Cross recurrence relations for r-Lah numbers

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Abstract

We give a new combinatorial interpretation of Lah and r-Lah numbers. We establish two cross recurrence relations: the first one, which uses an algebraic approach, is a recurrence relation of order two with rational coefficients; The second one uses a combinatorial proof and is a recurrence relation with integer coefficients. We also express r-Lah numbers in terms of Lah numbers. Finally, we give identities related to rising and falling factorial powers.

Keywords. Lah numbers, r-Lah numbers, recurrence relation, combinatorial interpretation.

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1 Introduction

For nonnegative integers n, k, r, the r-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_r$, see for instance [2], count the number of partitions of the set $\{1, 2, ..., n\}$ into k ordered lists with the restriction that the elements $1, 2, \ldots, r$ belong to distinct lists. Note that as for Lah numbers, the r-Lah numbers satisfy for $n > r$, the recurrence relation

$$\left\lfloor {n \atop k} \right\rfloor_r = (n + k - 1) \left\lfloor {n-1 \atop k} \right\rfloor_r + \left\lfloor {n-1 \atop k-1} \right\rfloor_r, \tag{1}$$

with $\left\lfloor {n \atop k} \right\rfloor_r = \delta_{k,r}$ if $n = r$ and $\left\lfloor {n \atop k} \right\rfloor_r = 0$ if $n < r$, where $\delta_{k,r}$ is the Kronecker symbol.

For $r = 1, 2, 3, 4$, we refer to Sloane's EOIS [5]: A008297, A143497, A143498, A143499.

The r-Lah numbers can be interpreted combinatorially as an intermediate situation between the r-Stirling numbers of both kinds. The classical explicit formula for Lah numbers is given by, see [3, 4],

$$\left\lfloor {n \atop k} \right\rfloor = \frac{(n-1)!}{(k-1)!}\binom{n}{k} = \frac{n!}{k!}\binom{n-1}{k-1}. \tag{2}$$

In section two, we give a combinatorial proof to the closed form of the r-Lah numbers. Section three presents two cross recurrence relations for $\left\lfloor {n \atop k} \right\rfloor_r$, one with rational coefficients using an algebraic approach and another one with integer coefficients using a bijective proof. The fourth section is devoted to expressing r-Lah numbers in terms of the classical Lah numbers with integer coefficients. In the last section, we give identities related to rising and falling factorial powers.

2 Counting Lah numbers and r-Lah numbers

Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ count the number of partitions of the set $\{1, 2, ..., n\}$ into k ordered lists. For doing, we select k elements from n, each elements corresponding to the start of one list. This gives $\binom{n}{k}$ possibilities. From the remaining $n-k$ elements, we pick a first element, which is denoted l_1.There are k possibilities to add it to the lists (following each start element list). Notice that l_1 can be considered as a cut point of the list in two parts (before l_1 and following l_1). The second one, denoted l_2 (from the $n-k$ remaining elements) has $(k+1)$ possibilities to belong to the k lists (because l_1 add a new possibility). Notice that l_2 also adds a new part. Now the third element, denoted l_3, has $(k+2)$ possibilities to belong to the k lists, and so on... The last element, denoted l_{n-k}, has $(k+(n-k-1)) = n-1$ possibilities to belong to the k lists. So, $\left\lfloor {n \atop k} \right\rfloor = \binom{n}{k}(k+1)(k+2)\cdots(n-1) = \binom{n}{k}\frac{(n-1)!}{(k-1)!}$. The same approach gives an explicit formula of the r-Lah numbers.

Theorem 1 *An explicit formula of r-Lah numbers is given by*

$$\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!}\binom{n-r}{k-r}, \quad (n \geq k \geq r). \tag{3}$$

Relation (3) can be easily verified by induction. Note that if we replace r by 1, we get the classical explicit formula for Lah numbers. Let us give a combinatorial proof to the above relation.

Proof. We put the numbers $1, 2, ..., r$ in the r first lists (one by list). The $n-r$ elements left are used to complete the $k-r$ empty lists. This can be done in $\binom{n-r}{k-r}$ ways. Each one of these $k-r$ elements is the start of the corresponding list which is not the case for the lists contains elements $1, 2, ..., r$ (we can put an element before and after the element first inserted). Then there are $2r + (k-r) = r + k$ possibilities to insert the next element in the k lists. As for the proof of the classical Lah numbers, the next one has $r + k + 1$ possibilities, and the last one has $r + k + (n-k-1) = r + n - 1$ possibilities. So, $\left\lfloor {n \atop k} \right\rfloor_r = \binom{n-r}{k-r}(r+k)(r+k+1)\cdots(r+n-1) = \frac{(n+r-1)!}{(k+r-1)!}\binom{n-r}{k-r}$. □

3 Cross recurrences for r-Lah numbers

The relation (1) gives a recurrence relation with fixed r. Bellow, we establish a cross recurrence relation with respect to r.

Theorem 2 *The r-Lah numbers satisfy the recurrence relation*

$$\left\lfloor {n \atop k} \right\rfloor_r = (k-r+1)\frac{(n+r-2)}{(k+r-1)}\left\lfloor {n-1 \atop k} \right\rfloor_{r-1} + \frac{(n+r-2)}{(k+r-2)}\left\lfloor {n-1 \atop k-1} \right\rfloor_{r-1}. \quad (4)$$

Proof. From relation (3), we have $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n-r)!}{(k-r)!}\binom{n+r-1}{k+r-1}$, and using Pascal's formula, we get $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n-r)!}{(k-r)!}\left(\binom{n+r-2}{k+r-1} + \binom{n+r-2}{k+r-2}\right)$, thus

$$\begin{aligned}
\left\lfloor {n \atop k} \right\rfloor_r &= \frac{(n-r)!}{(k-r)!}\left(\binom{n+r-2}{k+r-1} + \binom{n+r-2}{k+r-2}\right) \\
&= \frac{(n-r)!}{(k-r)!}\left(\frac{(n+r-2)}{(k+r-1)}\binom{n+r-3}{k+r-2} + \frac{(n+r-2)}{(k+r-2)}\binom{n+r-3}{k+r-3}\right) \\
&= (k-r+1)\frac{(n+r-2)}{(k+r-1)}\left\lfloor {n-1 \atop k} \right\rfloor_{r-1} + \frac{(n+r-2)}{(k+r-2)}\left\lfloor {n-1 \atop k-1} \right\rfloor_{r-1}.
\end{aligned}$$

□

Corollary 2.1 *For $r = 1$, the classical Lah sequence satisfies*

$$\left\lfloor {n \atop k} \right\rfloor = (n-1)\left\lfloor {n-1 \atop k} \right\rfloor + \frac{(n-1)}{(k-1)}\left\lfloor {n-1 \atop k-1} \right\rfloor. \quad (5)$$

The following theorem gives a nice cross recurrence relation for $\left\lfloor {n \atop k} \right\rfloor_r$. The main difference with (4) is that the coefficients are integers.

Theorem 3 *For any nonnegative integers $0 \le r \le k \le n$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{i=0}^{n-k}(i+1)!\binom{n-r}{i}\left\lfloor {n-i-1 \atop k-1} \right\rfloor_{r-1}. \quad (6)$$

Proof. Let us consider k lists. We suppose that the first list contains the element "1" and i $(0 \leq i \leq n-k)$ other elements chosen from the set $\{r+1,\ldots,n\}$. Thus, there are $\binom{n-r}{i}$ ways to choose the i elements, $(i+1)!$ ways to constitute the first list and $\left\lfloor {n-1-i \atop k-1} \right\rfloor_{r-1}$ ways to distribute the remaining $n-i-1$ elements into $k-1$ lists such that $2,3,\ldots,r$ are in distinct lists. We conclude by summing the product of the three terms. □

4 Expression of r-Lah numbers in terms of Lah numbers

Theorem 4 *For any nonnegative integers,* $0 \leq r \leq k \leq n$, *we have*

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{s=0}^{n-k} \sum_{i_1+\cdots+i_r=s} (i_1+1)!\cdots(i_r+1)!\binom{n-r}{i_1,\ldots,i_r,n-r-s}\left\lfloor {n-r-s \atop k-r} \right\rfloor. \tag{7}$$

Proof. Let us consider k lists. The r first lists contain respectively the elements $1,2,\ldots,r$,, and contain also $i_1,\ldots,i_r$ others elements, such that $i_1+\cdots+i_r = s$ $(0 \leq s \leq n-k)$. Then, there are $\binom{n-r}{i_1,i_2,\ldots,i_r,n-r-s} = \binom{n-r}{i_1}\binom{n-r-i_1}{i_2}\cdots\binom{n-r-i_1-\cdots-i_{r-1}}{i_r}$ ways to choose the $i_1,\ldots,i_r$ elements and $(i_1+1)!\cdots(i_r+1)!$ to constitute the r first lists. Now, it remains to distribute the $n-r-s$ elements into $k-r$ lists, which gives $\left\lfloor {n-r-s \atop k-r} \right\rfloor$ possibilities. Consequently, there are $(i_1+1)!\cdots(i_r+1)!\binom{n-r}{i_1,\ldots,i_r,n-r-s}\left\lfloor {n-r-s \atop k-r} \right\rfloor$ ways to constitute the k lists such that the first r lists contain respectively $i_1+1,\ldots,i_r+1$ elements. We conclude by summing. □

5 Relation between r-Lah numbers and rising-falling factorial powers

As an application of the r-Lah numbers, we give a generalized identity considering rising and falling factorial powers. This identity counts the number of ways to put $n+r$ elements on $x+r$ rails such that the first r elements must be on the r first rails respectively.

Theorem 5 *For any nonnegative integers,* $0 \leq r \leq n$, *we have*

$$(x+2r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor {n+r \atop k+r} \right\rfloor_r x^{\underline{k}}, \tag{8}$$

where $x^{\underline{k}} = x\,(x-1)\cdots(x-k+1)$ *and* $x^{\overline{k}} = x\,(x+1)\cdots(x+k-1)$.

Proof. We put the elements $1, 2, \ldots, r$ on the r first rails, also we place the n remaining elements on the $x+r$ rails. Thus we have $x+2r$ ways to place the $(r+1)$-th element, $x+2r+1$ ways to place the $(r+2)$-th element, and so on $x+2r+n-1$ ways to place the $(n+r)$ -th element. This gives $(x+2r)\,(x+2r+1)\cdots(x+2r+n-1) = (x+2r)^{\overline{n}}$ possibilities. For the right hand, for a given k $\;(0 \le k \le n)$, we have to constitute $k+r$ lists with $n+r$ elements to affect all of them in $x+r$ rails such that the r lists containing the r first elements respectively, must be in the r first rails. There are $\left\lfloor{n+r \atop k+r}\right\rfloor_r$ possibilities to constitute such lists. For the k remaining lists, we have x choices for the first one, $x-1$ for the second one, and so on, $x-k+1$ ways for the k-th list, which gives $x\,(x-1)\cdots(x-k+1) = x^{\underline{k}}$ possibilities. □

The following corollary gives us a symmetric version for Theorem 6 and its dual inverse formula.

Corollary 5.1 *For* $0 \le r \le n$, *we have*

$$(x+r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor{n+r \atop k+r}\right\rfloor_r (x-r)^{\underline{k}}, \qquad (9)$$

$$(x-r)^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n+k} \left\lfloor{n+r \atop k+r}\right\rfloor_r (x+r)^{\overline{k}}. \qquad (10)$$

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