.

Obviously, $|T| = 2^{12}\cdot 3^3 \cdot 5^2 \cdot 13$ and $v = |T : T_\alpha| = 2^6 \cdot 5^2$ or $2^5 \cdot 3$. By Proposition 3.1, we have $156 \mid 2^6 \cdot 5^2$ or $156 \mid 2^5 \cdot 3$, a contradiction.

Case (3) $T \cap M$ is a parabolic subgroup of T.

By Lemma 2.2, the parabolic subgroup of ${}^2F_4(q^2)$ is conjugate to P_1 or P_2. Then the order of parabolic subgroup is $q^{24}(q^2-1)^2(q^2+1)$ or $q^{24}(q^2-1)^2(q^4+1)$. We get $v = (q^4+1)(q^6+1)(q^{12}+1)$ or $(q^2+1)(q^6+1)(q^{12}+1)$. By Proposition 3.1, we have $39 \mid (q^{18}+q^{14}+q^{12}+q^8+q^6+q^2+1)$ or $39 \mid (q^{18}+q^{16}+q^{12}+q^{10}+q^6+q^4+1)$. But $q^{18}+q^{14}+q^{12}+q^8+q^6+q^2+1 \equiv 1 \,(mod\, 3)$ and $q^{18}+q^{16}+q^{12}+q^{10}+q^6+q^4+1 \equiv 1\,(mod\,3)$. Thus $3 \nmid (q^{18}+q^{14}+q^{12}+q^8+q^6+q^2+1)$ or $3 \nmid (q^{18}+q^{16}+q^{12}+q^{10}+q^6+q^4+1)$, a contradiction.

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